# ON PARABOLIC SUBGROUPS OF CHEVALLEY GROUPS OVER LOCAL RINGS 

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(Received October 22, 1974)

Introduction. Let $G$ be a Chevalley-Demazure group scheme associated with a connected complex semi-simple Lie group $G_{C}$ (as for definition, see [1] 1.1), $\Delta$ be the root system associated with $G$ and a maximal torus $T$ of $G$, and $R$ be a commutative ring with a unit. We shall fix a fundamental root system $\Pi$ of $\Delta$ once for all. Denote by $x_{\alpha}(t)$ the unipotent element of $G(R)$ associated with a root $\alpha$ of $\Delta$ and $t \in R$. Let $V(R)$ be the subgroup of $G(R)$ generated by $x_{\alpha}(t)$ for all negative roots $\alpha$ of $\Delta$ and all $t \in R$. Then a subgroup $P$ of $G(R)$ containing $V(R) T(R)$ is called a parabolic subgroup of $G(R)$ associated with $\Pi$. Following J. Tits, it is well known that if $R$ is a field, then the set of parabolic subgroups of $G(R)$ associated with $\Pi$ is lattice isomorphic to the family of subsets of $\Pi$.
N. S. Romanovskii [4] has given a discription of parabolic subgroups of $G L_{n}(R)$ for a local ring $R$. In this note, for a simple ChevalleyDemazure group scheme $G$ and a local ring $R$, we shall give a generalization of the Tits' theorem in the same situation as Romanovskii's result. The main theorem is stated in Section 1, and we shall prove our main theorem in Sections 2 and 3. The author wishes to express his hearty thanks to professor E. Abe for his many helpful comments and encouragement.

## 1. The statement of the main theorem.

1.1. Let $G$ be a Chevalley-Demazure group scheme and $R$ be a commutative ring with a unit. A collection of ideals $\left\{\mathfrak{N}_{\alpha}\right\}_{\alpha \in \Lambda}$ which corresponds bijectively to the set $\Delta$ of roots, is called a carpet of $R$ associated with $\Delta$. Furthermore, a carpet $\left\{\mathfrak{H}_{\alpha}\right\}_{\alpha \in \Delta}$ is called a permissible (resp. semi-permissible) carpet associated with ( $4, \Pi$ ), if the following conditions (1) and (2) (resp. (1) and (2')) are satisfied,
(1) for any roots $\alpha$ and $\beta$ of $\Delta$ such that $\alpha+\beta \in \Delta$

$$
\mathfrak{N}_{\alpha} \mathfrak{T V}_{\beta} \subset \mathfrak{N}_{\alpha+\beta}
$$

(2) for each negative root $\alpha$ of $\Delta, \mathfrak{N}_{\alpha}=R$,
(2') if $\mathfrak{U}_{\alpha}$ is a proper ideal of $R$, then $\mathfrak{Y}_{-\alpha}=R$.
1.2. Assume $G$ is simple. Let $R$ be a local ring, $\mathfrak{M}$ be the maximal ideal of $R, k$ be the residue class field $R / \mathfrak{M}$ and $\operatorname{ch}(k)$ be the characteristic of $k$. We shall set up the following assumptions,
( a ) $c h(k) \neq 2$ for any type of $G$,
(b) if $G$ are of types $A_{3}, B_{m}(m \geqq 2), C_{n}, D_{n}(n \geqq 3)$ and $F_{4}$, then $k \neq F_{3}$ where $F_{3}$ is a field with three elements and if $G$ is of type $G_{2}$, then $\operatorname{ch}(k) \neq 3$. Then our main theorem is the following.
1.3. Theorem. Let G be a simple Chevalley-Demazure group scheme and $R$ be a local ring. Assume $G$ and $R$ satisfy ( $a$ ) and (b) in 1.2. Let $P$ be a parabolic subgroup of $G(R)$ associated with $\Pi$ and denote $\mathfrak{N}_{\alpha}=\left\{t \in R \mid x_{\alpha}(t) \in P\right\}$ for each root $\alpha$ of $\Delta$. Then $\left\{\mathfrak{N}_{\alpha}\right\}_{\alpha \in \Delta}$ is a permissible carpet, and further, the mapping $\Psi: P \rightarrow\left\{\mathcal{R}_{\alpha}\right\}_{\alpha \in \Delta}$ is a bijection of the set of parabolic subgroups of $G(R)$ associated with $\Pi$ onto the set of permissible carpets associated with ( $\Lambda, \Pi$ ).

Remark. If $G$ is not simple, examining the proof of lemma in 2.5 , we can see that, if we assume $\operatorname{ch}(k) \neq 2$ and $\operatorname{ch}(k) \neq 3$ instead of (a) and (b) in 1.2 , our main theorem also holds.

