On Parameterized Path and Chordless Path Problems

Yijia Chen
BASICS, Department of Computer Science
Shanghai Jiaotong University
Shanghai 200030, China
yijia.chen@cs.sjtu.edu.cn

Jörg Flum
Abteilung für Mathematische Logik
Universität Freiburg
Eckerstr. 1, 79104 Freiburg, Germany
joerg.flum@math.uni-freiburg.de

Abstract

We study the parameterized complexity of various path (and cycle) problems, the parameter being the length of the path. For example, we show that the problem of the existence of a maximal path of length k in a graph $\mathcal{G}$ is fixed-parameter tractable, while its counting version is #W[1]-complete. The corresponding problems for chordless (or induced) paths are W[2]-complete and #W[2]-complete respectively. With the tools developed in this paper we derive the NP-completeness of a related classical problem, thereby solving a problem due to Hedetniemi [21].

1. Introduction

The problem of deciding whether a given graph $\mathcal{G} = (V, E)$ contains a path (or a cycle) of a given length k is among the most natural and easily stated algorithmic graph problems. It is therefore not surprising that its classical and its parameterized complexity have been studied extensively (e.g. in [2, 3, 25, 28, 29]). For example, for every fixed k, the problem can be solved in time $O(|V| \cdot |E|)$ (cf. [25]). However if the path length k is part of the input, then the problem is clearly NP-complete as it includes the Hamiltonian path problem. Nevertheless it is fixed-parameter tractable, even it is solvable in time $2^{O(k)} \cdot \|\mathcal{G}\|$ (cf. [29, 2]). In particular, the problem whether a graph has a path of length $\log |V|$ is solvable in polynomial time. Furthermore, in [28] it is implicitly shown that the problem whether a chordless (or, induced) path of length k exists is W[1]-complete. This result has been rediscovered in [20]. In [32] the computational complexity of determining the chromatic number of graphs without long chordless paths is discussed. Recently, the search for algorithms detecting chordless cycles (of odd length ≥ 5) has received much attention due to its relationship to Berge graphs and to the Strong Perfect Graph Theorem (cf. [6, 8, 10, 11, 26]).

Counting paths (or cycles) of length k is #W[1]-complete [16], that is, most likely there is no $f(k) \cdot n^c$-algorithm for counting paths of length k in a graph of size n for any computable function $f : \mathbb{N} \to \mathbb{N}$ and constant c (even though there is a $2^{O(k)} \cdot n^{2.376}$ algorithm for finding a cycle or a path of length k [3]). This was the first natural parameterized counting problem proven to be harder than its decision version (assuming FPT $\neq$ W[1]).

Recently, we studied the parameterized complexity of so-called maximality problems [5]. For example, given a graph $\mathcal{G}$ and a natural number k, the maximal independent set problem asks, if there is an independent set maximal with respect to set inclusion and of size k. It turned out that one can obtain quite general results linking the parameterized complexity of a problem and the parameterized complexity of the corresponding maximality problem. These results were obtained for the class of *Fagin-definable* problems. For example, the independent set problem is Fagin-definable, as there is a formula $\varphi(X)$ of first-order logic with the set variable X expressing that "X is an independent set," e.g.

$$\varphi(X) := \forall y \forall z \big((Xy \wedge Xz) \to \neg Eyz\big).$$

We address the problem of the Fagin-definability of the path problem and of the chordless path problem in Section 7. However we already remark here that the general theory developed in [5] does not yield a relevant upper bound for the parameterized complexity of the maximal path problem and the maximal chordless path problem, because the logical complexity of the corresponding formulas is too high.

In our study of maximality problems we realized that even the classical complexity of the problem (stated in [21]) of deciding whether a graph has a maximal chordless path of length $\leq k$ was unknown.

In this paper we systematically analyze the parameterized complexity of the decision and counting versions of chordless paths problems and of maximal (chordless) path problems. As a by-product we solve the open question of [21] just mentioned by showing the NP-completeness of the corresponding problem.

First we prove that the problem of the existence of a maximal path of length k in a given graph is fixed-parameter tractable, while its counting version is $\#$W$[1]$-complete. Thus we add a further example–admittedly not very different from the path problem–to the (short) list of nontrivial parameterized problems where the counting version is probably harder than the decision version.

In Section 4 we present a construction on the class of graphs introduced by Papadimitriou and Yannakakis [28] in order to reduce the independent set problem to the chordless path problem; they even show that the (corresponding "log-") problems are polynomially equivalent. In the terminology of parameterized complexity this means that the chordless path problem is W$[1]$-complete, a result reproven in [20]. We refine the construction of Papadimitriou and Yannakakis in various ways thereby getting among others:

– the problem of counting chordless paths of length k is #W$[1]$-complete under parsimonious reductions (in [31] the #W$[1]$-completeness under Turing reductions was shown);

– the maximal chordless path problem (Is there a maximal chordless path of length k?) is W$[2]$-complete.

We do not mention the construction and the listing version of the problems considered so far, as their complexity is related to the complexity of the decision version in the usual way (cf. [19, 5]). We summarize the results (the already known and the new ones) in a table.

	paths	cycles	chordless paths	chordless cycles
plain	FPT	FPT	W$[1]$-complete	W$[1]$-complete
counting plain	#W$[1]$-complete	#W$[1]$-complete	#W$[1]$-complete	#W$[1]$-complete
maximal	FPT	$\times\times\times$	W$[2]$-complete	$\times\times\times$
counting maximal	#W$[1]$-complete	$\times\times\times$	#W$[2]$-complete	$\times\times\times$

If in this table the decision version of a problem is not fixed-parameter tractable, then #W$[\ldots]$-completeness for the counting version means #W$[\ldots]$-completeness under parsimonious reductions. Clearly, for fixed-parameter tractable problems we only get #W$[\ldots]$-completeness under Turing reductions (otherwise FPT $=$ W$[1]$). Furthermore, concerning the $\times\times\times$ note that maximality problems make no sense for cycle problems.

In Section 6 we deal with the problem stated in [21] of the existence of a maximal chordless path of length $\leq k$. Besides its NP-completeness we can show, by a further refinement of the construction of Papadimitriou and Yannakakis, that the parameterized version is W$[2]$-complete.

Finally, in Section 7 we mention some open problems, particularly some concerning *holes*, that is, chordless cycles of length at least 4. Although we are not able to determine the exact complexity of detecting a hole of odd length $\leq k$, we derive some results for related problems. We thereby show that there is a polynomial time algorithm that, given a graph $\mathcal{G}$ outputs a hole in $\mathcal{G}$ of minimum length (and rejects if there is no hole).

2. Preliminaries

The set of natural numbers (that is, nonnegative integers) is denoted by $\mathbb{N}$. For a natural number n let $[n] := \{1, \ldots, n\}$.

2.1. Parameterized complexity. We assume that the reader is familiar with the basic notions of parameterized complexity theory (cf. [13, 17]). In particular, a parameterized problem is *fixed-parameter tractable* if it is solvable in time $f(k) \cdot p(n)$ for some computable function f and some polynomial p; here n is the length of the instance and k denotes its parameter. We denote by FPT the class of all fixed-parameter tractable problems. Recently (see [18]), special attention has received the subclass EPT of FPT consisting of the problems where the function f can be chosen in $2^{O(k)}$.

For parameterized (decision) problems $\mathcal{Q}$ and $\mathcal{Q}'$ we write $\mathcal{Q} \le^{\text{fpt}} \mathcal{Q}'$ if there is a (many-one) fpt reduction from $\mathcal{Q}$ to $\mathcal{Q}'$. We write $\mathcal{Q} \equiv^{\text{fpt}} \mathcal{Q}'$ if $\mathcal{Q} \le^{\text{fpt}} \mathcal{Q}'$ and $\mathcal{Q}' \le^{\text{fpt}} \mathcal{Q}$.

We also consider parameterized counting problems (cf. [16, 17] for detailed definitions). For such problems $\mathcal{F}$ and $\mathcal{F}'$ we write $\mathcal{F} \le^{\text{fpt}} \mathcal{F}'$ if there is an fpt parsimonious reduction from $\mathcal{F}$ to $\mathcal{F}'$ (that is, an fpt reduction preserving the values) and $\mathcal{F} \le^{\text{fpt-T}} \mathcal{F}'$ if there is an fpt Turing reduction from $\mathcal{F}$ to $\mathcal{F}'$. We write $\mathcal{F} \equiv^{\text{fpt-T}} \mathcal{F}'$ if $\mathcal{F} \le^{\text{fpt-T}} \mathcal{F}'$ and $\mathcal{F}' \le^{\text{fpt-T}} \mathcal{F}$.

2.2. First-order logic. A *vocabulary* τ is a finite set of relation symbols. Each relation symbol has an *arity*. A τ-*structure* $\mathcal{A}$ consists of a set A called the *universe*, which we assume to be finite, and an interpretation $R^{\mathcal{A}} \subseteq A^r$ of each r-ary relation symbol $R \in \tau$. For example, we view a *graph* as a structure $\mathcal{G} = (G, E^{\mathcal{G}})$, where E is a binary relation symbol and $E^{\mathcal{G}}$ is an irreflexive and symmetric binary relation on the set of vertices G. Nevertheless, often we denote the vertex set of a graph $\mathcal{G}$ by V and the edge set by E (instead of G and $E^{\mathcal{G}}$) and use the set notation $\{v, w\}$ for edges.

For a τ-structure $\mathcal{A}$ we denote by $\|\mathcal{A}\|$ its size, that is, the length of a string encoding $\mathcal{A}$ in a natural way. The number $\|\mathcal{A}\|$ will be within a polynomial factor of the term

$$|\tau| + |A| + \sum_{R \in \tau} |R^{\mathcal{A}}| \cdot \text{arity}(R).$$

Let $\mathcal{A}$ and $\mathcal{B}$ be structures of the same vocabulary τ. An *embedding* from $\mathcal{A}$ to $\mathcal{B}$ is a one-to-one mapping $h : A \to B$ such that for all $R \in \tau$, say, of arity r, and for all $(a_1, \ldots, a_r) \in A^r$,

$$(a_1, \ldots, a_r) \in R^{\mathcal{A}} \implies (h(a_1), \ldots, h(a_r)) \in R^{\mathcal{B}}.$$

Formulas of first-order logic of vocabulary τ are built up from atomic formulas $x = y$ and $Rx_1 \ldots x_r$ where $x, y, x_1, \ldots, x_r$ are variables and $R \in \tau$ is of arity r, using the boolean connectives and existential and universal quantification. For $t \ge 0$ let Π_t denote the class of all first-order formulas of the form

$$\forall x_{11} \ldots \forall x_{1k_1} \exists x_{21} \ldots \exists x_{2k_2} \ \ldots \ Qx_{t1} \ldots Qx_{tk_t} \ \psi,$$

where $Q = \exists$ if t is even and $Q = \forall$ otherwise, and where ψ is quantifier-free. In particular, Π_0 is the set QF of quantifier-free formulas. If $u \ge 1$ we denote by $\Pi^0_{t,u}$ the subclass of Π_t consisting of those formulas where $k_1, \ldots, k_t \le u$ (that is, *all* quantifier blocks have length $\le u$).

