On Parameterized Dissipation Inequalities and Receding Horizon Robust Control

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Abstract: This paper considers the standard input-to-state stability (ISS) inequality for discrete-time nonlinear systems, which involves a candidate Lyapunov function (LF) and a supply function that dictates the ISS gain of the system. To reduce conservatism, a set of parameters is assigned to both the LF and the supply function. A set-valued map, which generates admissible sets of parameters for each state and input, is defined such that the corresponding parameterized LF and supply function enjoy the standard ISS inequality. It is demonstrated that the so-obtained parameterized ISS inequality offers non-conservative analysis conditions, even when LFs and supply functions with a particular structure, such as quadratic forms, are considered. For bounded inputs, it is then shown how parameterized ISS inequalities can be used to synthesize a closed-loop system with an optimized envelope of trajectories. An implementation method based on receding horizon optimization is proposed, along with a recursive feasibility and complexity analysis. The advances provided by the proposed synthesis methodology are illustrated for a continuous stirred tank reactor.

Keywords: Discrete-time, Input-to-state stability, Lyapunov methods, Predictive control.

1. INTRODUCTION

The problem considered in this paper is robust stabilization, in the sense of input-to-state stability (ISS) (Sontag, 1990; Jiang and Wang, 2001), of discrete-time nonlinear systems that are possibly subject to constraints. Virtually all approaches to solve this problem reduce to the construction of a control ISS-Lyapunov function (or shortly, ILF), see, e.g., (Sontag, 1990; Krstić et al., 1995; Liberzon et al., 2002; Malisoff and Sontag, 2004) and the references therein. However, the construction of such functions for nonlinear systems is a very challenging problem. In particular, even linear systems with hard state/input constraints pose a non-trivial challenge to finding a non-conservative ILF. As such, it would be desirable to identify a non-conservative class of ISS Lyapunov functions that leads to a tractable implementation for nonlinear systems. As our interest lies mainly within the discrete-time domain, the following brief discussion on ISS is restricted to results for perturbed *discrete-time* systems.

The fact that a continuous Lyapunov function implies ISS on compact sets, see (Freeman and Kokotović, 2008) and (Lazar et al., 2010) for a proof of this conjecture, can be used to establish inherent ISS of an asymptotically stable system. However, for constrained systems, even the nominal stabilization problem is challenging. Apart from this, as pointed out in (Lazar and Heemels, 2008), or in (Chen and Scherer, 2006) in the linear $\mathcal{H}_\infty$ setting, it would be desirable to optimize the closed-loop ISS gain depending on the system trajectory, besides guaranteeing a common (optimal) ISS gain for all trajectories, which is the typical solution (Jiang and Wang, 2001). This feature was referred to in (Lazar and Heemels, 2008) as "optimized ISS". Therein, a solution to attain optimized ISS was proposed based on the explicit knowledge of a continuous Lyapunov function for the nominal, constrained system, which is difficult to obtain in general. To remove this impediment, this paper proposes a definition of a parameterized ILF (p-ILF), without a fixed structure, that is applicable to general discrete-time nonlinear systems. The term parameterized ILF denotes the fact that the ILF candidate is endowed with a set of parameters, not necessarily structured in a particular form (e.g., a matrix of certain dimensions), which can take multiple values within an admissible set that depends on each state. As such, the conditions for input-to-state stability can be formulated in terms of the set valued map that generates an admissible set of parameters for each state and disturbance. The conditions that define a p-ILF are time-invariant. The non-conservatism of the proposed p-ILFs, even with a fixed structure, is indicated by a converse theorem, which establishes that exponentially stable nonlinear systems admit a p-quadratic ILF.

Then, it is shown how the developed concept of parameterized ILF, or equivalently, parameterized ISS inequality, can be used in combination with receding horizon optimization to design an input-to-state stabilizing control law for constrained nonlinear systems subject to additive outer perturbations. At each time instant, the control scheme searches for a feasible set of control inputs and parameters, which define the ILF and the supply function for each state and disturbance, respectively, while minimizing the closed-loop ISS gain. Under the assumption that the disturbance input is bounded, besides the non-conservative nature of the parameterized conditions, the proposed receding horizon scheme enjoys several benefits: it guarantees an optimized envelope of closed-loop trajectories, it is recursively feasible (under reasonable assumptions) and it can be formulated as a single semidefinite program (SDP) for input affine nonlinear systems.

Some remarks that put the developed methodology in perspective with respect to existing robust model predictive control (MPC) schemes (Rawlings and Mayne, 2009), along with an illustration of the design procedure for a nonlinear model of a continuous stirred tank reactor complete the paper.

2. PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ define $\Pi_{\geq c} := \{k \in \Pi \mid k \geq c\}$ and similarly $\Pi_{\leq c}$, $\mathbb{R}_\Pi := \Pi$ and $\mathbb{Z}_\Pi := \mathbb{Z} \cap \Pi$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, let $\text{int}(\mathcal{S})$ denote the interior of $\mathcal{S}$. A polytope is a closed and bounded polyhedron. For a vector $x \in \mathbb{R}^n$, $[x]_i$ denotes the i-th element of x and $\|\cdot\|$ denotes an arbitrary p-norm, $p \in \mathbb{Z}_{\geq 1} \cup \infty$. Let $\|x\|_\infty := \max_{i=1,\ldots,n} |[x]_i|$ and $\|x\|_2 := \sqrt{\sum_{i=1}^n |[x]_i|^2}$, where $|\cdot|$ denotes the absolute value. For a sequence $\mathbf{w} := \{w(l)\}_{l \in \mathbb{Z}_+}$ with $w(l) \in \mathbb{R}^n$, $l \in \mathbb{Z}_+$, let $\|\mathbf{w}\| := \sup\{\|w(l)\| \mid l \in \mathbb{Z}_+\}$ and let $\mathbf{w}_{[k]} := \{w(l)\}_{l \in \mathbb{Z}_{[0,k]}}$. $\mathbf{0}$ denotes a sequence of vectors with all the elements equal to zero. For a matrix $Z \in \mathbb{R}^{m \times n}$, $[Z]_{ij}$ denotes the element in the i-th row and j-th column of Z. For a matrix $Z \in \mathbb{R}^{m \times n}$ let $\|Z\| := \sup_{x \neq 0} \frac{\|Zx\|}{\|x\|}$ denote its corresponding induced matrix norm. $I_n \in \mathbb{R}^{n \times n}$ denotes the n-th dimensional identity matrix. For a symmetric matrix $Z \in \mathbb{R}^{n \times n}$ let $Z \succ 0 (\succeq 0)$ denote that Z is positive definite (semi-definite). Moreover, $*$ is used to denote the symmetric part of a matrix, i.e., $\left[\begin{smallmatrix} a & b^\top \\ b & c \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & * \\ b & c \end{smallmatrix}\right]$.

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and $\lim_{s \to \infty} \varphi(s) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k \to \infty} \beta(s, k) = 0$.