Throughout the following section, let $G$ be a Chevalley-Demazure group scheme, and let $R$ be a local ring and $\mathfrak{M}$ be the maximal ideal of $R$. Denote by $\Delta$ the root system associated with $G$ and a maximal torus $T$ of $G$, by $\Pi$ a system of fundamental roots of $\Delta$, by $\Delta^{+}$(resp. $\Delta^{-}$) the set of positive (resp. negative) roots of $\Delta$. Let $S$ be a closed subset of $\Delta^{+}$and $\Omega=\left\{\mathscr{T}_{\alpha}\right\}_{\alpha \in \Delta}$ be a carpet of $R$ associated with $\Delta$. Then we denote by $U_{S}(\Re)$ the subgroup of $G(R)$ generated by $x_{\alpha}(t)$ for all $t \in \mathbb{N}_{\alpha}, \alpha \in S$. In particular, if $\mathfrak{N}_{\alpha}=\mathfrak{Y}$ for all $\alpha \in S$, we denote $U_{S}(\Omega)$ by $U_{S}(\mathfrak{F l})$, and if $S=\Delta^{+}$, denote $U_{S}(\mathfrak{H})$ by $U(\mathfrak{H})$. In the above notation, replacing $\Delta^{+}$by $\Delta^{-}$, we can construct $V_{S}(\Re), V_{S}(\mathfrak{H})$ and $V(\mathfrak{U})$ which are same as $U_{S}(\Re)$, $U_{S}(\mathfrak{Z})$ and $U(\mathfrak{Z})$ respectively.

## 2. Proof of injectivity.

2.1. Lemma. Let $R$ be a local ring in which 2 is invertible. Let $N$ be a subgroup of $G(R)$ normalized by the maximal torus $T(R)$. Then, for each root $\alpha$ of $\Delta, \mathfrak{Y}_{\alpha}=\left\{t \in R \mid x_{\alpha}(t) \in N\right\}$ is an ideal of $R$.

Proof. Assume $x_{\alpha}(t) \in N$, then it is sufficient to prove $x_{\alpha}(b t) \in N$ for any $b \in R$. Every element $b$ of $R$ can be written in the form

$$
b=\left(\frac{b+1}{2}\right)^{2}-\left(\frac{b-1}{2}\right)^{2}
$$

Thus it is sufficient to show that $x_{\alpha}\left(a^{2} t\right) \in N$ for any $a \in R$. If $a$ is invertible, setting $w_{\alpha}(\alpha)=x_{\alpha}(\alpha) x_{-\alpha}\left(-a^{-1}\right) x_{\alpha}(\alpha)$ and $h_{\alpha}(\alpha)=w_{\alpha}(\alpha) w_{\alpha}(-1)$, we have $h_{\alpha}(a) x_{\alpha}(t) h_{\alpha}(\alpha)^{-1}=x \alpha\left(\alpha^{2} t\right) \in N$. If $a$ is not invertible, then $a^{2}+1$, $a^{2}-1$ are invertible, and we have

$$
x_{\alpha}\left(a^{2} t\right)=x_{\alpha}\left(\left(\frac{a^{2}+1}{2}\right)^{2} t\right) x_{\alpha}\left(-\left(\frac{a^{2}-1}{2}\right)^{2} t\right)
$$

Thus our assertion can be reduced to the former (cf. [4] Lemma 1).
q.e.d.

For roots $\alpha$ and $\beta$ of $\Delta$, write $\langle\alpha, \beta\rangle=2(\alpha, \beta) /(\beta, \beta)$, and these are called Cartan integers where $(\alpha, \beta)$ is the scalar product of $\alpha$ and $\beta$. Then we have the following lemma.
2.2. Lemma. Let $\Delta$ be of rank $>1$. If $\alpha$ and $\beta$ are any positive roots of $\Delta$ and $\alpha \neq \beta$, then there exists $a$ root $\gamma \in \Delta$ such that
(a)

$$
\langle\alpha, \gamma\rangle= \pm 2,\langle\beta, \gamma\rangle=0
$$

or
(b)

$$
\langle\alpha, \gamma\rangle \equiv 1, \quad\langle\beta, \gamma\rangle \equiv 0(\bmod 2)
$$

Proof. If $(\alpha, \beta)=0$, then taking $\gamma=\alpha$, we have $\langle\alpha, \gamma\rangle=2$ and $\langle\beta, \gamma\rangle=0$. Suppose $(\alpha, \beta) \neq 0$, then we have $\alpha+\beta \in \Delta$. Let $\Delta_{2}$ be a subsystem of roots in $\Delta$ of rank 2 consisting of the roots $i \alpha+j \beta, i, j \in Z$, then our assurtion follows easily from the following tables of Cartan integers with respect to the roots of $\Delta_{2}$.
$\Delta_{2}$ : of type $A_{2} \quad \Delta_{2}^{+}=\left\{\alpha_{1}, \alpha_{2}, \alpha_{1}+\alpha_{2}\right\}$

|  | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{1}+\alpha_{2}$ |
| :---: | ---: | ---: | :---: |
| $\alpha_{1}$ | 2 | -1 | 1 |
| $\alpha_{2}$ | -1 | 2 | 1 |
| $\alpha_{1}+\alpha_{2}$ | 1 | 1 | 2 |