If φ is a first-order formula, we write $\varphi(x_1, \ldots, x_m)$ to indicate that the free variables in φ are $x_1, \ldots, x_m$. If $\mathcal{A}$ is a τ-structure, and $\varphi(x_1, \ldots, x_m)$ is a formula of vocabulary τ, then we let

$$\varphi(\mathcal{A}) := \big\{(a_1, \ldots, a_m) \in A^m \mid \mathcal{A} \models \varphi(a_1, \ldots, a_m)\big\}.$$

Thus $|\varphi(\mathcal{A})|$ is the number of tuples satisfying φ.

For a class Φ of first-order formulas we consider the decision version p-MC(Φ) and the counting version p-#MC(Φ) of the *parameterized model-checking problem* (by $|\varphi|$ we denote the length of the formula φ):

p-MC(Φ)	
Instance:	A structure $\mathcal{A}$ and a formula $\varphi(x_1, \dots, x_m) \in \Phi$.
Parameter:	$\lvert\varphi\rvert$.
Problem:	Is $\varphi(\mathcal{A}) \neq \emptyset$?

p-#MC(Φ)	
Instance:	A structure $\mathcal{A}$ and a formula $\varphi(x_1, \dots, x_m) \in \Phi$.
Parameter:	$\lvert\varphi\rvert$.
Problem:	Compute $\lvert\varphi(\mathcal{A})\rvert$.

We introduce the classes of the W-hierarchy (in a way suitable for our purposes):

Definition 1. Let $t \geq 1$.

- W$[t]$ is the class of parameterized (decision) problems fpt reducible to p-MC$(\Pi^0_{t-1,1})$.
- #W$[t]$ is the class of parameterized counting problems fpt parsimoniously reducible to p-#MC$(\Pi^0_{t-1,1})$.

We shall use the following well-known results (cf. [12] for (a), and [16] and [5] for (b)):

Theorem 2. *(a)* p-INDEPENDENT-SET *is* W[1]*-complete and* p-MAXIMAL-INDEPENDENT-SET *is* W[2]*-complete, both under fpt reductions.*

(b) p-#INDEPENDENT-SET *is* #W[1]*-complete and* p-#MAXIMAL-INDEPENDENT-SET *is* #W[2]*-complete, both under fpt parsimonious reductions.*

Here, given a graph $\mathcal{G}$ and $k \in \mathbb{N}$ (as parameter), the problem p(-MAXIMAL)-INDEPENDENT-SET asks whether there exists a (maximal with respect to set inclusion) independent set of size k in $\mathcal{G}$.

2.3. Path and Cycles. Let $k \in \mathbb{N}$. The *generic path* of length k is the graph

$$\mathcal{P}_k := \Big([k+1],\ \big\{\{i,j\} \mid i,j \in [k+1],\, j-i=1\big\}\Big).$$

A *path of length* k in a graph $\mathcal{G} = (V,E)$ is a subgraph of $\mathcal{G}$ that is isomorphic to $\mathcal{P}_k$. Thus, if $v_1, \dots, v_{k+1} \in V$ with $e_i := \{v_i, v_{i+1}\} \in E$ for all $i \in [k]$, then $\big(\{v_1, \dots, v_{k+1}\}, \{e_1, \dots, e_k\}\big)$ is counted as one path (and not as two, as the notations $v_1, v_2, \dots, v_{k+1}$ and $v_{k+1}, v_k, \dots, v_1$ might suggest). Nevertheless, in many contexts we denote this path by $P = v_1, \dots, v_{k+1}$ and say that the path *starts* in v_1 and *ends* in v_{k+1}. The vertices v_1, v_{k+1} are the *endvertices* of this path. A path $v_1, \dots, v_{k+1}$ is *maximal* if there is no $v \in V$ such that $v, v_1, \dots, v_{k+1}$ or $v_1, \dots, v_{k+1}, v$ is a path. A *chordless path* (or, *induced path*) of length k in $\mathcal{G}$ is an induced subgraph of $\mathcal{G}$ that is isomorphic to $\mathcal{P}_k$. It should be clear how the notion of a *maximal* chordless path is defined.

Sometimes we use formulations like "P is a maximal (chordless) path in $\mathcal{G} = (V,E)$ with endvertices in $F \subseteq V$." Here "maximal" ("maximal chordless") refers to all (chordless) paths in $\mathcal{G}$ and not to all (chordless) paths with endvertices in $F \subseteq V$.

Let $k \geq 3$. A *(chordless) cycle of length* k in a graph $\mathcal{G} = (V,E)$ is an (induced) subgraph of $\mathcal{G}$ that is isomorphic to the *generic cycle* of length k

$$\mathcal{C}_k := \Big([k],\ \big\{\{i,j\} \mid i,j \in [k],\, j-i \equiv 1 \bmod k\big\}\Big).$$

Part (a) of the following theorem has been shown in [29] and part (b) in [16]:

Theorem 3. *(a)* p-PATH *and* p-CYCLE *are fixed-parameter tractable, they are even in* EPT.

(b) p-#PATH *and* p-#CYCLE *are* #W$[1]$*-complete under fpt Turing reductions.*

Here, for example,

p-PATH	
Instance:	A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Is there a path in $\mathcal{G}$ of length k?

and

p-#PATH	
Instance:	A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Count the number of paths in $\mathcal{G}$ of length k.

3. Maximal Paths

This section is devoted to a proof of the following two results.

Theorem 4. *The problem*

p-MAXIMAL-PATH	
Instance:	A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Is there a maximal path in $\mathcal{G}$ of length k?

is fixed-parameter tractable, it is even in EPT.

Theorem 5. *The problem*

p-#MAXIMAL-PATH	
Instance:	A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Count the number of maximal paths in $\mathcal{G}$ of length k.

is #W$[1]$*-complete under fpt Turing reductions.*

Let $\mathcal{G} = (V, E)$ be a graph and $u, v \in V$. Consider a path P in $\mathcal{G}$ from the vertex u to the vertex v. Let $N^{\mathcal{G}}(u, v)$ be the set of neighbors of u or v, more precisely:

$$N^{\mathcal{G}}(u, v) := \big\{ w \in V \mid \{u, w\} \in E \text{ or } \{v, w\} \in E \big\}.$$

The following simple observation will be crucial.

Lemma 6. *Let P be a path from u to v in $\mathcal{G}$. Then*

$$P \textit{ is a maximal path} \iff P \textit{ contains all vertices in } N^{\mathcal{G}}(u, v).$$

The fixed-parameter tractability of p-MAXIMAL-PATH is obtained by a reduction to the embedding problem for a class of structures of bounded treewidth. The latter problem has been shown to be fixed-parameter tractable in [29] (cf. [2], too):

Theorem 7. *Let* C *be a decidable class of structures of bounded tree-width. Then*

p-EMB(C)	
Instance:	A structure $\mathcal{A} \in$ C and a structure $\mathcal{B}$.
Parameter:	$\|\mathcal{A}\|$.
Problem:	Does there exist an embedding from $\mathcal{A}$ to $\mathcal{B}$?

is in EPT.

Proof of Theorem 4: It suffices to show the following problem is in EPT:

p-POINT-MAXIMAL-PATH	
Instance:	A graph $\mathcal{G} = (V, E)$, $u, v \in V$, and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Is there a maximal path in $\mathcal{G}$ from u to v of length k?

Let C be the class of all generic paths (see Preliminaries) with three additional unary relations. Then all structures in C have treewidth ≤ 1 (recall that unary relations do not change the treewidth). So, by Theorem 7, it suffices to give an fpt algorithm with an oracle to p-EMB(C) solving p-POINT-MAXIMAL-PATH.

We denote the three unary relation symbols by R_f ("first element relation"), R_ℓ ("last element relation"), and R ("neighbor relation"). Consider an instance of p-POINT-MAXIMAL-PATH consisting of a graph $\mathcal{G} = (V, E)$, $u, v \in V$, and $k \in \mathbb{N}$. By Lemma 6, if $|N^{\mathcal{G}}(u, v)| > k + 1$, then $(\mathcal{G}, u, v, k)$ is a no instance. So we may assume that $|N^{\mathcal{G}}(u, v)| = m$ for some $m \leq k + 1$. We set $\mathcal{G}_{u,v} := (\mathcal{G}, R_f^{\mathcal{G}}, R_\ell^{\mathcal{G}}, R^{\mathcal{G}})$, where

$$R_f^{\mathcal{G}} := \{u\}, \;\; R_\ell^{\mathcal{G}} := \{v\}, \;\; R^{\mathcal{G}} := N^{\mathcal{G}}(u, v).$$

Recall that the generic path $\mathcal{P}_k$ of length k has vertex set $[k + 1]$. Again by Lemma 6, we see that there is a maximal path in $\mathcal{G}$ from u to v of length k if and only if for some unary relation $R^{\mathcal{P}_k}$ of cardinality $m \in [k + 1]$ there exists an embedding from $(\mathcal{P}_k, \{1\}, \{k + 1\}, R^{\mathcal{P}_k})$ to $\mathcal{G}_{u,v}$. Since there are at most 2^{k+1} many such subsets, we get our claim. □

A direct consequence of Theorem 4 is:

Corollary 8. log n-MAXIMAL-PATH *is in* PTIME, *where*

log n-MAXIMAL-PATH	
Instance:	A graph $\mathcal{G} = (V, E)$.
Problem:	Is there a maximal path in $\mathcal{G}$ of length log $\|V\|$?

However it is impossible to improve the log $|V|$ bound in Corollary 8 as exemplified by the following result.