Next, consider the discrete-time system

$$x(k+1) = \Phi(x(k), w(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state and $w(k) \in \mathbb{R}^d$ is the disturbance, at the discrete-time instant k, and $\Phi : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ is an arbitrary continuous map with $\Phi(0, 0) = 0$. For a given $x(0) \in \mathbb{R}^n$ and sequence $\mathbf{w} := \{w(k)\}_{k \in \mathbb{Z}_+}$, a sequence of states $\mathbf{x}(x(0), \mathbf{w}) := \{x(k)\}_{k \in \mathbb{Z}_+}$ is called a discrete trajectory of system (1). For given subsets $\mathbb{X}$ of $\mathbb{R}^n$ and $\mathbb{W}$ of $\mathbb{R}^d$, a sequence of subsets of $\mathbb{R}^n$, $\mathcal{E}(\mathbb{X}, \mathbb{W}) := \{\mathbb{E}(k)\}_{k \in \mathbb{Z}_+}$, with $\mathbb{E}(0) = \mathbb{X}$ and such that $x(k) \in \mathbb{E}(k)$ for all $k \in \mathbb{Z}_+$ and all $\mathbf{w}$ with $w(k) \in \mathbb{W}$ for all $k \in \mathbb{Z}_+$ is called the discrete envelope of all trajectories of system (1) with respect to initial conditions in $\mathbb{X}$ and inputs in $\mathbb{W}$. The corresponding nominal discrete envelope of all trajectories of system (1) is obtained as $\mathcal{E}(\mathbb{X}, \mathbf{0})$. The notation $\mathbf{x}(x(0), \mathbf{w}) \in \mathcal{E}(\mathbb{X}, \mathbb{W})$ will be used in the sense of the previous definition. The term discrete is omitted in what follows for brevity.

For any $\mathbb{W} \subseteq \mathbb{R}^d$, with a slight abuse of notation define $\Phi(x, \mathbb{W}) := \{\Phi(x, w) \mid w \in \mathbb{W}\}$.

Definition 1. A set $\mathbb{X} \subseteq \mathbb{R}^n$ is called (robustly) *positively invariant*, or shortly PI($\mathbb{W}$), for system (1) if for all $x \in \mathbb{X}$ it holds that $\Phi(x, \mathbb{W}) \subseteq \mathbb{X}$.

Definition 2. Let $\mathbb{X}$ with $0 \in \text{int}(\mathbb{X})$ be a subset of $\mathbb{R}^n$. We call system (1) input-to-state stable in $\mathbb{X}$ with respect to inputs in $\mathbb{W}$, or shortly, ISS($\mathbb{X}$,$\mathbb{W}$), if there exists a $\mathcal{KL}$-function $\beta(\cdot, \cdot)$ and a $\mathcal{K}$-function γ such that, for each $x(0) \in \mathbb{X}$ it holds that the corresponding state trajectory of (1) satisfies $\|x(k)\| \leq \beta(\|x(0)\|, k) + \gamma(\|\mathbf{w}_{[k-1]}\|)$, $\forall k \in \mathbb{Z}_{\geq 1}$. System (1) is called exponentially stable in $\mathbb{X}$, or shortly, ES($\mathbb{X}$), if for $\mathbf{w} = \mathbf{0}$ the above property holds with $\beta(s, k) := \theta \mu^k s$ for some $\theta \in \mathbb{R}_{\geq 1}$, $\mu \in \mathbb{R}_{[0,1)}$.

Theorem 3. (Jiang and Wang, 2001; Lazar, 2006) Let $\mathbb{X} \subseteq \mathbb{R}^n$ be a PI($\mathbb{W}$) set for (1) with $0 \in \text{int}(\mathbb{X})$. Furthermore, let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty, \sigma \in \mathcal{K}$, $\rho \in \mathbb{R}_{[0,1)}$ and let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a function such that:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \ \forall x \in \mathbb{X}, \tag{2a}$$

$$V(\Phi(x, w)) \leq \rho V(x) + \sigma(\|w\|), \quad \forall (x, w) \in \mathbb{X} \times \mathbb{W}. \tag{2b}$$

Then system (1) is ISS($\mathbb{X}$, $\mathbb{W}$).

A function V that satisfies (2) is called an *ISS Lyapunov function (ILF($\mathbb{X}$,$\mathbb{W}$))* and ρ is called the *rate of decrease* of V. Under the assumptions of Theorem 3, for any $c \in \mathbb{R}_+$ such that $\mathbb{X} \subseteq \mathbb{V}_c := \{x \mid V(x) \leq c\}$, it holds that $x(k) \in \mathbb{V}_{\rho^k c + \frac{1}{1-\rho} \sup_{w \in \mathbb{W}} \sigma(\|w\|)}$ for all $k \in \mathbb{Z}_+$, $x(0) \in \mathbb{X}$ and all $\mathbf{w}$ with $w(k) \in \mathbb{W}$ for all $k \in \mathbb{Z}_+$. Equivalently, it holds that $\mathcal{E}(\mathbb{X}, \mathbb{W}) \subseteq \mathcal{V}_c(\mathbb{X}, \mathbb{W})$, where

$$\mathcal{V}_c(\mathbb{X}, \mathbb{W}) := \left\{ \mathbb{V}_{\rho^k c + \frac{1}{1-\rho} \sup_{w \in \mathbb{W}} \sigma(\|w\|)} \right\}_{k \in \mathbb{Z}_+}. \tag{3}$$

This means that the envelope of trajectories of a system of the form (1) that admits a ILF($\mathbb{X}$,$\mathbb{W}$), i.e., V, will be contained within the discrete envelope generated by a family of sublevel sets of V, i.e., $\mathcal{V}_c(\mathbb{X}, \mathbb{W})$, for some suitable $c \in \mathbb{R}_+$.

The following remarks are in order, before proceeding to the next section. Firstly, notice that the closed-loop ISS gain of system (1) is explicitly dictated by the function σ, for a given α_1, α_2, ρ, see, e.g., (Lazar, 2006). As such, optimization of the closed-loop ISS gain can be formulated as optimization of the gain of the function σ. Secondly, notice that the problem of minimizing, in an element-wise sense, the Hausdorff distance between the envelope of trajectories of system (1) with respect to its nominal envelope of trajectories, i.e., $\mathcal{E}(\mathbb{X}, \mathbf{0}) \subseteq \mathcal{V}_c(\mathbb{X}, \mathbf{0})$, can also be formulated as optimization of the gain of the function σ.

3. PARAMETERIZED ISS LYAPUNOV FUNCTIONS

Let $\mathbb{P}$ and $\mathbb{Q}$ denote sets of parameter sets, where each parameter set (or element of $\mathbb{P}$, $\mathbb{Q}$) contains a finite number of parameters with an arbitrary structure, e.g., a parameter set or element in $\mathbb{P}$, $\mathbb{Q}$ can be a matrix of certain fixed dimensions. Let us now define a function $V : \mathbb{R}^n \times \mathbb{P} \to \mathbb{R}_+$, which is zero at zero for all elements in $\mathbb{P}$ and a function $\sigma : \mathbb{R}_+ \times \mathbb{Q} \to \mathbb{R}_+$ with $\sigma(\cdot, Q) \in \mathcal{K}$ for all $Q \in \mathbb{Q} \setminus \{0\}$. Next, let $(P_1, P_2) \in \mathbb{P} \times \mathbb{P} =: \mathbb{P}^2$, $Q \in \mathbb{Q}$ and consider the following inequalities for some $x \in \mathbb{X}$ and $w \in \mathbb{W}$:

$$\alpha_1(\|x\|) \leq V(x, P_1) \leq \alpha_2(\|x\|), \tag{4a}$$

$$V(\Phi(x, w), P_2) \leq \rho V(x, P_1) + \sigma(\|w\|, Q). \tag{4b}$$

Consider the set-valued map $\mathcal{P} : \mathbb{R}^n \times \mathbb{R}^d \rightrightarrows \mathbb{P} \times \mathbb{P} \times \mathbb{Q}$,

$$\mathcal{P}(x, w) := \{(P_1, P_2, Q) \in \mathbb{P}^2 \times \mathbb{Q} \mid \text{(4a) and (4b) hold}\}. \tag{5}$$