$\Delta_{2}$ : of type $B_{2} \quad \Delta_{2}^{+}=\left\{\alpha_{1}, \alpha_{2}, \alpha_{1}+\alpha_{2}, \alpha_{1}+2 \alpha_{2}\right\}$

|  | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{1}+\alpha_{2}$ | $\alpha_{1}+2 \alpha_{2}$ |
| :---: | ---: | ---: | :---: | :---: |
| $\alpha_{1}$ | 2 | -2 | 2 | 0 |
| $\alpha_{2}$ | -1 | 2 | 0 | 1 |
| $\alpha_{1}+\alpha_{2}$ | 1 | 0 | 2 | 1 |
| $\alpha_{1}+2 \alpha_{2}$ | 0 | 2 | 2 | 2 |


| $\Delta_{2}$ | : of type $G_{2}, \Delta_{2}^{+}=\left\{\alpha_{1}, \alpha_{2}, \alpha_{1}+\alpha_{2}, 2 \alpha_{1}+\alpha_{2}, 3 \alpha_{1}+\alpha_{2}, 3 \alpha_{1}+2 \alpha_{2}\right\}$ |  |  |  |  |  |
| :---: | ---: | ---: | ---: | ---: | :---: | :---: |
|  | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{1}+\alpha_{2}$ | $2 \alpha_{1}+\alpha_{2}$ | $3 \alpha_{1}+\alpha_{2}$ | $3 \alpha_{1}+2 \alpha_{2}$ |
| $\alpha_{1}$ | 2 | -1 | -1 | 1 | 1 | 0 |
| $\alpha_{2}$ | -3 | 2 | 3 | 0 | -1 | 1 |
| $\alpha_{1}+\alpha_{2}$ | 1 | 1 | 2 | 1 | 0 | 1 |
| $2 \alpha_{1}+\alpha_{2}$ | 1 | 0 | 1 | 2 | 1 | 1 |
| $3 \alpha_{1}+\alpha_{2}$ | 3 | -1 | 0 | 3 | 2 | 1 |
| $3 \alpha_{1}+2 \alpha_{2}$ | 0 | 1 | 3 | 0 | 1 | 2 |
|  |  |  |  |  |  | q.e.d. |

2.3. Corollary. Let $\Delta$ be a simple root system of type $A_{l} l \geqq 2$ $l \neq 3, E_{6}, E_{7}$ or $E_{8}$. Then for any positive root $\alpha$ and $\beta$ of $\Delta$, there exists a root $\gamma \in \Delta$ such that $\langle\alpha, \gamma\rangle \equiv 1,\langle\beta, \gamma\rangle \equiv 0(\bmod 2)$.

Proof. Assume $(\alpha, \beta) \neq 0$, then the subsystem $\Delta_{2}$ of $\Delta$ generated by $\alpha$ and $\beta$ is of type $A_{2}$. Thus our assertion can be checked by the table of Cartan integers of type $A_{2}$. Suppose $(\alpha, \beta)=0$. Since $\langle\alpha, \beta\rangle=$ ( $w \alpha, w \beta\rangle$ for any element $w$ of the Weyl group $W$, we may assume $\alpha=\alpha_{l}$ where $\Pi=\left\{\alpha_{1}, \cdots, \alpha_{l}\right\}, l \geqq 4$. $^{*)} \quad$ Thus $Z_{0}\left(\alpha_{l}\right)=\left\{\gamma \in \Delta \mid\left(\alpha_{l}, \gamma\right)=0\right\}$ is a simple subsystem of type $A_{l-2}, A_{5}, D_{6}$ or $E_{7}$, if $\Delta$ is of type $A_{l}, E_{6}, E_{7}$ or $E_{8}$ respectively (cf. M. R. Stein [5]). Therefore there exists an element $w$ of $W$ such that $w(\alpha)=\alpha_{l}, w(\beta)=\alpha_{l-2}$. Thus, there exists a subsystem $\Delta^{\prime}$ of type $A_{4}$ in which we may assume $\alpha=\alpha_{4}, \beta=\alpha_{2}$, where $\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}\right\}$ is a fundamental system of $4^{\prime}$. Taking $\gamma=\alpha_{1}+\alpha_{2}+\alpha_{3}$, we have

$$
\langle\alpha, \gamma\rangle \equiv 1, \quad\langle\beta, \gamma\rangle \equiv 0(\bmod 2)
$$

q.e.d.
2.4. Lemma. Let $\Re=\left\{\mathfrak{Y}_{\alpha}\right\}_{\alpha \in \Delta}$ be a carpet of $R$ associated with $\Delta$ such that $\mathfrak{U}_{\alpha} \mathfrak{X}_{\beta} \subset \mathfrak{U}_{\alpha+\beta}$ for $\alpha, \beta$ and $\alpha+\beta \in \Delta$, and $S$ be a closed subset of $\Delta^{+}$. Let $\beta_{1}, \cdots, \beta_{M}$ be any given ordering of $S$. Then each element
*) In the proof of above corollary, we shall set the fundamental root system as follows
( $A_{l}$ )
( $D_{l}$ )

( $E_{l}$ )


$$
l=6,7,8
$$

of $U_{s}(\Re)$ is expressed in the form

$$
x_{\beta_{1}}\left(s_{1}\right) \cdots x_{\beta_{M}}\left(s_{M}\right)
$$

where $s_{i} \in \mathfrak{U N}_{\beta_{i}} i=1, \cdots, M$.
Proof. Let $U^{\prime}$ be the set of elements expressible in the form as stated in the lemma. To prove our assertion, it is sufficient to show that $x_{\alpha}(t) U^{\prime} \subset U^{\prime}$ for any $x_{\alpha}(t), t \in \mathfrak{N}_{\alpha}$ and $\alpha \in S$. By the same way as in [1] 2.7, we can show this easily. q.e.d.
2.5. Lemma. Assume that $G$ is simple, and $G$ and $R$ satisfy (a) and (b) in 1.2. Let $N$ be a subgroup of $U(R)$ normalized by $T(R)$. If we express an element $x$ of $N$ in the form

$$
x=x_{\beta_{1}}\left(s_{1}\right) \cdots x_{\beta_{M}}\left(s_{M}\right)
$$

where $\beta_{1}<\cdots<\beta_{M}$ be any regular ordering of $\Delta^{+}$, then $x_{\beta_{i}}\left(s_{i}\right) \in N$ for $i=1,2, \cdots, M$.