Theorem 9. *The problem*

$\log^2 n$-MAXIMAL-PATH	
Instance:	A graph $\mathcal{G} = (V, E)$.
Problem:	Is there a maximal path in $\mathcal{G}$ of length $\log^2 \|V\|$?

is not in PTIME *unless the exponential time hypothesis* (ETH) *fails.*

In the proof we will use the following path extension construction: Let $\mathcal{G} = (V, E)$ be a graph and $d, \ell \in \mathbb{N}$. We obtain the graph $\mathcal{G}_{d,\ell} = (V_{d,\ell}, E_{d,\ell})$ from $\mathcal{G}$ by adding d many isolated vertices $a_1, \ldots, a_d$ and furthermore vertices $b_1, b_2, \ldots, b_\ell$ constituting a path and where b_1 is adjacent to all vertices in $\mathcal{G}$. More precisely:

$$\begin{aligned}
V_{d,\ell} &:= V \,\dot\cup\, \big\{a_i \mid i \in [d]\big\} \,\dot\cup\, \big\{b_i \mid i \in [\ell]\big\};\\
E_{d,\ell} &:= E \cup \big\{\{v, b_1\} \mid v \in V\big\}\\
&\qquad\quad \cup \big\{\{b_i, b_{i+1}\} \mid i \in [\ell - 1]\big\}.
\end{aligned}$$

One easily verifies for $k \geq 1$ and $\ell \geq k + 1$,

$$\mathcal{G} \text{ has a path of length } k \iff \mathcal{G}_{d,\ell} \text{ has a path of length } k + \ell.$$

Moreover we need the following fact (by $\log^2 m$ we mean $\lfloor \log^2 m \rfloor$):

Lemma 10. *Given $N \geq 64$ and $k \in \mathbb{N}$ with $k \leq \log^2 N$ we can compute in time polynomial in N a number $\ell \in \mathbb{N}$ with $\ell \leq N$ such that*

$$k + \ell = \log^2 (N + \ell).$$

Proof: The claim easily follows from the following two facts holding for $N \geq 64$:

- $\log^2 (N + m + 1) - \log^2 (N + m) \leq 1$ for all $m \geq 0$;
- $N > \log^2 (N + N)$. □

First we show:

Theorem 11. *If the problem $\log^2 n$-*MAXIMAL-PATH *is in* PTIME, *then* HAMILTONIAN-PATH *is solvable in time $2^{O(\sqrt{n})}$, where n is the number of vertices of the given graph.*

Proof: We assume that there is a polynomial time algorithm $\mathbb{A}$ that decides $\log^2 n$-MAXIMAL-PATH in polynomial time. Let $(\mathcal{G}, k)$ be an instance of HAMILTONIAN-PATH with $\mathcal{G} = (V, E)$. Without loss of generality we assume $n := |V| \geq 64$. We show how to use $\mathbb{A}$ to solve HAMILTONIAN-PATH in time $2^{O(\sqrt{n})}$.

We let $N > n$ be the smallest natural number satisfying $\log^2 N \geq 2n + 1 > n$. Clearly $N = 2^{O(\sqrt{n})}$. With the algorithm of Lemma 10 we compute an $\ell \leq N$ such that $n + \ell = \log^2 (N + \ell)$. Note that $n + 1 \leq \ell \leq N = 2^{O(\sqrt{n})}$. For the graph $\mathcal{G}_{N-n,\ell}$ we have: $\mathcal{G}$ has a Hamiltonian path, i.e., a path of length n if and only if $\mathcal{G}_{N-n,\ell}$ has a path P of length $n + \ell$ $(= \log^2 (N + \ell)) = \log^2 |V_{N-n,\ell}|$. Clearly such a path P must be a maximal path. So we simulate $\mathbb{A}$ on $\mathcal{G}_{N-n,\ell}$, which takes time

$$\|\mathcal{G}_{N-n,\ell}\|^{O(1)} = \big(N + \ell\big)^{O(1)} = 2^{O(\sqrt{n})}.$$ □

Proof of Theorem 9: Assume that $\log^2 n$-PATH $\in$ PTIME. Then by Theorem 11 HAMILTONIAN-PATH is solvable in time $2^{o(n)}$, where n is the number of vertices of the given graph. Then 3-SAT is solvable in time $2^{o(m)}$, where m is the number of clauses of the given propositional formula (this is seen, say, by the reduction of [27] from 3-SAT to HAMILTONIAN-PATH). Therefore, by the Sparsification Lemma [22], the problem 3-SAT is solvable in time $2^{o(n)}$, where n is the number of variables of the given propositional formula. But this just means that ETH fails. □

A similar proof yields:

Theorem 12. *The problem*

$\log^2 n$-PATH		
	Instance:	A graph $\mathcal{G} = (V, E)$.
	Problem:	Does $\mathcal{G}$ contain a path of length $\log^2 |V|$?

is not in PTIME *unless* ETH *fails.* [1]

The next two propositions will give a proof of Theorem 5. The following result also relies on Lemma 6.

Proposition 13. p-#MAXIMAL-PATH $\leq^{\text{fpt}}$ p-#MC(QF).

Proof: Let $(\mathcal{G}, k)$ with $\mathcal{G} = (V, E)$ be an instance of p-#MAXIMAL-PATH. For paths of length k we express the right hand side of the equivalence in Lemma 6 by the formula φ_k below thus getting: $u_1, \ldots, u_{k+1}$ is a maximal path in $\mathcal{G}$ if and only if $\mathcal{G} \models \varphi_k(u_1, \ldots, u_{k+1})$, where

$$\varphi_k(x_1, \ldots, x_{k+1}) := \bigwedge_{i,j \in [k+1], i<j} (\neg x_i = x_j \wedge E x_i x_{i+1}) \wedge \forall y \Big((E x_1 y \vee E x_{k+1} y) \to \bigvee_{i \in [k+1]} x_i = y \Big).$$

To get the desired reduction to p-#MC(QF), we have, among others, to get rid of the universal quantifier. For fixed x_1, x_{k+1} the universal quantifier essentially ranges over the neighbors of these two elements; since in the relevant cases there will be at most $k+1$ many we can replace the quantification by an iterated conjunction, more precisely:

We let $<$ be a binary and $T_1, \ldots, T_{k+1}$ ternary relation symbols, where, without loss of generality, we assume $k \geq 1$. We expand $\mathcal{G}$ to a structure

$$\mathcal{A} := \big(V, E, <^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_{k+1}^{\mathcal{A}}\big)$$

with the properties:

- $<^{\mathcal{A}}$ is an ordering of V;
- $T_1^{\mathcal{A}}, \ldots, T_{k+1}^{\mathcal{A}}$ are ternary relations on V such that for $u, v \in V$:
  - if $|N^{\mathcal{G}}(u, v)| > k+1$ or $|N^{\mathcal{G}}(u, v)| = 0$, then for *all* $i \in [k+1]$ there is *no* $w \in V$ with $(u, v, w) \in T_i^{\mathcal{A}}$;
  - if $1 \leq |N^{\mathcal{G}}(u, v)| = m \leq k+1$, then there are $w_1, \ldots, w_m$ such that $N^{\mathcal{G}}(u, v) = \{w_1, \ldots, w_m\}$, for all $i \in [m]$
  $$\big\{w \mid w \in V \text{ and } (u, v, w) \in T_i^{\mathcal{A}}\big\} = \{w_i\}$$
  and for i with $m < i \leq k+1$
  $$\big\{w \mid w \in V \text{ and } (u, v, w) \in T_i^{\mathcal{A}}\big\} = \{w_m\}.$$

Then $u_1, \ldots, u_{k+1}$ is a maximal path in $\mathcal{G}$ if and only if there are $v_1, \ldots, v_{k+1}$ with $\mathcal{A} \models \psi_k(u_1, \ldots, u_{k+1}, v_1, \ldots, v_{k+1})$, where $\psi_k(x_1, \ldots, x_{k+1}, y_1, \ldots, y_{k+1})$ is the formula

$$\bigwedge_{i,j \in [k+1],\, i<j} (\neg x_i = x_j \wedge E x_i x_{i+1}) \wedge \bigwedge_{i \in [k+1]} T_i x_1 x_{k+1} y_i \wedge \bigwedge_{i \in [k+1]} \bigvee_{j \in [k+1]} y_i = x_j.$$

[1] This problem was raised in [2], and Alon informed us that he was already aware of Theorem 12 before we got the result.

Moreover, in the positive case, the $v_1, \ldots, v_{k+1}$ are uniquely determined by $u_1, \ldots, u_{k+1}$. In order to count a path $u_1, \ldots, u_{k+1}$ only once (and not twice as the notations $u_1, \ldots, u_{k+1}$ and $u_{k+1}, \ldots, u_1$ might suggest) we consider the formula $\chi_k(x_1, \ldots, x_{k+1}, y_1, \ldots, y_{k+1}) := \psi_k \wedge x_1 < x_{k+1}$. Then

$$\Big|\big\{P \mid P \text{ is a maximal path in } \mathcal{G} \text{ of length } k\big\}\Big| = \Big|\big\{(u_1, \ldots, u_{k+1}, v_1, \ldots, v_{k+1}) \mid \mathcal{A} \models \chi_k(\bar{u}, \bar{v})\big\}\Big|,$$

that is, $|\chi_k(\mathcal{A})|$ is the number of maximal paths in $\mathcal{G}$ of length k. This gives the desired reduction from p-#MAXIMAL-PATH to p-#MC(QF). □

Proposition 14. p-#PATH $\leq^{\text{fpt}}$ p-#MAXIMAL-PATH.

Proof: This is quite straightforward. Let $(\mathcal{G}, k)$ with $\mathcal{G} = (V, E)$ and $k \geq 1$ be an instance of p-#PATH. Let $V^* := \{v^* \mid v \in V\}$ be a disjoint copy of V. We set $\mathcal{G}' = (V', E')$ with

$$V' := V \cup V^* \quad \text{and} \quad E' := E \cup \big\{\{v, v^*\} \mid v \in V\big\}.$$

Then:

(a) Every maximal path in $\mathcal{G}'$ of length ≥ 2 starts and ends in V^*.