For any $(x, w) \in \mathbb{X} \times \mathbb{W}$, $\mathcal{P}(x, w) \neq \emptyset$ denotes the fact that there exists at least one triplet $(P_1, P_2, Q) \in \mathbb{P}^2 \times \mathbb{Q}$ that satisfies (4). To distinguish between the outputs of $\mathcal{P}$ we will use $[\mathcal{P}(x, w)]_i$, $i \in \mathbb{Z}_{[1,3]}$ to denote the set where the i-th component of a triplet $(P_1, P_2, Q) \in \mathbb{P}^2 \times \mathbb{Q}$ that satisfies (4) takes values. With a slight abuse of notation we will use $P(x)$, $Q(w)$ to denote any $P_1 \in [\mathcal{P}(x, w)]_1$ and $Q \in [\mathcal{P}(x, w)]_3$, respectively.

Definition 4. A function $V(x, P(x))$ with $P(x) \in [\mathcal{P}(x)]_1$ is called a parameterized ISS Lyapunov function with respect to

$\mathbb{X}$ and $\mathbb{W}$ (p-ILF($\mathbb{X}$,$\mathbb{W}$)) for system (1) if $\mathbb{X}$ is PI($\mathbb{W}$) and there exists a function σ such that:

$$\mathcal{P}(x,w) \neq \emptyset, \quad \forall (x,w) \in \mathbb{X} \times \mathbb{W}, \tag{6a}$$

$$[\mathcal{P}(x,w)]_2 \cap [\mathcal{P}(\Phi(x,w),v)]_1 \neq \emptyset, \quad \forall (x,w,v) \in \mathbb{X} \times \mathbb{W}^2. \tag{6b}$$

In the nominal case, i.e., when $\mathbb{W} = \{0\}$, the above definition recovers the definition of a (weak) parameterized Lyapunov function (Lazar and Gielen, 2010), which is consistent with the standard definition of a ILF (Jiang and Wang, 2001).

Theorem 5. Let $\mathbb{X} \subseteq \mathbb{R}^n$ be a PI($\mathbb{W}$) set for (1) with $0 \in \operatorname{int}(\mathbb{X})$, for some $\mathbb{W} \subseteq \mathbb{R}^d$. Suppose that system (1) admits a p-ILF($\mathbb{X}$,$\mathbb{W}$). Then system (1) is ISS($\mathbb{X}$,$\mathbb{W}$).

The proof of Theorem 5 follows standard arguments (Jiang and Wang, 2001; Lazar, 2006) and is omitted for brevity.

To illustrate the relaxation with respect to the standard ISS result presented in the previous section, notice that now the sublevel sets of the p-ILF V can be different for each state, as they can have different Minkowski functions (Luenberger, 1969), i.e., $\mathbb{V}_c^{P(x)} := \{x \in \mathbb{R}^n \mid V(x, P(x)) \leq c\}$, $c \in \mathbb{R}_+$. Moreover, the envelope of trajectories will now satisfy $\mathcal{E}(\mathbb{X},\mathbb{W}) \subseteq \mathcal{V}_c^p(\mathbb{X},\mathbb{W})$, where

$$\mathcal{V}_c^p(\mathbb{X},\mathbb{W}) := \left\{ \mathbb{V}^{P(x(k))}_{\rho^k c + \sum_{i=0}^{k} \sigma(\|w(i)\|, Q(w(i)))} \right\}_{k \in \mathbb{Z}_+}. \tag{7}$$

Alternatively, one can also use the equivalent (for $r(0) = c$) characterization $\mathcal{E}(\mathbb{X},\mathbb{W}) \subseteq \mathcal{V}_r^p(\mathbb{X},\mathbb{W})$, where

$$\mathcal{V}_r^p(\mathbb{X},\mathbb{W}) := \left\{ \mathbb{V}^{P(x(k))}_{r(k)} \right\}_{k \in \mathbb{Z}_+}, \ r(0) \in \mathbb{R}_+, \mathbb{X} \subseteq \mathbb{V}^{P(x(0))}_{r(0)},$$
$$r(k) = \rho r(k-1) + \sigma(\|w(k-1)\|, Q(w(k-1))), \tag{8}$$

for all $k \in \mathbb{Z}_{\geq 1}$. Note that the above characterizations of the envelope of trajectories that corresponds to a p-ILF indicate that besides guaranteeing ISS in the standard sense, a p-ILF offers much more freedom, which can be used to synthesize input-to-state stabilizing control laws that also deliver the optimized ISS property, as it will be shown in the next section.

The following converse result reveals the non-conservatism of p-ILFs, even when a particular structure is imposed. We will consider perhaps the most popular type of structure for candidate Lyapunov and supply functions, i.e., a p-quadratic-ILF defined as $V(x, P(x)) := x^\top P(x) x$ and a quadratic supply defined as $\sigma(\|w\|, Q(w)) := \|Q^{\frac{1}{2}}(w) w\|_2^2 = w^\top Q(w) w$, with $P(x) \in [\mathcal{P}(x,w)]_1$, $Q(w) \in [\mathcal{P}(x,w)]_3$, $\mathcal{P}(x,w) \subseteq \mathbb{P}^2 \times \mathbb{Q}$ for all (x,w), where $\mathbb{P} \subseteq \mathbb{R}^{n \times n}$ and $\mathbb{Q} \subseteq \mathbb{R}^{d \times d}$. In what follows, the class of systems (1) is restricted to $\Phi(x,w) := \Phi(x,0) + w$, $w \in \mathbb{W} \subseteq \mathbb{R}^n$ with $0 \in \operatorname{int}(\mathbb{W})$, i.e., the case of outer additive disturbances with $d = n$ is considered.

Theorem 6. Let $\mathbb{X} \subseteq \mathbb{R}^n$ be a compact PI($\mathbb{W}$) set for system (1) with $0 \in \operatorname{int}(\mathbb{X})$ and suppose that the nominal system corresponding to (1), i.e., $x(k+1) = \Phi(x(k), 0)$, $k \in \mathbb{Z}_+$, is ES($\mathbb{X}$). Then, system (1) admits a p-quadratic-ILF($\mathbb{X}$,$\mathbb{W}$).

The proof of Theorem 6 makes use of the standard converse result in (Jiang and Wang, 2002) to establish the existence of a generic continuous Lyapunov function which is then employed to construct an admissible p-quadratic-ILF($\mathbb{X}$,$\mathbb{W}$). The details of the proof are omitted for brevity. For a similar construction in the nominal case see (Lazar and Gielen, 2010).

4. RECEDING HORIZON ROBUST CONTROL USING P-QUADRATIC-ILFS

The focus of this section is the design of a tractable optimization problem that implements the search for a p-ILF for discrete-time systems. To this end, several simplifying assumption will be made, although the theoretical concepts and ISS results apply in fact to general discrete-time nonlinear systems and p-ILF candidates.