Proof. For a unit element $u$ of $R$, we have

$$
\left[h_{r}(u), x_{\beta}(t)\right]=x_{\beta}\left(\left(u^{\langle\beta, \gamma\rangle}-1\right) t\right)
$$

where $[a, b]=a b a^{-1} b^{-1}$ for $a, b \in G(R)$. If there exists $\gamma \in \Delta$ such that $\left\langle\beta_{1}, \gamma\right\rangle \equiv 1,\left\langle\beta_{2}, \gamma\right) \equiv 0(\bmod 2)$, then by 2.4 , we obtain the following,

$$
\begin{aligned}
{\left[h_{r}(-1), x\right]=} & {\left[h_{r}(-1), x_{\beta_{1}}\left(s_{1}\right)\right]^{x_{\beta_{1}}\left(s_{1}\right)}\left[h_{r}(-1), x_{\beta_{2}}\left(s_{2}\right)\right] } \\
& \cdots{ }^{x_{\beta_{1}}\left(s_{1}\right) \cdots x_{\beta_{M-1}}\left(s_{M-1}\right)}\left[h_{r}(-1), x_{\beta_{M}}\left(s_{M}\right)\right] \\
= & x_{\beta_{1}}\left(-2 s_{1}\right) x_{\beta_{3}}\left(s_{3}^{\prime}\right) \cdots x_{\beta_{M}}\left(s_{M}^{\prime}\right) \in N
\end{aligned}
$$

where ${ }^{a} y=a y a^{-1}$. If $u$ and $u^{2}-1$ are units of $R$ and there exists a root $\gamma \in \Delta$ such that $\left\langle\beta_{1}, \gamma\right\rangle= \pm 2,\left\langle\beta_{2}, \gamma\right\rangle=0$, then we have the following,

$$
\begin{aligned}
{\left[h_{r}(u), x\right]=} & {\left[h_{r}(u), x_{\beta_{1}}\left(s_{1}\right)\right]^{x_{\beta_{1}}\left(s_{1}\right)}\left[h_{r}(u), x_{\beta_{2}}\left(s_{2}\right)\right] } \\
& \ldots{ }^{\beta_{\beta_{1}}\left(s_{1}\right) \cdots x_{\beta_{M-1}}\left(s_{M-1}\right)}\left[h_{r}(u), x_{\beta_{M}}\left(s_{M}\right)\right] \\
= & x_{\beta_{1}}\left(\left(u^{ \pm 2}-1\right) s_{1}\right) x_{\beta_{3}}\left(s_{3}^{\prime}\right) \cdots x_{\beta_{M}}\left(s_{M}^{\prime}\right) \in N .
\end{aligned}
$$

By Lemma 2.2, its Corollary 2.3 and the assumptions (a) and (b) in 1.2, we can see easily that, repeating the above process, we obtain $x_{\beta_{1}}\left(v s_{1}\right) \in N$ for some unit element $v$ of $R$. Thus by 2.1 , we have $x_{\beta_{1}}\left(s_{1}\right) \in N$. By induction on the indices $i$ of roots $\beta_{i}$, we have $x_{\beta_{i}}\left(s_{i}\right) \in N$ for $i=1,2$, $\cdots, M$.
q.e.d.
2.6. Proposition. Let $P$ be a parabolic subgroup of $G(R)$. Then $P$ is generated by the elements of $B(R)$ and $P \cap U(R)$ where $B(R)=$ $V(R) T(R)$.