(b) Let $P = u_1, \ldots, u_k$ with $u_1, \ldots, u_k \in V$. Then

$$P \text{ is a path in } \mathcal{G} \iff u_1^*, P, u_k^* \text{ is a maximal path in } \mathcal{G}'.$$

Hence, $(\mathcal{G}, k) \to (\mathcal{G}', k+2)$ is an fpt parsimonious reduction from p-#PATH to p-#MAXIMAL-PATH. □

Proof of Theorem 5: By Proposition 13 we have p-#MAXIMAL-PATH $\in$ #W[1] and by Proposition 14 and Theorem 3 (b) the problem p-#MAXIMAL-PATH is #W[1]-hard under fpt Turing reductions. □

4. Chordless Paths and Chordless Cycles

In this section we introduce an operation on graphs relating the independent sets of a graph $\mathcal{G}$ with the chordless paths of the image of $\mathcal{G}$ under the operation. This operation is due to Papadimitriou and Yannakakis [28]. We will use it and refinements of it to derive essentially all hardness results in Sections 4–6. In this section we apply it to get the W[1]-hardness of the chordless path problem and the chordless cycle problem (these results are already implicit in [28]); moreover, we use it to show that the corresponding counting problems are #W[1]-hard under fpt parsimonious reductions (this improves [31] where these problems are shown to be #W[1]-hard under fpt Turing reductions).

More precisely, the operation of [28] acts on graphs and positive natural numbers: Let $\mathcal{G} = (V, E)$ be a graph and $k \geq 1$. The vertex set of the graph $\mathcal{G}(k) = (V(k), E(k))$ essentially consists of k copies of V. Each copy will be a clique in $\mathcal{G}(k)$. Two distinct copies are linked according to $\mathcal{G}$. Finally there are three[2] additional vertices for

[2]For the purpose of the proofs in this section two additional elements would suffice but we want to use the same construction for the proof of Theorem 18.

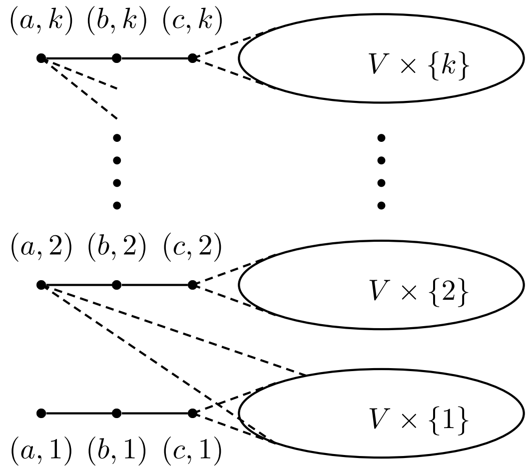


Figure 1.

each $i \in [k]$, which for $i \geq 2$ allow a further transition from the $(i-1)$th copy to the ith copy (cf. Figure 1). More precisely:

$$\begin{aligned}
V(k) &:= \big(V \,\dot{\cup}\, \{a,b,c\}\big) \times [k];\\
E(k) &:= \bigcup_{i\in[k]} \{\{(u,i),(v,i)\} \mid u,v \in V \text{ and } u \neq v\}\\
&\quad \cup \bigcup_{1\leq i<j\leq k} \big\{\{(u,i),(v,j)\} \mid u,v \in V \text{ and } (u = v \text{ or } \{u,v\} \in E)\big\}\\
&\quad \cup \bigcup_{i\in[k]} \Big(\big\{\{(a,i),(b,i)\},\{(b,i),(c,i)\}\big\} \cup \big\{\{(c,i),(u,i)\} \mid u \in V\big\}\Big)\\
&\quad \cup \bigcup_{1\leq i\leq k-1} \big\{\{(u,i),(a,i+1)\} \mid u \in V\big\}.
\end{aligned}$$

For each $i \in [k]$, we call

$$S_i := \big(V \cup \{a,b,c\}\big) \times \{i\}$$

the *ith slice of* $\mathcal{G}(k)$. Clearly $S_i \setminus \{(a,i),(b,i)\}$ is a clique.

The crucial observation of [28] relating the independent sets of $\mathcal{G}$ with chordless paths in $\mathcal{G}(k)$ reads as follows:

Lemma 15. *Let $\mathcal{G} = (V,E)$ be a graph and $k \geq 1$. Furthermore let $\mathcal{G}(k)$ be the graph just constructed. For $u_1,\dots,u_k \in V$ we set*

$$\begin{aligned}
P(u_1,\dots,u_k) := (a,1),(b,1),(c,1),(u_1,1),&(a,2),(b,2),(c,2),(u_2,2),(a,3)\dots,\\
&\dots,(u_{k-1},k-1),(a,k),(b,k),(c,k),(u_k,k).
\end{aligned}$$

Then:

(a) $P(u_1,\dots,u_k)$ is a path of length $4k-1$ in $\mathcal{G}(k)$.

(b) $P(u_1,\dots,u_k)$ is a chordless path in $\mathcal{G}(k)$ if and only if $\{u_1,\dots,u_k\}$ is an independent set of $\mathcal{G}$ of size k.

(c) *Every chordless path of length* $4k-1$ *in* $\mathcal{G}(k)$ *has the form* $P(u_1,\dots,u_k)$ *for suitable* $u_1,\dots,u_k \in V$.[3]

Proof: Part (a) is clear. Part (b) is easy: By construction of $\mathcal{G}(k)$ the path $P(u_1,\dots,u_k)$ is not a chordless one if and if only if for some $i,j\in[k]$ with $i\neq j$ the vertices (u_i,i) and (u_j,j) are adjacent in $\mathcal{G}(k)$. But this means that $u_i=u_j$ or $\{u_i,u_j\}\in E$. In the first case the set $\{u_1,\dots,u_k\}$ has size less than k and in second one it is not an independent set.

We turn to part (c) and assume that P is an arbitrary chordless path of length $4k-1$ in $\mathcal{G}(k)$. We show that it must have the form $P(u_1,\dots,u_k)$ for some $u_1,\dots,u_k\in V$.

Claim 1. For $i\in[k]$

$$|P\cap S_i|=4,\quad (a,i),(b,i)\in P\quad\text{and}\quad |P\cap(V\times\{i\})|\geq 1.$$

In fact, since $S_i\setminus\{(a,i),(b,i)\}$ is a clique, it contains at most 2 vertices from P. Therefore S_i contains *at most* 4 vertices from P. As P contain $4k$ vertices, it must contain exactly four vertices from each slice S_i and hence at least one from $V\times\{i\}$. $\dashv$

Claim 2. $(c,1)\in P$.

By Claim 1, we have $(a,1),(b,1)\in P$. But $(a,1)$ is only adjacent to $(b,1)$ and hence must be an endvertex of P. The point $(c,1)$ is the only further neighbor of $(b,1)$, hence $(c,1)\in P$. $\dashv$

As just remarked $(a,1)$ is an endvertex of P, so we may assume that P starts in $(a,1)$.

Claim 3. There are $u_1,\dots,u_k\in V$ such that $P=P(u_1,\dots,u_k)$.

Let $P=v_1,\dots,v_{4k}$. By Claim 1 and Claim 2 we already know that $v_1=(a,1),\ v_2=(b,1),\ v_3=(c,1)$, and $v_4=(u_1,1)$ for some $u_1\in V$.

Now we show for $i\in[k-1]$, if

$$v_{4(i-1)+1},v_{4(i-1)+2},v_{4(i-1)+3},v_{4i}=(a,i),(b,i),(c,i),(u_i,i)$$

for some $u_i\in V$, then

$$v_{4i+1},v_{4i+2},v_{4i+3},v_{4(i+1)}=(a,i+1),(b,i+1),(c,i+1),(u_{i+1},i+1)$$

for some $u_{i+1}\in V$. Clearly, this yields the claim. $\dashv$

By Claim 1, $(a,i+1),(b,i+1)\in P$, and since $(a,i+1)$ is adjacent to $(u_i,i)=v_{4i}$, we have

$$v_{4i+1}=(a,i+1).$$

Similarly, as $(a,i+1)$ and $(b,i+1)$ are adjacent, we get $v_{4i+2}=(b,i+1)$. Hence, $v_{4i+3}=(c,i+1)$. Finally, by Claim 1, there exits some $u_{i+1}\in V$ with $(u_{i+1},i+1)\in P$. As $(u_{i+1},i+1)$ is adjacent to $(c,i+1)$ $(=v_{4i+3})$ in $\mathcal{G}(k)$, we conclude that $v_{4i+4}=(u_{i+1},i+1)$. $\square$

As already mentioned the following result is implicit in [28] and explicit in [20].

Theorem 16. *The problems*

[3] Recall that the path $P(u_1,\dots,u_k)$ is the same as the corresponding path starting in (v_k,k) and ending in $(a,1)$.

p-CHORDLESS-PATH	
Instance:	A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Does $\mathcal{G}$ have a chordless path of length k?

and

p-CHORDLESS-CYCLE	
Instance:	A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Does $\mathcal{G}$ have a chordless cycle of length k?

are W[1]*-complete under fpt reductions.*

Proof: We show the W[1]-hardness of p-CHORDLESS-PATH by an fpt reduction from the W[1]-hard problem p-INDEPENDENT-SET (cf. Theorem 2 (a)). Let $\mathcal{G} = (V, E)$ be a graph and $k \in \mathbb{N}$. We assume $k \geq 1$. By the previous lemma we see that

$$\mathcal{G} \text{ has an independent set of size } k \iff \mathcal{G}(k) \text{ has a chordless path of length } 4k - 1.$$

This gives the desired reduction.

The W[1]-hardness of p-CHORDLESS-CYCLE is again shown by an fpt reduction from p-INDEPENDENT-SET. Let $\mathcal{G} = (V, E)$ be a graph and $k \in \mathbb{N}$. We assume $k \geq 2$. The graph $\mathcal{G}(k)'$ is obtained from the graph $\mathcal{G}(k)$ by adding an edge from $(a, 1)$ to every vertex of $V \times \{k\}$. Then $P(u_1, \ldots, u_k)$ is a cycle in $\mathcal{G}(k)'$ for $u_1, \ldots, u_k \in V$. As in the previous proof one shows:

$$\mathcal{G} \text{ has an independent set of size } k \iff \mathcal{G}(k)' \text{ has a chordless cycle of length } 4k.$$

Membership of p-CHORDLESS-PATH in W[1] is witnessed by the fpt reduction to p-MC(QF) given by $(\mathcal{G}, k) \mapsto (\mathcal{G}, \varphi_k)$ with

$$\varphi_k(x_1, \ldots, x_{k+1}) := \Big(\bigwedge_{i \in [k]} Ex_i x_{i+1} \wedge \bigwedge_{i,j \in [k+1], i \neq j} \neg x_i = x_j \wedge \bigwedge_{1 \leq i < j \leq k} \neg Ex_i x_{j+1} \Big).$$

It should be obvious how the formula φ_k has to be modified for p-CHORDLESS-CYCLE. □

In addition, we show for the counting versions.