To begin with, the class of systems considered in this section is of the form

$$\begin{aligned} x(k+1) &= \phi(x(k), u(k), w(k)), \\ &= f(x(k)) + g(x(k)) u(k) + w(k), \ k \in \mathbb{Z}_+, \end{aligned} \tag{9}$$

where $x(k) \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $u(k) \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control input, $w(k) \in \mathbb{W} \subseteq \mathbb{R}^n$ is the disturbance input and $\phi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is an arbitrary continuous map with $\phi(0,0,0) = 0$.

Assumption 7. The sets $\mathbb{X}$, $\mathbb{U}$, $\mathbb{W}$ are polytopes, $0 \in \operatorname{int}(\mathbb{X})$ and $0 \in \operatorname{int}(\mathbb{U})$. □

Definition 8. A set $\mathbb{X} \subseteq \mathbb{R}^n$ is called robust constrained control invariant with respect to $\mathbb{U}$, $\mathbb{W}$ (or shortly CCI($\mathbb{X}$,$\mathbb{U}$,$\mathbb{W}$)) for system (9) if for all $x \in \mathbb{X}$, $\exists u \in \mathbb{U}$ such that $\phi(x, u, \mathbb{W}) \subseteq \mathbb{X}$.

Above, $\phi(x, u, \mathbb{W})$ is defined similarly as $\Phi(x, \mathbb{W})$.

Assumption 9. $\mathbb{X} \subseteq \mathbb{R}^n$ is a CCI($\mathbb{X}$,$\mathbb{U}$,$\mathbb{W}$) set for the discrete-time system (9). □

Consider also a p-quadratic-ILF candidate of the form:

$$V(x, P(x)) := x^\top P(x) x, \ P(x) \succ 0, \forall x,$$

with supply function of the form:

$$\sigma(\|w\|, Q(w)) := w^\top Q(w) w, \ Q(w) \succeq 0, \forall w. \tag{10}$$

Notice that although the above supply function candidate does not necessarily satisfy $\sigma(\cdot, Q(w)) \in \mathcal{K}$ for all $Q(w)$, as it is allowed that $Q(w) = 0$ for some $w \in \mathbb{W} \setminus \{0\}$, it always enjoys the class $\mathcal{K}_\infty$ upper bound $a\|w\|_2^2$ with $a \in \mathbb{R}_{\geq \lambda_{\max}(Q(w))} \cap \mathbb{R}_{>0}$. As such, if inequality (4b) holds with a function σ as in (10), then it also holds with $\sigma(\|w\|) := a\|w\|_2^2 \in \mathcal{K}_\infty$.

Next, define $\mathcal{L} := \mathbb{Z}_{[1,L]}$, $L \in \mathbb{Z}_{\geq n+1}$, let $\{w^l\}_\mathcal{L}$ denote the set of vertices of the polytope $\mathbb{W}$ and let $\mathcal{Q} := \{Q_{jl}\}_{(j,l) \in \mathcal{L}^2}$ with $Q_{jl} \in \mathbb{R}^{n \times n}$ be a set of symmetric matrices such that ${w^j}^\top Q_{jl} w^l \geq 0$ for all $(j,l) \in \mathcal{L}^2$. For any $w \in \mathbb{W}$ let $\mu_l \in \mathbb{R}_{[0,1]}$ with $\sum_{l \in \mathcal{L}} \mu_l = 1$ be such that $w = \sum_{l \in \mathcal{L}} \mu_l w^l$ and consider the following matrix equation:

$$\begin{aligned} &\left(\sum_{l \in \mathcal{L}} \mu_l w^l \right)^\top Q(w) \left(\sum_{l \in \mathcal{L}} \mu_l w^l \right) \\ &= \sum_{l \in \mathcal{L}} \mu_l^2 {w^l}^\top Q_{ll} w^l + \sum_{l \in \mathcal{L}} \sum_{j \in \mathcal{L}, j \neq l} \mu_l \mu_j {w^j}^\top Q_{jl} w^l. \end{aligned} \tag{11}$$

Notice that for $\{Q_{ij}\}_{(i,j \in \mathcal{L}^2)} \in \mathcal{Q}$, the set of solutions of the above equation is non-empty for all $w \in \mathbb{W}$. Also, observe that for $w \neq 0$ the left hand term in the above equation is a positive real number obtained as the sum of positive terms and possibly, negative terms, as $Q(w) \succeq 0$, while the right hand term is a positive real number obtained as the sum of positive terms only. As such, the right hand term is not just a matching expression of the left hand term.

Let $\phi_\mathrm{n}(1, x(k), u(0|k))$ denote the 1-step ahead predicted nominal state calculated at time $k \in \mathbb{Z}_+$, i.e., $\phi_\mathrm{n}(1, x(k), u(0|k)) =$

$f(x(k))+g(x(k))u(0|k)$. For brevity, $\phi_{\rm n}(1,x(k))$ will be used to denote $\phi_{\rm n}(1,x(k),u(0|k))$ in what follows. Let $\gamma\in\mathbb{R}_{>0}$, $\Gamma\in\mathbb{R}_{\geq\gamma}$ and consider the following set of inequalities:

$$\phi_{\rm n}(1,x(k))+w^l\in\mathbb{X},\ \forall l\in\mathcal{L},\ u(0|k)\in\mathbb{U}, \tag{12}$$

$${w^j}^\top Q_{jl}(1|k)w^l\geq 0,\quad \forall(j,l)\in\mathcal{L}^2, \tag{13}$$

$$\begin{aligned} &x(k)^\top(P(x(k))-\gamma I_n)x(k)\geq 0,\\ &x(k)^\top(\Gamma I_n-P(x(k)))x(k)\geq 0, \end{aligned}\tag{14}$$

$$Z(1|k)-\Gamma^{-1}I_n\succeq 0,\ \gamma^{-1}I_n-Z(1|k)\succeq 0, \tag{15}$$

$$\begin{pmatrix}\rho x(k)^\top P(x(k))x(k)+{w^j}^\top Q_{jl}(1|k)w^l & *\\ \phi_{\rm n}(1,x(k))+w^p & Z(1|k)\end{pmatrix}\succeq 0,\quad \forall(j,l,p)\in\mathcal{L}^3, \tag{16a}$$

$$P(x(k))=Z(1|k-1)^{-1},\quad \forall k\in\mathbb{Z}_{\geq 1}, \tag{16b}$$

and

$$\begin{pmatrix}\rho x(k)^\top P(x(k))x(k) & *\\ \phi_{\rm n}(1,x(k)) & Z(1|k)\end{pmatrix}\succeq 0. \tag{17}$$

In what follows assume that the current state $x(k)$, the constants γ,Γ, the polytopes $\mathbb{X}$, $\mathbb{W}$, $\mathbb{U}$ and the set of vertices $\{w^l\}_{l\in\mathcal{L}}$ are known. Consider the following optimization problems.

Problem 10. At time $k\in\mathbb{Z}_+$ infimize

$$\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{L}}{w^j}^\top Q_{jl}(1|k)w^l \tag{18}$$

over the set of unknown variables $\{Q_{ij}\}_{(i,j)\in\mathcal{L}^2}$, $Z(1|k)$, $u(0|k)$ (and $P(x(0))$ at $k=0$) subject to the set of inequalities (12), (13), (15), (16), (17) (and (14) at $k=0$).