Proof. Let $\tilde{\phi}$ be a group homomorphism $G(R) \rightarrow G(k)$ induced by the natural ring homomorphism $\phi: R \rightarrow k=R / \mathfrak{M}$, then $\tilde{\phi}(P)=P^{\prime}$ is a parabolic subgroup of $G(R)$ and by Tits' theorem, we have that $P^{\prime}$ is generated by $B(k)$ and $x_{\alpha}(1)$ for all root $\alpha \in I$ where $I$ is a subset of $\Pi$. On the other hand, since $\operatorname{Ker} \phi=V(\mathfrak{M}) T(\mathfrak{M}) U(\mathbb{M})$ (cf. [1] 3.3), we can choose generators of $P$ among the elements of $B(R)$ and $U(R)$. q.e.d.
2.7. Proof of injectivity. Let $P$ be a parabolic subgroup and set up $\mathfrak{U}_{\alpha}=\left\{t \in R \mid x_{\alpha}(t) \in P\right\}$ for each root $\alpha$ of $\Delta$. By 2.1 and the definition of $P$, it is clear that $\mathscr{U}_{\alpha}$ is an ideal of $R$ for each $\alpha \in \Delta$, and $\mathscr{N}_{\alpha}=R$ for each negative roots $\alpha$ of $\Delta$. On the other hand, let $\alpha$ and $\beta$ be roots of $\Delta$ such that $\alpha+\beta \in \Delta$, and $\Delta_{2}$ be the subsystem of $\Delta$ of rank $=2$ generated by $\alpha$ and $\beta$. If $\alpha>0$ and $\beta<0$, then there is an element $w$ of the Weyl group $W_{2}$ of $\Delta_{2}$ such that $w(\alpha)>0$ and $w(\beta)>0$. Thus, by the commutator relations for $x_{\alpha}(t), t \in \mathfrak{H}_{\alpha}, \alpha \in \Delta^{+}$(cf. [1] 2.2) and by 2.5, we see easily $\mathfrak{U}_{\alpha} \mathfrak{V t}_{\beta} \subset \mathfrak{U}_{\alpha+\beta}$ for any roots $\alpha$ and $\beta$ of $\Delta$. That is, $\left\{\mathfrak{N}_{\alpha}\right\}_{\alpha \in \Delta}$ is a permissible carpet. From 2.4, 2.5 and 2.6, it is clear that $P$ is generated by $x_{\alpha}(t), t \in \mathfrak{N}_{\alpha}, \alpha \in \Delta$ and $T(R)$. Namely, the mapping $\Psi:\{P\} \rightarrow\left\{\left\{\mathfrak{N}_{\alpha}\right\}_{\alpha \in \Delta}\right\}$ is injective.
q.e.d.
3. Proof of surjectivity. Now in order to prove that the mapping $\Psi$ in 2.7 is surjective, we shall first prove the following lemmas.
3.1. Let $\Re=\left\{\mathscr{H}_{\alpha}\right\}_{\alpha \in \Delta}$ be a permissible carpet of $R$ with respect to ( $\Delta, \Pi$ ). Setting $\Delta_{\Omega}^{\prime}=\left\{\alpha \in \Delta \mid \mathfrak{N}_{\alpha}=R\right\}$ and $\Delta_{\Omega}^{\prime \prime}=\left\{\alpha \in \Delta \mid \mathfrak{N}_{\alpha} \neq R\right\}$, we have that i) $\Delta_{\Omega}^{\prime} \supset \Delta^{-}$and $\Delta_{\Omega}^{\prime \prime} \subset \Delta^{+}$, ii) $\Delta_{\Omega}^{\prime}$ and $\Delta_{\rho}^{\prime \prime}$ are closed, iii) $\Delta_{\Omega}^{\prime \prime}$ is an ideal of $\Delta^{+}$, that is, if $\alpha \in \Delta_{\Omega}^{\prime \prime}, \beta \in \Delta^{+}$and $\alpha+\beta \in \Delta$, then $\alpha+\beta \in \Delta_{\Omega}^{\prime \prime}$. Thus we can see easily the following lemma.
3.2. Lemma. Using the same notation as in 3.1, we have the following.
i) Let $\alpha$ be a positive root, then $\alpha \in \Delta_{\Omega}^{\prime \prime}$ if and only if there exists a root $\alpha_{i} \in \Delta_{\Omega}^{\prime \prime} \cap \Pi$ such that $n_{i} \neq 0$ for $\alpha=n_{1} \alpha_{1}+\cdots+n_{l} \alpha_{l}$ where $\Pi=\left\{\alpha_{1}, \quad, \alpha_{l}\right\}$,
ii) we set $Z^{+} \Delta_{\Omega}^{\prime}=\left\{n_{1} \beta_{1}+\cdots+n_{s} \beta_{s} \mid n_{i} \in Z^{+}, \beta_{i} \in \Delta_{\Omega}^{\prime}\right\}$ and $Z^{+} \Delta_{\Omega}^{\prime \prime}=$ $\left\{m_{1} \gamma_{1}+\cdots+m_{t} \gamma_{t} \mid m_{j} \in Z^{+}, \gamma_{j} \in \Delta_{\Omega}^{\prime \prime}\right\}$ where $Z^{+}$is the set of positive rational integers. Then we have $Z^{+} \Delta_{\Omega}^{\prime} \cap Z^{+} \Delta_{\Omega}^{\prime \prime}=\varnothing$.
3.3. Lemma. Let $\Re=\left\{\mathfrak{N}_{\alpha}\right\}_{\alpha \in \Delta}$ be a permissible carpet of $R$ associated with $(\Delta, I I)$ and $\Delta_{s}^{\prime}, \Delta_{\Omega}^{\prime \prime}$ be the same as in 3.1. Let $Q$ be a subgroup of $G(R)$ generated by $x_{\alpha}(t)$ for all $t \in R, \alpha \in \Delta_{\Omega}^{\prime}$ and elements of $T(R)$. Then we have $Q \cap U_{d_{g}^{\prime \prime}}(R)=\{1\}$.