Theorem 17. *p-#CHORDLESS-PATH and p-#CHORDLESS-CYCLE are $\#$W[1]-complete under fpt parsimonious reductions.*

Proof: Membership in $\#$W[1] is shown by straightforward fpt parsimonious reductions to p-#MC(QF) (as it is done for p-#PATH and p-#CYCLE in Lemma 14.31 of [17]).

For the #W[1]-hardness we reduce the #W[1]-hard problem p-#INDEPENDENT-SET (cf. Theorem 2 (b)) to our problems. Essentially we use the reductions for the decision problems presented in the previous proofs, but now we have to make sure that the reductions preserve the number of solutions, that is, are parsimonious. Note that, by Lemma 15, every independent set $I := \{u_1, \ldots, u_k\}$ of size k gives rise to $k!$ many chordless paths in $\mathcal{G}(k)$, namely for each ordering $u'_1, \ldots, u'_k$ of the elements of I to the path $P(u'_1, \ldots, u'_k)$. We illustrate the necessary changes for p-#CHORDLESS-PATH. The proof for p-#CHORDLESS-CYCLE is similar.

Let $\mathcal{G} = (V, E)$ be a graph and $k \geq 1$. We fix an ordering $<^V$ of V. We add to the graph $\mathcal{G}(k)$ for all $u, v \in V$ with $v <^V u$, which are not adjacent in $\mathcal{G}$, and all $i, j \in [k]$ with $i < j$ an edge between the vertex (u, i) (of the ith copy) and the vertex (v, j) (of the jth copy), thereby obtaining the graph $\mathcal{G}(<^V, k)$. Then it is easy to see that for $u_1, \ldots, u_k \in V$

$$P(u_1, \ldots, u_k) \text{ is a chordless path in } \mathcal{G}(<^V, k) \iff \begin{array}{l}\{u_1, \ldots, u_k\} \text{ is an independent set in } \mathcal{G} \text{ and} \\ u_1 <^V \ldots <^V u_k.\end{array}$$

From this equivalence one gets that the number of independent sets of $\mathcal{G}$ of size k and the number of chordless paths of $\mathcal{G}(<^V, k)$ of length $4k - 1$ coincide. $\Box$

5. Maximal Chordless Paths

While in Section 3 we have seen that p-MAXIMAL-PATH is fixed-parameter tractable, we now show:

Theorem 18. p-MAXIMAL-CHORDLESS-PATH *is* W[2]*-complete under fpt reductions.*

Before giving the proof, we introduce an auxiliary construction, which ensures that maximal (chordless) paths of length $\leq k$ have endvertices in a given set F: Let $\mathcal{G} = (V, E)$ be a graph, $F \subseteq V$, and $k \in \mathbb{N}$. We obtain the graph $\mathcal{G}_{F,k}$ from $\mathcal{G}$ by adding to every vertex not in F a path of length $k + 1$ of new vertices, e.g.: $\mathcal{G}_{F,k} = (V_{F,k}, E_{F,k})$ with

$$V_{F,k} := V \,\dot{\cup}\, \big((V \setminus F) \times [k + 1]\big);$$

$$E_{F,k} := E \cup \Big\{ \big\{v, (v, 1)\big\} \mid v \in V \setminus F \Big\} \cup \Big\{ \big\{(v, i), (v, i + 1)\big\} \mid i \in [k] \text{ and } v \in V \setminus F \Big\}.$$

Then one easily verifies:

Lemma 19. *(a) Let P be a maximal chordless path in $\mathcal{G}$ with endvertices in F. Then P is also a maximal chordless path in $\mathcal{G}_{F,k}$.*

(b) Let P be a maximal chordless path in $\mathcal{G}_{F,k}$ of length $\leq k$. Then P is a maximal chordless path in $\mathcal{G}$ with endvertices in F (in particular, all vertices of P are in V).

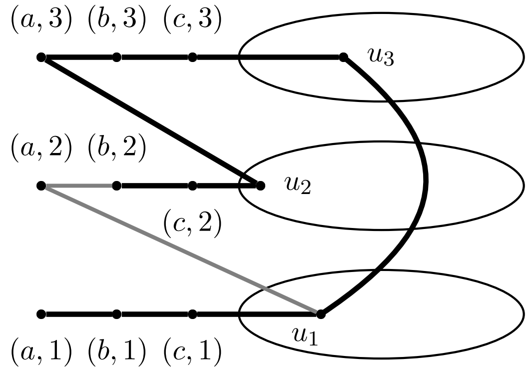
Proof of Theorem 18: The membership of p-MAXIMAL-CHORDLESS-PATH is shown by a straightforward reduction to p-MC($\Pi^0_{1,1}$): For each $k \in \mathbb{N}$ there is a $\Pi^0_{1,1}$-formula $\psi_k(x_1, \ldots, x_{k+1})$ expressing that $x_1, \ldots, x_{k+1}$ is a maximal chordless path of length k, for example:

$$\psi_k(x_1, \ldots, x_{k+1}) := \Big(\bigwedge_{i \in [k]} E x_i x_{i+1} \wedge \bigwedge_{i,j \in [k+1], i \neq j} \neg x_i = x_j \wedge \bigwedge_{1 \leq i < j \leq k} \neg E x_i x_{j+1} \Big) \wedge$$
$$\forall y \Big(\Big(E x_1 y \rightarrow \bigvee_{2 \leq i \leq k+1} (x_i = y \vee E x_i y) \Big) \wedge \Big(E x_{k+1} y \rightarrow \bigvee_{1 \leq i \leq k} (x_i = y \vee E x_i y) \Big) \Big).$$

To show the W[2]-hardness of p-MAXIMAL-CHORDLESS-PATH we present an fpt reduction from the W[2]-complete problem p-MAXIMAL-INDEPENDENT-SET (cf. Theorem 2 (a)).

So let $(\mathcal{G}, k)$ with $\mathcal{G} = (V, E)$ be an instance of p-MAXIMAL-INDEPENDENT-SET and assume $k \geq 1$. Recall the definition of $\mathcal{G}(k + 1)$. Let $I = \{u_1, \ldots, u_k\}$ be an independent set of size k in $\mathcal{G}$. Then:

$$P(u_1, \ldots, u_k), (a, k + 1), (b, k + 1), (c, k + 1) \text{ is a maximal chordless path in } \mathcal{G}(k + 1)$$
$$\iff I \text{ is a maximal independent set of } \mathcal{G}.$$



A path of length $4k+2$ (for $k=2$), the grey edges witness the maximality at the endvertex $(b,2)$.

Figure 2.

In fact, if $I \dot\cup \{u\}$ is an independent set in $\mathcal{G}$, then, by Lemma 15 (b), $P(u_1,\dots,u_k,u)$ is a chordless path in $\mathcal{G}(k{+}1)$ extending $P(u_1,\dots,u_k),(a,k{+}1),(b,k{+}1),(c,k{+}1)$. Conversely, if $P(u_1,\dots,u_k),(a,k{+}1),(b,k{+}1),(c,k+1),v$ for some vertex v in $\mathcal{G}(k+1)$ is a chordless path, then the vertex v is adjacent to $(c,k+1)$ and thus must have the form $v=(u,k+1)$ with $u \in V$. But then, again by Lemma 15 (b), $\{u_1,\dots,u_k,u\}$ is an independent set extending I.

Furthermore, one easily verifies:

– Every chordless path of length $4k+2$ starting in $(a,1)$ and ending in $(c,k+1)$ must have the form $P(u_1,\dots,u_k),(a,k+1),(b,k+1),(c,k+1)$, where $\{u_1,\dots,u_k\}$ is an independent set of $\mathcal{G}$.

In general, there might be (and in general will be) maximal chordless paths of length $4k+2$ of a different form in $\mathcal{G}(k+1)$ (cf. Figure 2). To get rid of them instead of $\mathcal{G}(k+1)$ we consider the graph $(\mathcal{G}(k+1))_{F,4k+2}$ for $F=\{(a,1),(c,k+1)\}$. Putting all together we see:

$$\mathcal{G} \text{ has a maximal independent set of size } k \iff (\mathcal{G}(k+1))_{F,4k+2} \text{ has a maximal chordless path of length } 4k+2. \qquad \Box$$

For the counting version we get:

Theorem 20. *p-#MAXIMAL-CHORDLESS-PATH is $\#$W[2]-complete under fpt parsimonious reductions.*

Proof: Membership in $\#$W[2] is shown by a parsimonious fpt reduction to p-#MC$(\Pi^0_{1,1})$: Given an instance $(\mathcal{G},k)$ of p-#MAXIMAL-CHORDLESS-PATH, we add an ordering $<^{\mathcal{G}}$ of the set of vertices of $\mathcal{G}$ and assign to $(\mathcal{G},k)$ the instance $((\mathcal{G},<^{\mathcal{G}}),\chi_k)$ of p-#MC$(\Pi^0_{1,1})$ with $\chi_k(x_1,\dots,x_{k+1}) := \psi_k(x_1,\dots,x_{k+1}) \wedge x_1 < x_{k+1}$, where ψ_k is as in the preceding proof. As in the proof of Proposition 13, the conjunct $x_1 < x_{k+1}$ ensures that the sequences $x_1,\dots,x_{k+1}$ and $x_{k+1},\dots,x_1$ are not counted as two paths.

We know that p-#MAXIMAL-INDEPENDENT-SET is $\#$W[2]-hard under fpt parsimonious reductions (cf. Theorem 2 (b)). The reduction from p-MAXIMAL-INDEPENDENT-SET to p-MAXIMAL-CHORDLESS-PATH of the preceding proof does not preserve the number of solutions, as one maximal independent set of size k yields various maximal chordless path of length $4k+2$. Nevertheless adding an ordering and edges as in the proof of Theorem 17 one gets an fpt parsimonious reduction from p-#MAXIMAL-INDEPENDENT-SET to p-#MAXIMAL-CHORDLESS-PATH; hence, the latter problem is $\#$W[2]-hard under fpt parsimonious reductions. $\Box$

6. The inflationary version

Let $\mathcal{Q}$ be any of the parameterized path or cycle (decision) problems considered so far. Then we define the parameterized problem $\mathcal{Q}_{\leq}$, the inflationary version of $\mathcal{Q}$, by

$$(\mathcal{G}, k) \in \mathcal{Q}_{\leq} \iff \text{for some } k' \leq k\text{:} \quad (\mathcal{G}, k') \in \mathcal{Q}.$$

As an immediate consequence of the definition we get:

(i) If $\mathcal{Q} \in$ FPT, then $\mathcal{Q}_{\leq} \in$ FPT.