Problem 11. At time $k\in\mathbb{Z}_+$ infimize

$$\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{L}}{w^j}^\top Q_{jl}(1|k)w^l \tag{19}$$

over the set of unknown variables $\{Q_{ij}\}_{(i,j)\in\mathcal{L}^2}$, $Z(1|k)$, $u(0|k)$ (and $P(x(0))$ at $k=0$) subject to the set of inequalities (12), (13), (15), (16) (and (14) at $k=0$).

The first observation about the above problems is that in both cases, due to the inequalities (13), the infimum is a minimum. Moreover, as the cost is a linear function of the matrix variables $\{Q_{ij}(1|k)\}_{(i,j)\in\mathcal{L}^2}$ and all inequalities are either linear inequalities or linear matrix inequalities in the unknown variables at each $k\in\mathbb{Z}_+$, a solution to either Problem 10 or Problem 11 can be found by solving a single SDP problem at each $k\in\mathbb{Z}_+$. Also, notice that the number of inequalities in (13) and (16) can be significantly reduced by setting $Q_{jl}(1|k)=Q_{lj}(1|k)$ for all $(j,l)\in\mathcal{L}^2, j\neq l$, as then ${w^j}^\top Q_{jl}(1|k)w^l={w^l}^\top Q_{lj}(1|k)w^j$ due to symmetry of the elements of $\mathcal{Q}$.

To simplify the exposition of the following results, rather then introducing a difference inclusion that corresponds to (9) in closed-loop with the set of feasible control inputs for each $x(k)$ defined by (12)-(16), consider a control law $u_f:\mathbb{R}^n\to\mathbb{U}$ that selects for each state an arbitrary element of the admissible set of inputs that corresponds to (12)-(16). Let $\mathcal{Q}^*(k):=\{Q^*_{ij}(1|k)\}_{(i,j)\in\mathcal{L}^2}$ denote a set of matrices that attains the optimum in (18), or (19), and let $Q^*(w(k))$ denote a solution that satisfies the corresponding equation (11). In what follows we will implicitly make use of the fact that for each $\mathcal{Q}^*(k)$, $k\in\mathbb{Z}_+$, (11) admits a solution for all $w\in\mathbb{W}\setminus\{0\}$.

Theorem 12. Suppose that Assumption 9 holds, $0\in\mathbb{W}$ and the inequalities (15) and (17) (and (14) at time $k=0$) are recursively feasible in $\mathbb{X}$. Then Problem 10 is recursively feasible in $\mathbb{X}$ and the corresponding closed-loop system, i.e.,

$$x(k+1)=\phi(x(k),u_f(x(k)),w(k)),\quad k\in\mathbb{Z}_+, \tag{20}$$

is ISS($\mathbb{X}$,$\mathbb{U}$,$\mathbb{W}$). Moreover, system (20) enjoys the optimized ISS property in the sense that $\mathcal{E}(\mathbb{X},\mathbb{W})\subseteq\mathcal{V}^p_r(\mathbb{X},\mathbb{W})$, where

$$\begin{aligned}&\mathcal{V}^p_r(\mathbb{X},\mathbb{W}):=\left\{\mathbb{V}^{P(x(k))}_{r(k)}\right\}_{k\in\mathbb{Z}_+},\ r(0)\in\mathbb{R}_+,\mathbb{X}\subseteq\mathbb{V}^{P(x(0))}_{r(0)},\\ &r(k)=\rho r(k-1)+w(k-1)^\top Q^*(w(k-1))w(k-1),\\ &\qquad\forall k\in\mathbb{Z}_{\geq 1}.\end{aligned}\tag{21}$$

Proof. Let us first establish recursive feasibility of Problem 10. Notice that inequality (16) is equivalent, via the Schur complement with:

$$\begin{aligned}&\rho x(k)^\top P(x(k))x(k)+{w^j}^\top Q_{jl}(1|k)w^l\\ &\geq(\phi_{\rm n}(1,x(k))+w^p)^\top Z(1|k)^{-1}(\phi_{\rm n}(1,x(k))+w^p).\end{aligned}\tag{22}$$

Let

$$\begin{aligned}\Upsilon(1|k):=&\sup_{u(0|k)\in\mathbb{U},P(x(k)),Z(1|k),p\in\mathcal{L}}\{x^p(1|k)^\top Z(1|k)^{-1}x^p(1|k)\\ &-\rho x(k)^\top P(x(k))x(k)\},\end{aligned}$$

where $x^p(1|k):=\phi_{\rm n}(1,x(k))+w^p$ and the supremum is a maximum for all $x(k)\in\mathbb{X}$ due to boundedness of $\mathbb{X}$, $\mathbb{U}$, $\mathcal{L}$, inequalities (15), (16b) (and (14) at $k=0$) and continuity of ϕ. Then, the set of matrices $\mathcal{Q}$ with

$$w^jQ_{jl}(1|k)w^l\geq\max\{\Upsilon(1|k),0\},\quad\forall(j,l)\in\mathcal{L}^2$$

satisfies (16) and (13) for all $k\in\mathbb{Z}_+$. Also, observe that (12) is recursively feasible by Assumption 9 and it implies that $x(k)\in\mathbb{X}$ for all $x(0)\in\mathbb{X}$ and all $\mathbf{w}$ with $w(k)\in\mathbb{W}$ for all $k\in\mathbb{Z}_+$. As such, together with recursive feasibility of (15) and (17) (and (14) at time $k=0$) it follows that Problem 10 is recursively feasible.

Next, let us prove that $V(x(k),P(x(k)))=x(k)^\top P(x(k))x(k)$ is a p-quadratic-ILF for system (20). The fact that $P(x(k))=Z(1|k-1)^{-1}$ for all $k\in\mathbb{Z}_{\geq 1}$ together with inequalities (15) and inequality (14) at time $k=0$ implies that V satisfies inequality (4a) with $\alpha_1(s):=\gamma s^2$ and $\alpha_2(s):=\Gamma s^2$. Then, let $w(k)=\sum_{p\in\mathcal{L}}\mu_p(k)w^p\neq 0$ for some $\mu_p(k)\in\mathbb{R}_{[0,1]}$ with $\sum_{p\in\mathcal{L}}\mu_p(k)=1$ for all $k\in\mathbb{Z}_+$. By multiplying the inequality (16) with $\mu_p(k)$, summing up and applying the Schur complement, yields

$$\begin{aligned}&\rho x(k)^\top P(x(k))x(k)+{w^j}^\top Q_{jl}(1|k)w^l\\ &\geq(\phi_{\rm n}(1,x(k))+w(k))^\top Z(1|k)^{-1}(\phi_{\rm n}(1,x(k))+w(k)),\end{aligned}\tag{23}$$

for all $(j,l)\in\mathcal{L}^2$ and all $w(k)\in\mathbb{W}\setminus\{0\}$. Multiplying the above inequality with $\mu_l^2(k)$ for $j=l$ and with $\mu_l(k)\mu_j(k)$ for $(j,l)\in\mathcal{L}^2, j\neq l$, summing up and using the fact that $\sum_{l\in\mathcal{L}}\mu_l^2(k)+\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{L},j\neq l}\mu_l(k)\mu_j(k)=\left(\sum_{l\in\mathcal{L}}\mu_l(k)\right)\left(\sum_{l\in\mathcal{L}}\mu_l(k)\right)=1$ yields:

$$\begin{aligned}&\rho x(k)^\top P(x(k))x(k)\\ &+\sum_{l\in\mathcal{L}}\mu_l^2(k){w^l}^\top Q_{ll}w^l+\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{L},j\neq l}\mu_l(k)\mu_j(k){w^j}^\top Q_{jl}w^l\\ &\geq(\phi_{\rm n}(1,x(k))+w(k))^\top Z(1|k)^{-1}(\phi_{\rm n}(1,x(k))+w(k)),\end{aligned}\tag{24}$$

for all $(j, l) \in \mathcal{L}^2$ and all $w(k) \in \mathbb{W} \setminus \{0\}$. Then, using relation (11) and (16b) yields that:

$$V(\phi(x(k), u_f(x(k)), w(k)), P(x(k+1))) - \rho V(x(k), P(x(k))) \leq w(k)^\top Q^*(w(k))w(k), \quad (25)$$

for all $w(k) \in \mathbb{W} \setminus \{0\}$ and all $k \in \mathbb{Z}_+$. As such, observing that (17) is equivalent by Schur complement with

$$V(\phi(x(k), u_f(x(k)), 0), P(x(k+1))) \leq \rho V(x(k), P(x(k))), \quad (26)$$

implies that V is a p-quadratic-ILF for system (20) with supply function $\sigma(\|w(k)\|) := a\|w(k)\|_2^2$ and

$$a \in \mathbb{R}_{\geq \sup_{k\in\mathbb{Z}_+}\{\lambda_{\max}(Q^*(w(k)))\}} \cap \mathbb{R}_{>0},$$

where it is straightforward to establish that the supremum is a maximum, similarly as done for $\Upsilon(1|k)$. Hence, by Theorem 5 system (20) is ISS($\mathbb{X}$,$\mathbb{U}$,$\mathbb{W}$). The last claim follows directly by applying recursively inequality (25), which, by inequality (26), holds for all $w(k) \in \mathbb{W}$. □

Recursive feasibility of Problem 10 hinges on recursive feasibility of the "nominal stabilization" inequalities (15) and (17), which ensure that V is a parameterized quadratic Lyapunov function for the nominal system $x(k+1) = \phi_n(x(k), u_f(x(k)))$, $k \in \mathbb{Z}_+$. A detailed treatment of nominal stabilization using parameterized Lyapunov functions and receding horizon control was presented recently in (Lazar and Gielen, 2010). The interested reader can find therein sufficient conditions for recursive feasibility of (15) and (17), which involve a rather technical set of assumptions and thus, they are omitted here.

Theorem 13. Suppose that Assumption 9 holds and $0 \notin \mathbb{W}$. Then Problem 11 is recursively feasible in $\mathbb{X}$ and the envelope of trajectories of the corresponding closed-loop system, i.e.,

$$x(k+1) = \phi(x(k), u_f(x(k)), w(k)), \quad k \in \mathbb{Z}_+, \quad (27)$$

satisfies $\mathcal{E}(\mathbb{X}, \mathbb{W}) \subseteq \mathcal{V}_r^p(\mathbb{X}, \mathbb{W})$, where

$$\mathcal{V}_r^p(\mathbb{X}, \mathbb{W}) := \left\{\mathbb{V}_{r(k)}^{P(x(k))}\right\}_{k\in\mathbb{Z}_+}, \; r(0) \in \mathbb{R}_+, \mathbb{X} \subseteq \mathbb{V}_{r(0)}^{P(x(0))},$$
$$r(k) = \rho r(k-1) + w(k-1)^\top Q^*(w(k-1))w(k-1), \quad \forall k \in \mathbb{Z}_{\geq 1}. \quad (28)$$

The proof of the above theorem parallels, *mutatis mutandis*, the proof of Theorem 12 and is omitted for brevity. This result applies to the case of persistent disturbances and offers an a priori guarantee of recursive feasibility, as (17) is no longer required. Notice that ISS is no longer obtained in this case, but this is not an issue, as $0 \notin \mathbb{W}$. However, an explicit optimized envelope of trajectories is provided for the closed-loop system, which implies the standard corresponding property of ultimate boundedness, see, e.g., (Rawlings and Mayne, 2009).

A number of remarks with respect to Problem 10 and the corresponding control law are in order, before proceeding with the presentation of an illustrative example.

Remark 14. Given an arbitrary ISS closed-loop system (Jiang and Wang, 2001) that admits a quadratic, possibly time-varying control ISS Lyapunov function $\tilde{V}$ and supply function $\tilde{\sigma}$, respectively, with corresponding envelope of trajectories $\tilde{\mathcal{E}}(\mathbb{X}, \mathbb{W}) \subseteq \tilde{\mathcal{V}}_r^p(\mathbb{X}, \mathbb{W})$, it could be established via optimality of the supply attained with Problem 10 that $\mathcal{V}_r^p(\mathbb{X}, \mathbb{W}) \subseteq \tilde{\mathcal{V}}_r^p(\mathbb{X}, \mathbb{W})$. In other words, the worst case element-wise Hausdorff distance between the envelope of trajectories and the nominal envelope provided by the controller obtained via Problem 10 would either match or outperform the one provided by any other, arbitrary ISS controller that yields a quadratic ILF and supply function, respectively, for the closed-loop system. A formal proof of this conjecture makes the object of future research. □

Remark 15. Similarly as achieved by the tube-based robust MPC methodology, see, e.g., (Mayne et al., 2005; Raković, 2009; Rawlings and Mayne, 2009), the proposed parameterized ISS inequality generates a closed-loop trajectory that is kept as close as possible to the *nominal* closed-loop trajectory for all admissible disturbances. More precisely, the proposed receding horizon robust control scheme minimizes at each time instant the Hausdorff distance between the next element (i.e., a sublevel set of V) of the envelope of closed-loop trajectories and the corresponding element of the *nominal* envelope. In tube-based MPC a similar objective is pursued by constraining the perturbed trajectory to lie within an envelope consisting of a fixed (time wise) approximation of the minimal robustly positively invariant (mRPI) set (Raković et al., 2005) that is RPI. The flexibility of the tube-based design can be improved by parameterizing the tube center, radius or the vertices of the fixed RPI set, which was only recently studied in (Kouvaritakis et al., 2010), for linear systems. Such a parameterization corresponds to the optimized ISS schemes for nonlinear systems provided in (Lazar and Heemels, 2008), (Lazar et al., 2009), which rely on a known, fixed control Lyapunov function with a sublevel set that has a flexible center or radius; see also (Lazar, 2009), where the notion of a flexible Lyapunov function was introduced. However, the parameterized ISS inequality proposed in this paper allows for complete freedom, within the specified family of quadratic functions, as it generates the Minkowski function (Luenberger, 1969) of the envelope section for each state along the trajectory. In the tube-based setting this would amount to including the construction of the approximation of the mRPI set in the corresponding "on-line" optimization problem. □