Proof. Let $\mathscr{J}_{C}$ be a simple Lie algebra over the complex field $C$,
$(d \rho, V)$ be a faithful representation of $\mathscr{S}_{C}$ with $\mathscr{S}_{C}$-module $V$ over $C$, $\left\{H_{\alpha_{1}}, \cdots, H_{\alpha_{l}}, X_{\alpha}, \alpha \in \Delta\right\}$ be a Chevalley basis of $\mathbb{E}_{C}$ and $V_{Z}$ be an admissible lattice of $V$, then for any root $\alpha$ and $t \in R$, we can construct a unipotent automorphism $x_{\alpha}(t)=\exp t d \rho\left(X_{\alpha}\right)$ of $V_{z} \otimes R$. Let $\{\lambda\}$ be the set of weights of $d \rho$ and denote by $V^{\lambda}$ a weight space associated with a weight $\lambda$, then we have $V_{z}=\sum_{\lambda} \oplus V_{z}^{\lambda}$ where $V_{z}^{\lambda}=V_{z} \cap V^{\lambda}$. For any element $v \in V_{z}^{2}, d \rho\left(X_{\alpha}\right) v \in V_{z}^{\lambda+\alpha}$ (cf. Steinberg [6]). Therefore, for each $x_{\alpha}(t)$,

$$
\begin{aligned}
x_{\alpha}(t) v & =v+t d \rho\left(X_{\alpha}\right) v+t^{2} \frac{1}{2} d \rho(X) v+\cdots \\
& =v+u
\end{aligned}
$$

where $u \in \sum_{i=1} \oplus V_{z}^{\lambda+i \alpha} \otimes R$. On the other hand, for any $h(\chi) \in T(R)$ and $v \in V_{z}^{\lambda}, h(\chi) v=c_{\lambda, \chi} v$ for some $c_{\lambda, x} \in R$. Thus for any $x \in Q \cap U_{d_{\Omega}^{\prime \prime}}(R)$ and $v \in V_{z}^{\lambda}$, we have $x v \in c v+\sum_{\left.\mu \in Z^{+}\right\lrcorner_{\Omega}^{\prime}} \oplus V_{z}^{\lambda+\mu} \otimes R$ and $x v \in v+\sum_{\nu \in Z^{+} \Delta_{g}^{\prime \prime}} \oplus$ $V^{\lambda+\nu} \otimes R$, thus by 3.2 ii , $x v=v$. Since $\lambda$ can be chosen arbitrary, we have $x=1$.
q.e.d.
3.4. Proposition. Let $\mathfrak{R}=\left\{\mathfrak{N}_{\alpha}\right\}_{\alpha \in \Delta}$ be a permissible carpet, and use the same notation as in 3.3. Then $U_{d_{\beta}^{\prime}}(\Omega) Q$ is a subgroup of $G(R)$.

Proof. To prove the proposition, it is sufficient to show the following,
(a)

$$
x_{\alpha}(t) U_{d_{\Omega}^{\prime \prime}}(\Omega) \subset U_{d_{\Omega}^{\prime \prime}}(\Omega) Q
$$

for all $t \in \mathfrak{V}_{\alpha}, \alpha \in \Delta$. Assume $\alpha \in \Delta^{+}$. Since $\Delta_{\Omega}^{\prime \prime}$ is an ideal of $\Delta^{+}, U_{U_{\Omega}^{\prime \prime}}(\Omega)$ is a normal subgroup of $U(R)$, thus (a) holds. For a negative root $\alpha$ of $\Delta$, (a) follows from the following two lemmas.
3.5. Lemma. Let $\Re=\left\{\mathscr{H}_{\alpha}\right\}_{\alpha \in \Delta}$ be a semi-permissible carpet. Set $\Delta_{\Omega}^{\prime+}=\Delta^{+} \cap \Delta_{\Omega}^{\prime}$ and $\Delta_{\Omega}^{\prime \prime+}=\Delta^{+} \cap \Delta_{\Omega}^{\prime \prime}$. Then we have

$$
\begin{equation*}
x_{-\alpha}(t) U_{d_{\Omega}^{\prime \prime}}+(\Re) \subset U_{d_{\Omega}^{\prime \prime}+}(\Omega) U_{d_{\Omega}^{\prime}+}(\mathfrak{M}) x_{-\alpha}(R) T(R) \tag{b}
\end{equation*}
$$

for any $\alpha \in \Pi$ and $t \in \mathscr{H}_{-\alpha}$.
Proof. By 2.4, any element $x$ of $U_{s_{\Omega^{\prime \prime}}+}(\Re)$ is expressed by the form

$$
x=x_{\beta_{1}}\left(s_{1}\right) \cdots x_{\beta_{M}}\left(s_{M}\right)
$$

where $\left\{\beta_{1}, \cdots, \beta_{M}\right\}=\Delta_{\Omega^{\prime \prime}}^{\prime+}$ and $s_{i} \in \mathfrak{X}_{\beta_{i}}, i=1, \cdots, M$. Set up $x_{i}=x_{\beta_{i}}\left(s_{i}\right)$ $\cdots x_{\beta_{M}}\left(s_{M}\right)$. Then we shall prove (b) by induction on $i$. If $\beta_{i-1} \neq \alpha$, we have

$$
\begin{aligned}
x_{-\alpha}(t) x_{i-1} & =x_{-\alpha}(t) x_{\beta_{i-1}}\left(s_{i-1}\right) x_{i} \\
& =x_{\beta_{i-1}}\left(s_{i-1}\right) \prod_{j, k>0} x_{-j \alpha+k \beta_{i-1}}\left(c_{j, k} t^{j} s_{i-1}^{k}\right) x_{-\alpha}(t) x_{i}
\end{aligned}
$$