As

$$\#\mathcal{Q}_{\leq}(\mathcal{G}, k) = \sum_{i=0}^{k} \#\mathcal{Q}(\mathcal{G}, i) \quad \text{and} \quad \#\mathcal{Q}(\mathcal{G}, k) = \#\mathcal{Q}_{\leq}(\mathcal{G}, k) - \#\mathcal{Q}_{\leq}(\mathcal{G}, k-1),$$

we have:

(ii) $\#\mathcal{Q}_{\leq} \equiv^{\text{fpt-T}} \#\mathcal{Q}$.

Theorem 21. *(a)* p-PATH$_{\leq}$, p-CYCLE$_{\leq}$, p-MAXIMAL-PATH$_{\leq}$, p-CHORDLESS-PATH$_{\leq}$, *and* p-CHORDLESS-CYCLE$_{\leq}$ *are fixed-parameter tractable and the corresponding counting problems are* #W$[1]$*-complete under fpt Turing reductions.*

(b) p-#MAXIMAL-CHORDLESS-PATH$_{\leq}$ *is* #W$[2]$*-complete under fpt Turing reductions.*

Proof: The claims for the first three decision problems in part (a) follow from (i). For p-CHORDLESS-PATH$_{\leq}$ note that $(\mathcal{G}, k)$ is a positive instance for every graph with nonempty vertex set. Finally, for p-CHORDLESS-CYCLE$_{\leq}$ the claim follows from the equivalence:

$$(\mathcal{G}, k) \in p\text{-CHORDLESS-CYCLE}_{\leq} \iff (\mathcal{G}, k) \in p\text{-CYCLE}_{\leq}.$$

For the direction from right to left note that by drawing a chord in a cycle we obtain two new cycles. If none of them is chordless, we proceed till we get a chordless one.

By previous results the #W$[1]$-hardness and the #W$[2]$-hardness under fpt Turing reductions of the counting problems in (a) and (b), respectively, follow from (ii); membership in #W$[1]$ and #W$[2]$, respectively, is shown by adapting the proofs for the non-inflationary versions. We do this for the problem in (b).

We show p-#MAXIMAL-CHORDLESS-PATH$_{\leq} \in \#$W$[2]$ by an fpt parsimonious reduction to the problem p-#MC$(\Pi^0_{1,1})$. A first choice would consist in using the formula

$$\bigvee_{\ell \leq k} \big(\psi_\ell(x_1, \ldots, x_{\ell+1}) \wedge x_1 < x_{\ell+1}\big), \tag{1}$$

where

$$\psi_\ell(x_1, \ldots, x_{\ell+1}) := \Big(\bigwedge_{i \in [\ell]} E x_i x_{i+1} \wedge \bigwedge_{i,j \in [\ell+1], i \neq j} \neg x_i = x_j \wedge \bigwedge_{1 \leq i < j \leq \ell} \neg E x_i x_{j+1} \Big) \wedge$$
$$\forall y \Big(\Big(E x_1 y \to \bigvee_{2 \leq i \leq \ell+1} (x_i = y \vee E x_i y) \Big) \wedge \Big(E x_{\ell+1} y \to \bigvee_{1 \leq i \leq \ell} (x_i = y \vee E x_i y) \Big) \Big).$$

(The formula ψ_ℓ was used in the proof of Theorem 18 and expresses that $x_1, \ldots, x_{\ell+1}$ is a maximal chordless path of length ℓ and the formula $(\psi_\ell \wedge x_1 < x_{\ell+1})$ was used in the proof of Theorem 20 to count the maximal chordless paths of length ℓ.) There are two problems: First, we have to find a Π_1-formula equivalent to (1), where the length of the universal block is independent of k (we even claimed that we can get a block of length 1). Moreover, we have to fix the value of $x_{\ell+2}, \dots, x_k$ in paths of length ℓ. In fact, we can take the following formula $\rho_k(x_1, \dots, x_{k+1})$ (note that for vertices $x_1, \dots, x_{k+1}$ there is at most one $\ell \le k$ such that all iterated conjunctions in the first line are satisfied):

$$\forall y \bigvee_{\ell \le k} \Big(\bigwedge_{i \in [\ell]} E x_i x_{i+1} \wedge \bigwedge_{i,j \in [\ell+1], i \neq j} \neg x_i = x_j \wedge \bigwedge_{1 \le i < j \le \ell} \neg E x_i x_{j+1} \wedge \bigwedge_{\ell+1 < i \le k+1} x_i = x_{\ell+1} \wedge$$
$$\Big(\Big(E x_1 y \rightarrow \bigvee_{2 \le i \le \ell+1} (x_i = y \vee E x_i y) \Big) \wedge \Big(E x_{\ell+1} y \rightarrow \bigvee_{1 \le i \le \ell} (x_i = y \vee E x_i y) \Big) \Big) \wedge x_1 < x_{\ell+1} \Big).$$

Now it is easy to verify that $|\rho_k(\mathcal{G})|$ is the number of maximal chordless paths of lengths $\le k$ of a given graph $\mathcal{G}$. $\square$

But what is the complexity of p-MAXIMAL-CHORDLESS-PATH$_{\le}$? Since the fpt parsimonious reduction from p-#MAXIMAL-CHORDLESS-PATH$_{\le}$ to p-#MC$(\Pi^0_{1,1})$ of the preceding proof induces an fpt reduction of the corresponding decision problems, we know that p-MAXIMAL-CHORDLESS-PATH$_{\le} \in$ W$[2]$. We even show:

Theorem 22. *p-MAXIMAL-CHORDLESS-PATH$_{\le}$ is W$[2]$-complete under fpt reductions.*

Remark 23. The proof of Theorem 22 will yield the NP-completeness of the classical problem

MAXIMAL-CHORDLESS-PATH$_{\le}$	
Instance:	A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Problem:	Is there a maximal chordless path in $\mathcal{G}$ of length $\le k$?

This answers an open problem of S.M. Hedetniemi [21].

In view of Theorem 2 (a) to get Theorem 22 it suffices to show:

Proposition 24. p-MAXIMAL-INDEPENDENT-SET $\le^{\text{fpt}}$ p-MAXIMAL-CHORDLESS-PATH$_{\le}$.

Proof: Let $(\mathcal{G}, k)$ with $\mathcal{G} = (V, E)$ be an instance of p-MAXIMAL-INDEPENDENT-SET. From the proof of Theorem 18 we know that the graph $\mathcal{G}(k+1)$ has the following properties:

(a) Let $I = \{u_1, \dots, u_k\}$ be a maximal independent set of size k in $\mathcal{G}$. Then $P(u_1, \dots, u_k), (a, k+1), (b, k+1), (c, k+1)$ is a maximal chordless path in $\mathcal{G}(k+1)$ of length $4k+2$.

(b) Every maximal chordless path of length $4k+2$ in $\mathcal{G}(k+1)$ starting in $(a, 1)$ and ending in $(c, k+1)$ must have the form $P(u_1, \dots, u_k), (a, k+1), (b, k+1), (c, k+1)$, where $\{u_1, \dots, u_k\}$ is a maximal independent set of size k in $\mathcal{G}$.

In order to give a reduction of p-MAXIMAL-INDEPENDENT-SET to p-MAXIMAL-CHORDLESS-PATH$_{\le}$, we will extend $\mathcal{G}(k+1)$ to a graph $\mathcal{G}[k+1]$ by adding vertices (called e_t below) in such a way that every maximal chordless path between $(a, 1)$ and $(c, k+1)$ will have the form mentioned in (b).

We set $F := \big\{(a, 1), (c, k+1)\big\}$. We let $\mathcal{G}[k+1] = (V[k+1], E[k+1])$ be the graph with

$$V[k+1] := V(k+1) \,\dot\cup\, \big\{e_t \;\big|\; t \in [k] \text{ or } t \in \big(\{a, b, c\} \times [k+1]\big) \setminus F\big\},$$
$$E[k+1] := E(k+1) \cup \big\{\{(a,1), e_t\} \mid e_t \in V[k+1]\big\}$$
$$\cup \big\{\{e_t, t\} \mid t \in \big(\{a, b, c\} \times [k+1]\big) \setminus F\big\}$$
$$\cup \big\{\{e_t, (u, t)\} \mid t \in [k] \text{ and } u \in V\big\}.$$

Since $\mathcal{G}(k+1)$ is an induced subgraph of $\mathcal{G}[k+1]$, we have:

(i) If $b_1, \dots, b_m \in V(k+1)$ and if every vertex in $V[k+1] \setminus V(k+1)$, that is, every e_t has a neighbor among $b_1, \dots, b_m$ in $\mathcal{G}[k+1]$, then

$$(a_1, 1), b_1, \dots, b_m, (c, k+1) \text{ is a maximal chordless path in } \mathcal{G}(k+1) \iff (a_1, 1), b_1, \dots, b_m, (c, k+1) \text{ is a maximal chordless path in } \mathcal{G}[k+1].$$

Furthermore we show:

(ii) Every maximal chordless path P of $\mathcal{G}[k+1]$ starting in $(a, 1)$ and ending in $(c, k+1)$ is contained in $\mathcal{G}(k+1)$. Moreover, all vertices in $\{a, b, c\} \times [k+1]$ occur in P.

(iii) Every maximal chordless path P of $\mathcal{G}(k+1)$ starting in $(a, 1)$ and ending in $(c, k+1)$ and containing all vertices of $\{a, b, c\} \times [k+1]$ has length $4k+2$.

For (ii) let P be a maximal chordless path of $\mathcal{G}[k+1]$ starting in $(a, 1)$ and ending in $(c, k+1)$. First we show that every $t \in (\{a, b, c\} \times [k+1]) \setminus F$ is in P. Assume $t \notin P$. Then $e_t \in P$, since otherwise e_t, P would be a chordless path extending P, a contradiction. Then e_t has two neighbors in P, which must be the two neighbors of e_t in $\mathcal{G}[k+1]$, namely $(a, 1)$ and t; hence $t \in P$.