5. ILLUSTRATIVE EXAMPLE

Consider the model of a continuous stirred tank reactor (Seborg et al., 2004), which is a benchmark example for nonlinear MPC, see, e.g., (Mahmood and Mhaskar, 2008; Mhaskar et al., 2006; Magni et al., 2001). In this reactor an irreversible, first-order exothermic reaction of the form $A \rightarrow B$ takes place. The mathematical model of this reaction is highly nonlinear, i.e.,

$$\begin{aligned}\dot{C}_A &= \frac{F}{V}(C_{A0s} - C_A) - k_0 e^{\frac{-E}{RT_R}} C_A + \frac{F}{V}\Delta C_A,\\ \dot{T}_R &= \frac{F}{V}(T_{A0s} - T_R) - \frac{-\Delta H}{\rho c_p} k_0 e^{\frac{-E}{RT_R}} C_A + \frac{Q}{\rho c_p V},\end{aligned} \quad (29)$$

where C_A denotes the concentration of the reactant A and T_R denotes the temperature in the reactor. The control inputs are Q, the heat that is added to the reactor, and ΔC_A, the change in the inlet concentration of the reactant A. The values of all process parameters can be found in Table 1. The objective is to input-to-state stabilize the reactor at an open-loop unstable equilibrium point, i.e., $(C_A, T_R) = (0.7, 392.7)$, while respecting the state and input constraints, i.e., $(C_A, T_R) \in \mathbb{R}_{[0.41,0.73]} \times \mathbb{R}_{[392.3,398.3]}$ and $(\Delta C_A, Q) \in \mathbb{R}_{[-1,1]} \times \mathbb{R}_{[-480,480]}$, respectively. Additionally, effective disturbance rejection is desirable. The model is discretized using the forward Euler discretization method with sampling period $t_s = 0.5$ seconds and discrete state vector $x(k) = (C_A(kt_s), T_R(kt_s))^\top$, which yields a system of the form (9) with

$$f(x(k)) = x(k) + \begin{bmatrix} \frac{F}{V}(C_{A0s} - [x(k)]_1) - k_0 e^{\frac{-E}{R[x(k)]_2}}[x(k)]_1 \\ \frac{F}{V}(T_{A0s} - [x(k)]_2) - \frac{-\Delta H}{\rho c_p} k_0 e^{\frac{-E}{R[x(k)]_2}}[x(k)]_1 \end{bmatrix} t_s,$$

and $g(x(k)) = \begin{bmatrix} \frac{F}{V} & 0 \\ 0 & \frac{1}{\rho c_p V} \end{bmatrix}$. Furthermore, the additive disturbance satisfies $w(k) \in \mathbb{W} := \mathbb{R}_{[-0.012, 0.012]} \times \mathbb{R}_{[0.12 \times 0.12]}$ for all $k \in \mathbb{Z}_+$. To implement Problem 10, a straightforward coordinate change was performed to translate the equilibrium point in zero and then a backward transformation was employed to obtain the applied control action.

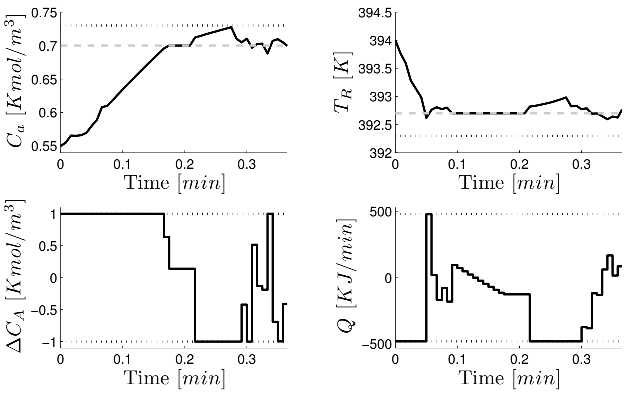

Fig. 1. Simulation results of the method proposed in this paper (——), the constraints ($\cdots$) and the desired equilibrium values (— — —) for system (9).

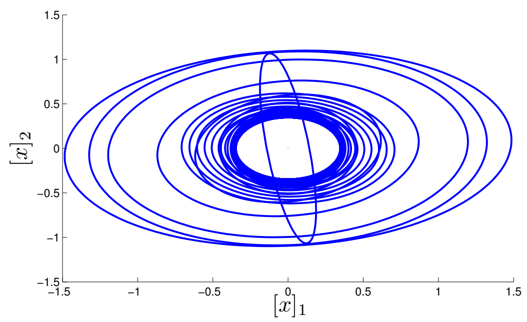

Fig. 2. The sublevel sets $\{\mathbb{V}_1^{P(x(k))}\}_{k \in \mathbb{Z}_{[0,45]}}$.

Problem 10 with $\rho = 0.8$, $\gamma = 0.01$ and $\Gamma = 100$ was used to calculate a control action for system (9) with initial condition $(C_A(0), T_R(0)) = (0.55, 394)$. The disturbance profile is chosen randomly during the first 10 samples, equal to zero for the next 15 samples, then $[w(k)]_2 = 0.12$ and $[w(k)]_1 = 0.012$ during 8 samples and, during the final 12 samples of the simulation, the disturbance is again chosen randomly. Figure 1 shows the state trajectories and control input values as a function of time. In Figure 2, a plot of the evolution of the sublevel sets of the p-quadratic-ILF, i.e., $\{\mathbb{V}_c^{P(x(k))}\}_{k \in \mathbb{Z}_{[0,45]}}$ for $c = 1$, is given. This indicates that the freedom of adapting the Minkowski function of each set that forms the envelope of closed-loop trajectories is fully exploited by the proposed controller. Figure 3 shows the trajectory envelope corresponding to the p-quadratic-ILF (in gray). To indicate the effectiveness of the developed controller in terms of disturbance rejection, in the same plot the envelope $\mathcal{W} := \{\mathbb{S}(k)\}_{k \in \mathbb{Z}_+}$ with $\mathbb{S}(k) = \mathbb{W}$ for all $k \in \mathbb{Z}_+$, is plotted in red. The actual trajectory (denoted by $- \times -$) and the realization of the disturbance $w(k)$ are also shown in Figure 3.

Table 1. Process parameters

$V = 0.1$	m^3
$R = 8.314$	$KJKmol^{-1}K^{-1}$
$C_{A0s} = 1$	$Kmolm^{-3}$
$T_{A0s} = 310$	K
$\Delta H = -4.78 \times 10^4$	$KJKmol^{-1}$
$k_0 = 72 \times 10^9$	min^{-1}
$E = 8.314 \times 10^4$	$KJKmol^{-1}$
$c_p = 0.239$	$KJKg^{-1}K^{-1}$
$\rho = 1000$	Kgm^{-3}
$F = 100 \times 10^3$	$m^3 min^{-1}$

The semi-definite programming problem to be solved at each time instant consisted of a linear matrix inequality of dimension 122×122. This dimension can be further reduced by setting $Q_{jl}(1|k) = Q_{lj}(1|k)$ for $(j, l) \in \mathcal{L}^2, j \neq l$, as explained earlier in this section. Still, in this form the SDP solver SeDuMi (Sturm, 2001) managed to always solve Problem 10 within the allowed sampling interval. Moreover, at each time instant $k \in \mathbb{Z}_+$ a solution to (11) was obtained. It is worth to mention that all of the MPC solutions for stabilizing the considered example, which were referred to at the beginning of this subsection, do not consider disturbances and still, they require solving on-line a nonlinear optimization problem with the $fmincon$ solver of Matlab.