where $-j \alpha+k \beta_{i-1}>0$ and $c_{j, k} t^{j} s_{i-1}^{k} \in \mathfrak{N}_{-j \alpha+k \beta_{i-1}} \cap \mathfrak{M}$. If $\beta_{i-1}=\alpha$ then

$$
x_{-\alpha}(t) x_{i-1}=x_{-\alpha}(t) x_{\alpha}\left(s_{i-1}\right) x_{i}=x_{\alpha}(v) x_{-\alpha}(w) x_{i} z
$$

where $v \in \mathfrak{N}_{\alpha}$ and $z \in T(R)$. Therefore by 2.4 and the assumption of the induction, we have

$$
x_{-\alpha}(t) x_{i-1} \in U_{d_{\Omega}^{\prime \prime}+}(\Omega) U_{d_{g}^{\prime}+}(\mathfrak{M}) x_{-\alpha}(R) T(R) .
$$

3.6. Lemma. We use the same notation as in 3.5. For a given negative root $-\alpha$, we assume that

$$
x_{-\alpha}(t) U_{d_{\Omega^{\prime}}^{\prime \prime}}(\Re) \subset U_{a_{\Omega^{\prime \prime}}^{\prime}+}(\Re) U_{A_{\Omega}^{\prime},}(\mathfrak{M}) V(\mathfrak{M}) T(R) x_{-\alpha}(R)
$$

for all semi-permissible carpet $\mathfrak{R}=\left\{\mathfrak{N}_{\alpha}\right\}_{\alpha \in \Delta}$. Then, for any element $w$ of the Weyl group associated with $\Delta$ such that $w(\alpha)>0$, we have

$$
x_{w(-\alpha)}(t) U_{d_{\Omega^{\prime},}^{\prime}}\left(\Omega^{\prime}\right) \subset U_{d_{\Omega^{\prime}}^{\prime}+}\left(\Omega^{\prime}\right) U_{d_{\Omega^{\prime}}^{\prime}}+(\mathfrak{M}) V(\mathfrak{M}) x_{w(-\alpha)}(R) T(R),
$$

where $\Omega^{\prime}=\left\{\mathfrak{X}_{\alpha}^{\prime}\right\}_{\alpha \in \Delta}$ is any semi-permissible carpet.
Proof. Denote by $w_{o}$ the reflection with respect to hyperplane orthogonal to a root $\sigma$, then for any element $w$ of the Weyl group $W$ such that $w(\alpha)>0$, we can choose an element $w^{\prime}$ of $W$ as follows i) $w(\alpha)=w^{\prime}(\alpha)$ ii) $w^{\prime}=w_{\alpha_{1}} w_{\alpha_{2}} \cdots w_{\alpha_{L}}$ where $\alpha_{i} \in \Pi, i=1,2, \cdots, L$, and $w_{\alpha_{j}} w_{\alpha_{j+1}} \cdots w_{\alpha_{L}}(\alpha)>0$ for $1 \leqq j \leqq L$. Therefore, without-loss of generality, we may assume $w=w_{o}$ for some $\sigma \in \Pi$. Let $\Re^{\prime}=\left\{\mathfrak{U}_{\alpha}^{\prime}\right\}_{\alpha \in \Delta}$ be any semi-permissible carpet. For each element $x$ of $U_{\alpha_{\Omega^{\prime}},}\left(\AA^{\prime}\right)$ we write $x=x_{\beta_{1}}\left(s_{1}\right) \cdots x_{\beta_{N}}\left(s_{N}\right)$ where $\left\{\beta_{1}, \cdots, \beta_{N}\right\}=\Delta_{\Omega^{\prime}}^{\prime \prime+}, s_{i} \in \mathfrak{Z}_{\beta_{i}}^{\prime}(i=1, \cdots, N)$ and $\beta_{i} \neq \sigma, i=1, \cdots, N-1$. Now, taking the conjugation of $x_{w(\alpha)}(t) x$ with $w(1)$, we have

$$
(*) \quad x_{-\alpha}( \pm t) w(1) x w(1)^{-1}=x_{-\alpha}( \pm t) x_{w\left(\beta_{1}\right)}\left( \pm s_{1}\right) \cdots x_{w\left(\beta_{N}\right)}\left( \pm s_{N}\right)
$$

where $w\left(\beta_{j}\right)>0$ for $i=1, \cdots, N-1$ and $w\left(\beta_{N}\right)<0($ resp. $>0)$ if $\beta_{N}=\sigma$ (resp. $\beta_{N} \neq \sigma$ ). Setting $\mathfrak{N}_{\beta}^{\prime}=\mathfrak{N}_{w(\beta)}$, we have semi-permissible carpet $\left\{\mathfrak{N}_{T}\right\}_{r \in \Lambda}$. First assume $\beta_{N}=\sigma$. Then, using the assumption of this lemma, (*) is equal to

$$
\begin{gathered}
(* *) \quad x_{w\left(\beta_{1}\right)}\left(v_{1}\right) \cdots x_{w\left(\beta_{N-1}\right)}\left(v_{N-1}\right) x_{r_{1}}\left(u_{1}\right) \cdots x_{r_{M}}\left(u_{M}\right) x_{r_{M+1}}\left(u_{M+1}\right) \\
\cdots x_{r_{T}}\left(u_{T}\right) x_{-\alpha}\left(t^{\prime}\right) x_{-o}\left(s_{N}^{\prime}\right) z
\end{gathered}
$$