Now assume that some vertex in $V[k+1] \setminus V(k+1)$, that is, some e_t occurs in P. Then P must start by $(a, 1), e_t$. As $(b, 1)$ is a neighbor of $(a, 1)$, this shows that $(b, 1) \notin P$, a contradiction by what we have already shown.

We turn to a proof of (iii). So let P be a maximal chordless path of $\mathcal{G}(k+1)$ starting in $(a, 1)$ and ending in $(c, k+1)$ and containing all vertices of $\{a, b, c\} \times [k+1]$. Thus P ends with $(b, k+1), (c, k+1)$. As $(c, k+1)$ is a neighbor of $(u, k+1)$ for all $u \in V$, we therefore get $\big(V \times \{k+1\}\big) \cap P = \emptyset$. On the other hand, for $i \in [k]$ the vertex (c, i) is a vertex of P with two neighbors in P, one being (b, i). Therefore $|\big(V \times \{i\}\big) \cap P| = 1$.

Thus P contains $4k+3$ vertices and hence has length $4k+2$. This finishes the proof of (iii).

Now we claim:

$$\mathcal{G} \text{ has a maximal independent set of size } k \iff \mathcal{G}[k+1] \text{ has a maximal chordless path of length} \le 4k+2 \text{ with endvertices } (a, 1) \text{ and } (c, k+1).$$

In fact, assume first that $\mathcal{G}$ has a maximal independent set $\{u_1, \dots, u_k\}$ of size k. Then, by (a), the sequence $P(u_1, \dots, u_k), (a, k+1), (b, k+1), (c, k+1)$ is a maximal chordless path of length $4k+2$ in $\mathcal{G}(k+1)$, and hence, by (i), in $\mathcal{G}[k+1]$.

Conversely, let P be a maximal chordless path in $\mathcal{G}[k+1]$ of length $\le 4k+2$ starting in $(a, 1)$ and ending in $(c, k+1)$. By (ii), P is a path in $\mathcal{G}(k+1)$ and hence, a maximal chordless path in $\mathcal{G}(k+1)$; furthermore, by (ii) and (iii), it has length $4k+2$. Now the claim follows from (b).

From the previous equivalence we get by Lemma 19,

$$\mathcal{G} \text{ has a maximal independent set of size } k \iff (\mathcal{G}[k+1])_{F, 4k+2} \text{ has a maximal chordless path of length} \le 4k+2,$$

which yields the desired reduction. □

Arguing as in the proof of Theorem 17, one obtains the corresponding reduction for the counting versions:

Proposition 25. p-#MAXIMAL-INDEPENDENT-SET $\le^{\text{fpt}}$ p-#MAXIMAL-CHORDLESS-PATH$_{\le}$.

By Theorem 21 (b) we know that p-#MAXIMAL-CHORDLESS-PATH$_{\le}$ is #W[2]-complete under fpt Turing reductions. Now we get:

Theorem 26. p-#MAXIMAL-CHORDLESS-PATH$_{\leq}$ *is* #W$[2]$*-complete under fpt parsimonious reductions.*

Proof: Proposition 25 together with Theorem 2 (b) yields the #W$[2]$-hardness under fpt parsimonious reductions. $\square$

7. Further Questions and Fagin-Definability

We have seen that p-#PATH $\leq^{\text{fpt}}$ p-#MAXIMAL-PATH and that both problems are #W$[1]$-complete under fpt Turing reductions. Is p-#MAXIMAL-PATH $\leq^{\text{fpt}}$ p-#PATH? If not (say, under the assumption FPT $\neq$ W$[1]$), then this would reflect that also for the path problem the transition to the corresponding maximality version increases the complexity, a phenomenon we have seen for various other problems in [5] and in this paper.

A *strong embedding* from a graph $\mathcal{G} = (V, E)$ to a graph $\mathcal{G}' = (V', E')$ is an embedding h from $\mathcal{G}$ to $\mathcal{G}'$ with the additional property:

$$\text{for all } u, v \in V: \qquad \Big(\{h(u), h(v)\} \in E' \Rightarrow \{u, v\} \in E \Big).$$

Clearly, $\mathcal{G}$ contains a chordless path of length k if and only if the generic path $\mathcal{P}_k$ is strongly embeddable in $\mathcal{G}$; and it contains a chordless cycle if and only if the generic cycle $\mathcal{C}_k$ is strongly embeddable.

Let C be an infinite decidable class of graphs. What are necessary and sufficient conditions on C such that the *parameterized strong embedding problem for* C

p-STRONG-EMB(C)	
Instance:	A graph $\mathcal{G} \in \mathrm{C}$ and a graph $\mathcal{G}'$.
Parameter:	$\|\mathcal{G}\|$.
Problem:	Does there exist a strong embedding from $\mathcal{G}$ to $\mathcal{G}'$?

is not fixed-parameter tractable?

7.1. Holes. A *hole* in a graph is a chordless cycle of length at least 4. A hole is *even* (*odd*) if it has even (odd) length. In the last time problems concerning holes have received much attention as they are related to the Strong Perfect Graph Theorem ("A graph is perfect if it contains neither an odd hole nor the complement of an odd hole"), which has been proven recently [6, 8]. We mention some results and open problems: It is not known whether there is a polynomial time algorithm deciding if a graph has an odd hole, while the questions whether a graph contains a hole and whether it contains an even hole are solvable in polynomial time (cf. [24, 9, 7]). Nevertheless, the complexity of finding an even hole of minimum length in a graph is still open as far as we know.

Clearly, as p-CHORDLESS-CYCLE is W$[1]$-complete, so is p-HOLE (defined in the obvious way). But the open problem just mentioned leads to the question: What are the complexities of p-HOLE$_{\leq}$, p-EVEN-HOLE$_{\leq}$, and p-ODD-HOLE$_{\leq}$, where for example

p-ODD-HOLE$_{\leq}$	
Instance:	A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Does $\mathcal{G}$ contain a hole of odd length $\leq k$?

It is easy to show that p-HOLE$_{\leq}$ $\in$ FPT by reducing it to p-EMB(C) for a suitable class C of bounded treewidth. But even the corresponding classical problem is polynomial time solvable (we include a proof as we did not find this result in the literature). Our statement is an immediate consequence of:

Theorem 27. *There is a polynomial time algorithm that, given a graph $\mathcal{G}$ outputs a hole in $\mathcal{G}$ of minimum length (and rejects, if there is no hole). In fact, there is an algorithm that takes time quadratic in $\|\mathcal{G}\|$.*

Proof: Our proof uses ideas from [26]. A cycle $v_0, \dots, v_{k-1}$ in a graph $\mathcal{G}$ is *2-chordless* if $k \geq 4$ and $\{v_i, v_{(i+2) \mod k}\} \notin E$ for $0 \leq i \leq k-1$. Clearly, any hole is 2-chordless and any 2-chordless cycle of minimum length is a hole. Hence:

Claim 1: Let C be a cycle in $\mathcal{G}$. Then C is a hole of minimum length if and only if C is a 2-chordless cycle of minimum length.

For a graph $\mathcal{G} = (V, E)$ we define a *directed* graph $\mathcal{G}^{\text{dir}} = (V^{\text{dir}}, E^{\text{dir}})$ by

$$V^{\text{dir}} := \big\{(u,v) \mid \{u,v\} \in E\big\}; \quad E^{\text{dir}} := \big\{((u,v),(v,w)) \mid u,v,w \text{ is a chordless path in } \mathcal{G}\big\}.$$

The following is immediate:

Claim 2: If $v_1, v_2, \dots, v_k$ is a 2-chordless cycle in $\mathcal{G}$, then

$$(v_1,v_2),(v_2,v_3),\dots,(v_{k-1},v_k),(v_k,v_1)$$

is a directed cycle in $\mathcal{G}^{\text{dir}}$.

Moreover, we show:

Claim 3: If $D = (v_1,v_2),(v_2,v_3),\dots,(v_{k-1},v_k),(v_k,v_1)$ is a directed cycle in $\mathcal{G}^{\text{dir}}$ of minimum length, then $v_1, \dots, v_k$ is a 2-chordless cycle in $\mathcal{G}$.

Proof: As $\{v_1, v_3\} \notin E$, we see that $k \geq 4$. Thus it suffices to show $v_1, \dots, v_k$ are pairwise distinct. Assume there are $1 \leq i < j \leq k$ with $v_i = v_j$. We choose such i, j with minimum $j - i$. Then $v_i, v_{i+1}, \dots, v_{j-1}$ are pairwise distinct and $j > i + 2$. Even $j > i + 3$, otherwise, $j - 1 = i + 2$ and thus $\{v_{i+2}, v_i\} \in E$ (as $\{v_{j-1}, v_j\} \in E$), a contradiction. As D has minimum length, $(v_i,v_{i+1}),(v_{i+1},v_{i+2}),\dots,(v_{j-1},v_i)$ is not a directed cycle in $\mathcal{G}^{\text{dir}}$. Therefore $\{v_{j-1}, v_{i+1}\} \in E$. We choose i', j' with $i+1 \leq i' < i'+1 < j' \leq j-1$ and minimum $j' - i'$ such that $\{v_{i'}, v_{j'}\} \in E$. Then $j' > i' + 2$ and thus $v_{i'}, v_{i'+1}, \dots, v_{j'}$ is a 2-chordless cycle. By Claim 2, $(v_{i'},v_{i'+1}),(v_{i'+1},v_{i'+2})\dots,(v_{j'-1},v_{j'}),(v_{j'},v_{i'})$ is a directed cycle in $\mathcal{G}^{\text{dir}}$, which is shorter than D, a contradiction. $\dashv$

Claim 2 and Claim 3 yield:

Claim 4: Let $v_1, \dots, v_k \in V$. Then

$$v_1, v_2, \dots, v_k \text{ is a 2-chordless cycle in } \mathcal{G} \text{ of minimum length } \iff$$
$$(v_1,v_2),(v_2,v_3),\dots,(v_{k-1},v_k),(v_k,v_1) \text{ is a directed cycle in } \mathcal{G}^{\text{dir}} \text{ of minimum length.}$$

Note that any directed cycle in $\mathcal{G}^{\text{dir}}$ must be of the form $(v_1,v_2),(v_2,v_3),\dots,(v_{k-1},v_k),(v_k,v_1)$ for suitable $k \geq 4$ and $v_1, \dots, v_k \in V$. Therefore, by Claim 1 and Claim 4, to prove our theorem, it suffices to show that there is a polynomial time algorithm that computes in a given directed graph a cycle of minimum length. But the existence of such an algorithm is well-known. (In fact, for this purpose for each pair of distinct vertices u and v of a directed graph $\mathcal{H} = (V, E)$ with $(v, u) \in E$ we compute the length of a path from u to v of minimum length.) $\Box$

We do not know what the complexities of the problems p-EVEN-HOLE$_\leq$ and p-ODD-HOLE$_\leq$ are. We call a cycle in a graph $\mathcal{G}$ *triangle-free* if the graph induced by $\mathcal{G}$ on each three distinct vertices of the cycle is not a clique. As any chord of a triangle-free cycle of odd length divides this cycle into two new triangle-free cycles, one of them being of odd length, too, we get:

$$p\text{-ODD-HOLE}_\leq \equiv^{\text{fpt}} p\text{-TRIANGLE-FREE-ODD-CYCLE}_\leq,$$

where the latter problem is defined in the obvious way. We are able to determine the complexity of a related problem:

Theorem 28. *The problem*

p-TRIANGLE-FREE-CYCLE	
Instance:	A graph $\mathcal{G}$ and $k \geq 1$.
Parameter:	k.
Problem:	Does $\mathcal{G}$ contain a triangle-free cycle of length k?

is W[1]*-complete under fpt reductions.*

Proof: The membership of the problem in W[1] is shown by a straightforward reduction to p-MC(QF). We turn to a proof of its W[1]-hardness. Again, we show it by a reduction from the independent set problem. We shall need the following consequence of Ramsey's Theorem.