Furthermore, the proposed control scheme manages to input-to-state stabilize the system, as it can be observed when the disturbance is set equal to zero (i.e., the states converge to the equilibrium for zero disturbance), while it provides "optimized ISS", as it can be observed from the extremely tight envelope of trajectories, with respect to the disturbance set-envelope $\mathcal{W}$. At all times the specified state and input constraints are satisfied non-trivially.

6. CONCLUSIONS

This paper has provided results on existence and synthesis of parameterized ISS Lyapunov functions for discrete-time nonlinear systems that are possibly subject to constraints. A p-ILF was defined by assigning a finite set of parameters to a standard ISS Lyapunov function, which can take different values for each state and disturbance input. It was demonstrated that the so-obtained p-ILFs offer non-conservative analysis conditions, even when functions with a particular structure, such as quadratic forms, are considered. Furthermore, a method for synthesizing p-ILFs for discrete-time nonlinear systems was proposed. For bounded inputs, it was shown how parameterized ISS inequalities can be used to synthesize a closed-loop system with an optimized envelope of trajectories. An implementation method based on receding horizon optimization was presented, along with a recursive feasibility and complexity analysis. The advances provided by the proposed synthesis methodology were illustrated for a nonlinear model of a continuous stirred tank reactor.

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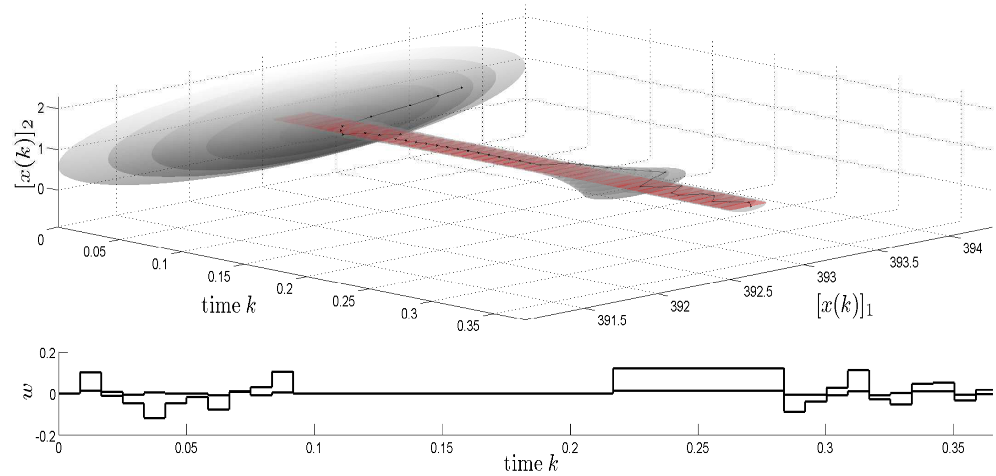

Fig. 3. The closed-loop trajectory envelope (in gray), disturbance set-envelope $\mathcal{W}$ (in red) and the actual trajectory ($- \times -$).

REFERENCES

Chen, H. and Scherer, C.W. (2006). Moving horizon $\mathcal{H}_\infty$ control with performance adaption for constrained linear systems. *Automatica*, 42, 1033–1040.

Freeman, R.A. and Kokotović, P.V. (2008). *Robust nonlinear control design*. Birkäuser Boston. Reprint of the 1996 edition.

Jiang, Z.P. and Wang, Y. (2001). Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37, 857–869.

Jiang, Z.P. and Wang, Y. (2002). A converse Lyapunov theorem for discrete-time systems with disturbances. *Systems & Control Letters*, 45, 49–58.

Kouvaritakis, B., Cannon, M., and Yadlin, R. (2010). On the centres and scalings of robust and stochastic tubes. In *UKACC International Conference on Control*. Conventry, United Kingdom.

Krstić, M., Kanellakopoulos, I., and Kokotović (1995). *Nonlinear and adaptive control design*. John Wiley & Sons, Inc. New York.

Lazar, M. (2006). *Model predictive control of hybrid systems: Stability and robustness*. Ph.D. thesis, Eindhoven University of Technology, The Netherlands.

Lazar, M. (2009). Flexible control Lyapunov functions. In *American Control Conference*, 102–107. St. Louis, MO.

Lazar, M. and Gielen, R.H. (2010). On parameterized Lyapunov and control Lyapunov functions for discrete-time systems. In *49th IEEE Conference on Decision and Control*, 3264–3270. Atlanta, Georgia, USA.

Lazar, M. and Heemels, W.P.M.H. (2008). Optimized input-to-state stabilization of discrete-time nonlinear systems with bounded inputs. In *American Control Conference*. Seattle.

Lazar, M., Heemels, W.P.M.H., and Jokić, A. (2009). Self-optimizing robust nonlinear model predictive control. In *Nonlinear Model Predictive Control*, volume 384 of *Lecture Notes in Control and Information Sciences*, 27–40. Springer-Verlag.

Lazar, M., Heemels, W.P.M.H., and Teel, A.R. (2010). Further input-to-state stability subtleties for discrete-time systems. In *UKACC International Conference on Control*. Coventry, UK.

Liberzon, D., Sontag, E.D., and Wang, Y. (2002). Universal construction of feedback laws achieving ISS and integral-ISS disturbance attenuation. *Systems & Control Letters*, 46, 111–127.

Luenberger, D.G. (1969). *Optimization by vector space methods*. John Wiley & Sons, Inc., Chichester, United Kingdom.

Magni, L., De Nicolao, G., Magnani, L., and R., S. (2001). A stabilizing model-based predictive control algorithm for nonlinear systems. *Automatica*, 37, 1351–1362.

Mahmood, M. and Mhaskar, P. (2008). Enhanced stability regions for model predictive control of nonlinear process systems. *Process Systems Engineering*, 54(6), 1487–1498.

Malisoff, M. and Sontag, E.D. (2004). Asymptotic controllability and input-to-state stabilization: The effect of actuator errors. In *Optimal Control, Stabilization, and Nonsmooth Analysis*, volume 301 of *Lecture Notes in Control and Information Sciences*, 155–171. Springer-Verlag, New York.

Mayne, D.Q., Seron, M.M., and Raković, S.V. (2005). Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41, 219–224.

Mhaskar, P., El-Farra, N.H., and Christofides, P.D. (2006). Stabilization of nonlinear systems with state and control constraints using Lyapunov-based predictive control. *Systems & Control Letters*, 55, 650–659.

Raković, S.V. (2009). Set theoretic methods in model predictive control. In *Nonlinear Model Predictive Control*, volume 384 of *Lecture Notes in Control and Information Sciences*, 41–54. Springer-Verlag.

Raković, S.V., Kerrigan, E.C., Kouramas, K., and Mayne, D.Q. (2005). Invariant approximations of the minimal robust positively invariant set. *IEEE Transactions on Automatic Control*, 50, 406–410.

Rawlings, J.B. and Mayne, D.Q. (2009). *Model Predictive Control: Theory and Design*. Nob Hill Publishing.

Seborg, D.E., Edgar, T.F., and Mellichamp, D.A. (2004). *Process dynamics and control*. John Wiley & Sons, Hoboken, NJ.

Sontag, E.D. (1990). Further facts about input to state stabilization. *IEEE Transactions on Automatic Control*, AC–35, 473–476.

Sturm, J.F. (2001). SeDuMi: Matlab toolbox for solving optimization problems over symmetric cones. Toolbox available online at: http://sedumi.mcmaster.ca/.