where $\gamma_{1}, \cdots, \gamma_{m}$ are positive, $\gamma_{\mu+1}, \cdots, \gamma_{T}$ are negative, $v_{i} \in \mathfrak{N}_{w\left(\beta_{i}\right)} i=1$, $\cdots, N-1, u_{j} \in \mathfrak{A}_{r_{j}} \cap \mathfrak{M}, j=1, \cdots, T, s_{N}^{\prime} \in \mathfrak{M} \cap \mathfrak{U}_{w\left(\beta_{N}\right)}$ and $z \in T(R)$. By 2.4, we may assume $\gamma_{M}=\sigma$ and $\gamma_{M+1}=-\sigma$, and (**) is equal to

$$
\begin{gathered}
(* * *) \quad x_{w\left(\beta_{1}\right)}\left(v_{1}\right) \cdots x_{w\left(\beta_{N-1}\right)}\left(v_{N-1}\right) x_{r_{1}}\left(u_{1}\right) \cdots x_{-o}(v) x_{o}(y) x_{r_{M+2}}\left(u_{M+2}^{\prime}\right) \\
\cdots x_{r_{T}}\left(u_{T}^{\prime}\right) x_{-\alpha}\left(t^{\prime \prime}\right) z^{\prime}
\end{gathered}
$$

where $v \in \mathfrak{N}_{-\sigma} \cap \mathfrak{M}, y \in \mathfrak{N}_{\sigma} \cap \mathfrak{M}, u_{j}^{\prime} \in \mathfrak{U}_{r_{j}} j=M+2, \cdots, T$, and $z^{\prime} \in T(R)$. Taking the conjugation of the above form with $w(1)$ again, and using 2.4, we have

$$
x_{w(-\alpha)}(t) x \in U_{d_{\Omega}^{\prime \prime}+}(\Re) U_{\Delta_{\Omega}^{\prime}+}(\mathfrak{M}) V(\mathfrak{M}) x_{w(-\alpha)}(R) T(R) .
$$

If $\beta_{N} \neq \sigma$, we can prove our assertion by the same way as above without calculation (***).
q.e.d.
3.7. Lemma. Let $\left\{\mathfrak{T}_{\alpha}\right\}_{\alpha \in \Delta}$ be a permissible carpet of $R$ with respect to $(\Delta, \Pi)$, and let $\Delta_{\Omega}^{\prime}, \Delta_{\Omega}^{\prime \prime}$ and $Q$ be same as in 3.1 and 3.3 respectively. Then we have $Q \cap U(R)=U_{S_{\Omega^{\prime}}}(R)$.

Proof. To prove our lemma, it is sufficient to show $Q \cap U(R) \subset$ $U_{d_{\Omega^{\prime}}}(R)$. If $x \in Q \cap U(R)$, then by 2.4, we have $x=y z$ where $y \in U_{\Lambda_{\Omega^{\prime}}}(R)$, $z \in U_{d_{\alpha^{\prime}}+}(R)$. Since $z \in Q$, we have $y \in Q \cap U_{d_{\rho_{j}^{\prime}}}(R)$, and by $3.3, y=1$. Therefore we have $x \in U_{d_{s^{\prime}}}(R)$.
q.e.d.
3.8. Proof of surjectivity. Let $\Omega=\left\{\mathfrak{R}_{\alpha}\right\}_{\alpha \in\lrcorner}$ be any permissible carpet of $R$ associated with ( $\Delta, \Pi$ ), and $P$ be a parabolic subgroup of $G(R)$ generated by $x_{\alpha}(t)$ for all $t \in \mathfrak{N}_{\alpha}, \alpha \in \Delta$ and elements of $T(R)$. Set $\mathfrak{V}_{\alpha}^{\prime}=\left\{t \in R \mid x_{\alpha}(t) \in P\right\}$. Then, to show our assertion, it is sufficient to prove that $\mathfrak{U}_{\alpha}=\mathfrak{Y}_{\alpha}^{\prime}$ for all roots $\alpha \in \Delta$. It is clear $\mathfrak{N}_{\alpha} \subset \mathfrak{H}_{\alpha}^{\prime}$. If $\alpha<0$, then $\mathfrak{N}_{\alpha}=\mathfrak{Y}_{\alpha}^{\prime}=R$. In order to prove $\mathfrak{N}_{\alpha} \supset \mathfrak{U}_{\alpha}^{\prime}$ for $\alpha>0$, from 2.5, it is sufficient to show $P \cap U(R)=U_{\Delta^{+}}(\Omega)$. By 3.4 , we have $P=U_{d_{\Omega^{\prime}}^{\prime}}(\Re) Q$, thus, for any $x \in P \cap U(R), x=y z$ where $y \in U_{d_{\Omega}^{\prime}}(\Omega), z \in Q$, and from 3.7, $z \in U(R) \cap Q \subset U_{\Delta^{+}}(\Omega)$, therefore $x \in U_{\Delta^{+}}(\Re)$, that is $P \cap U(R) \subset U_{\Delta^{+}}(\Omega)$. On the other hand, it is clear that $P \cap U(R) \supset U_{\Delta^{+}}(\Omega)$. Thus we have $P \cap U(R)=U_{\Lambda^{+}}(\Omega)$.

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