> There is a computable function $h : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$ every graph $\mathcal{G} = (V, E)$ with $|V| \geq h(k)$ contains a clique or an independent set of size k. In particular, for $k \geq 3$ every triangle-free cycle contains an independent set of size k.[4]

The following construction is inspired by [23]. Let $\mathcal{G} = (V, E)$ be a graph and $k, \ell \geq 1$. We define a graph $\mathcal{G}(k, \ell) = (V(k, \ell), E(k, \ell))$. Basically $\mathcal{G}(k, \ell)$ consists of three layers. The lowest layer is the original graph $\mathcal{G}$. The second layer consists of $2k$ vertices $s_{1,1}, s_{1,2}, \dots, s_{k,1}, s_{k,2}$, all adjacent to every vertex in $\mathcal{G}$. Finally the top layer, for each $i \in [k]$, contains vertices $b_{i,1}, b_{i,2}, \dots, b_{i,\ell}$ such that

$$P_i := s_{i,2}, b_{i,1}, b_{i,2}, \dots, b_{i,\ell}, s_{i+1,1}$$ [5]

is a path of length $\ell + 1$ connecting $s_{i,2}$ and $s_{i+1,1}$. More precisely:

$$\begin{array}{rcl} V(k,\ell) & := & V \ \dot\cup \ \big\{s_{i,j} \mid i \in [k] \text{ and } j \in [2]\big\} \ \dot\cup \ \big\{b_{i,j} \mid i \in [k], \text{ and } j \in [\ell]\big\}, \\ E(k,\ell) & := & E \ \cup \ \big\{\{s_{i,j}, v\} \mid i \in [k], j \in [2] \text{ and } v \in V\big\} \\ & & \quad \cup \ \big\{\{s_{i,2}, b_{i,1}\} \mid i \in [k]\big\} \ \cup \ \big\{\{b_{i,\ell}, s_{i+1,1}\} \mid i \in [k]\big\} \\ & & \quad \cup \ \big\{\{b_{i,j}, b_{i,j+1}\} \mid i \in [k] \text{ and } j \in [\ell - 1]\big\}. \end{array}$$

Then

$$|V(k,\ell)| = |V| + 2k + \ell \cdot k. \tag{2}$$

The proof of the following claim is immediate.

Claim 1: If $\{v_1, \dots, v_k\}$ is an independent set of size k in $\mathcal{G}$, then

$$v_1, P_1, v_2, P_2, \dots, v_k, P_k$$

is a triangle-free cycle of length $(3 + \ell) \cdot k$ in $\mathcal{G}(k, \ell)$.

We prove the following converse of Claim 1:

Claim 2: Let $k \geq 2$ and

$$\ell := \max\left\{1, \left\lceil \frac{h(k)}{k} \right\rceil - 3\right\}.$$

If $\mathcal{G}(k, \ell)$ contains a triangle-free cycle of length $(3 + \ell) \cdot k$, then $\mathcal{G}$ contains an independent set of size k.

Proof: Let C be a triangle-free cycle of length $(3 + \ell) \cdot k$ in $\mathcal{G}(k, \ell)$. By (2) we know that

$$|C \cap V| \geq k.$$

We distinguish two cases.

[4] For example as h one can take the function $k \mapsto 2^{2k}$. Better bounds are known (see [4]).

[5] Here and in the following for $i = k$ we read s_{i+1} as s_1.

$C \not\subseteq V$**:** Since C is a cycle, it must contain a vertex $s_{i,j}$. Therefore, $C \cap V$ must be an independent set, otherwise, C is not triangle-free.

$C \subseteq V$**:** Then C is a triangle-free cycle in $\mathcal{G}$. Since $\ell = \max\{1, \left\lceil \frac{h(k)}{k} \right\rceil - 3\}$, we have

$$|C| = (3 + \ell) \cdot k \geq h(k).$$

By the consequence of Ramsey's Theorem quoted above, we see that $\mathcal{G}$ contains an independent set of size k. $\dashv$

By Claim 1 and Claim 2 we see that for $\ell := \max\{1, \left\lceil \frac{h(k)}{k} \right\rceil - 3\}$

$$(\mathcal{G}, k) \mapsto \big(\mathcal{G}(k, \ell), (3 + \ell) \cdot k\big)$$

is an fpt reduction of p-INDEPENDENT-SET to p-TRIANGLE-FREE-CYCLE. $\square$

7.2. Fagin-definability. We close by taking up the question of the Fagin-definability of paths and cycle problems mentioned in the Introduction.

Let $\varphi(Z)$ be a first-order formula of vocabulary τ with a relation variable X, say, of arity r. Furthermore, let C be a class of τ-structures. On C the formula $\varphi = \varphi(Z)$ *Fagin-defines* the problem:

p-WD$_\varphi$(C)	
Instance:	A structure $\mathcal{A} \in$ C and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Is there a subset $S \subseteq A^r$ with $\lvert S\rvert = k$ such that $\mathcal{A} \models \varphi(S)$?

Let GRAPH be the class of graphs. Let an *ordered graph* $(\mathcal{G}, <^{\mathcal{G}})$ consist of a graph $\mathcal{G} = (V, E)$ and an ordering $<^{\mathcal{G}}$ of V and let GRAPH$_<$ be the class of ordered graphs. If we consider the path problem p-PATH as a problem on ordered graphs we denote it by p-PATH$_<$. The following proposition contains three non-definability results for the path problem, which are immediate consequences of known (nontrivial) results, and one definability result. Similar results hold for the chordless path problem and for the (chordless) cycle problem.

Proposition 29. *(a) There is no first-order formula $\varphi(Z)$ with a set variable Z (that is, with unary Z) such that* p-PATH $= p$-WD$_\varphi$(GRAPH).

(b) There is no Π_1-formula $\varphi(X)$ with a relation variable X of arbitrary arity r such that p-PATH $= p$-WD$_\varphi$ (GRAPH).

(c) There is no first-order formula $\varphi(Z)$ with a set variable Z such that p-PATH$_<$ $= p$-WD$_\varphi$(GRAPH$_<$).

(d) There is a first-order formula $\varphi(Z)$, even a Π_2-formula, with a binary Z such that

$$p\text{-PATH}_< = p\text{-WD}_\varphi(\text{GRAPH}_<).$$

Proof: (a) Assume that p-PATH $= p$-WD$_\varphi$(GRAPH) for some $\varphi(Z)$ with a set variable Z. Then the sentence of monadic second-order logic

$$\exists Z(\varphi(Z) \wedge \forall x Zx)$$

would axiomatize the class of graphs with a hamiltonian path. This contradicts a result of [14].

Part (b) is easily shown by using the well-known fact that Π_1-formulas are preserved under substructures. Part (c) follows as (a) but now using the corresponding result from [30].

(d) Let $(\mathcal{G}, <^{\mathcal{G}})$ be an ordered graph with $\mathcal{G} = (V, E)$. We can assume that $V = [n]$ and that $<^{\mathcal{G}}$ is the natural order of V. We show that there is a Π_2-formula $\varphi(Z)$ with binary Z such that

(i) If $k \in \mathbb{N}$ and $v_1, \ldots, v_{k+1}$ is a path in $\mathcal{G}$, then $(\mathcal{G}, <^{\mathcal{G}}) \models \varphi(S)$ for

$$S := \big\{(1, v_1), (2, v_2), \ldots, (k, v_k), (k+1, v_{k+1})\big\}.$$

(ii) If $(\mathcal{G}, <^{\mathcal{G}}) \models \varphi(S)$, then there is a $k \in \mathbb{N}$ such that $|S| = k+1$ and there is an $i \in [n]$ with $i + k \leq n$ and a path $v_1, \ldots, v_{k+1}$ in $\mathcal{G}$ such that

$$S = \big\{(i, v_1), (i+1, v_2), \ldots, (i+k-1, v_k), (i+k, v_{k+1})\big\}.$$

Then, clearly p-PATH$_<$ $= p$-WD$_\varphi$(GRAPH$_<$). As $\varphi(Z)$ we can take a Π_2-formula equivalent to

$$\exists x \exists y Zxy \wedge \mathit{funct}(Z) \wedge \mathit{dom\text{-}segm}(Z) \wedge \mathit{rg\text{-}path}(Z).$$

Here *funct*(Z) is a Π_1-formula expressing that Z is (the graph of) a function, *dom-int*(Z) a Π_2-formula expressing that the domain of Z is a segment (in $[n]$) and *rg-path*(Z) a Π_2-formula expressing that the range of Z is a path. For example, we can set

$$\mathit{funct}(Z) := \forall x \forall u \forall v \big((Zxu \wedge Zxv) \rightarrow u = v\big)$$

and

$$\mathit{dom\text{-}segm}(Z) := \forall x \forall y \forall z \forall u \forall v \exists w \big((Zxu \wedge Zyv \wedge (x < z < y)) \rightarrow Zzw\big).$$

We leave it to the reader to write down a Π_2-formula *rg-path*(Z). □

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