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On Parametric Model Order Reduction by Matrix Interpolation

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***Abstract*— A general framework for model order reduction is proposed for high-order parameter-dependent, linear time-invariant systems. The procedure is based on matrix interpolation and consists of six steps. At first a set of high-order nonparametric systems is computed for different parameter vectors. The resulting local high-order systems are then reduced by a projection-based reduction method. Thereby, proper right and left subspaces for the reduced systems are calculated. Next the bases of the right subspaces of the reduced systems are adapted and the bases of the left subspaces are adjusted. For that the concept of duality is introduced. Finally, the precomputed matrices of the local systems are interpolated in a matrix manifold with an interpolation method. In this paper the six steps of the algorithm and the degrees of freedom which arise therein are presented. Furthermore, advantages and difficulties in the selection of the degrees of freedom are pointed out. It is additionally shown that two existing methods for parametric model order reduction by matrix interpolation are special cases of the proposed general procedure as they – often implicitly – determine a limiting selection of the degrees of freedom.**

I. INTRODUCTION

With increasing demands on accuracy for optimization, simulation or control, the modeling of complex systems delivers large-scale systems of ordinary differential equations. In order to reduce the computational effort, methods of model order reduction can be applied to replace a large-scale model by a low-order model. Several methods of model order reduction are known to approximate the transfer behavior of the large-scale system [1].

For many engineering applications the high-order systems depend on parameters, for example geometry or material parameters. Methods of parametric model order reduction (pMOR) reduce the order of a large-scale system and at the same time preserve the parametric dependencies.

In [2] a first approach in pMOR was presented. It approximates the high-order model by matching moments of the transfer function with regard to the Laplace variable s and to one parameter. Basically, it was a generalization of the moment matching method to parametric systems with one linear parametric dependency. This method was enhanced in [3] to be applicable to the multiple parameter case. However, it suffers from the curse of dimensionality as the order of the reduced system increases rapidly with the number of parameters.

Besides, several interpolation based methods were proposed. They sample the parameter space and obtain a discrete set of high-order systems for different parameter vectors. The original system does not need to be analytically given. It is sufficient if it is available locally for the set of parameter values. This is for example the case if the original system is modeled by a FEM program. The local high-order models are reduced individually to obtain a set of low-order models.

In [4] a procedure applying interpolation of transfer functions of the local low-order models was presented. This method is based on the Truncated Balanced Realization of the local models and inherits its beneficial properties like stability-preservation for the interpolated system. However, the order of the interpolated reduced system increases with the number of local models.

Another possibility is the interpolation of the system matrices of local low-order models. In [5], [7] a method applying matrix interpolation with weighting functions to the reduced models was presented. A similar approach was proposed in [10], [11], [12] which uses element-wise interpolation in the tangent space of matrix manifolds. Both methods have in common that the order of the interpolated reduced system is independent of the number of local systems. Additionally, they can capture mode crossing and mode veering.

This paper focuses on a general method for pMOR by matrix interpolation. In section II a short introduction of projection-based model order reduction is given. In section III pMOR by matrix interpolation is arranged in a general framework, followed by the integration of the two existing methods into the framework in section IV. Numerical results are presented in section V.

II. PRELIMINARIES AND STATE OF THE ART

A. Linear time-invariant dynamical systems

In this paper, a linear time-invariant (LTI) dynamical system in descriptor form is considered:

$$G(\mathbf{p}) : \begin{cases} \mathbf{E}(\mathbf{p})\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p})\mathbf{x}(t) + \mathbf{B}(\mathbf{p})\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}(\mathbf{p})\mathbf{x}(t), \end{cases} \tag{1}$$

where $\mathbf{E}(\mathbf{p}) \in \mathbb{R}^{n\times n}$, $\mathbf{A}(\mathbf{p}) \in \mathbb{R}^{n\times n}$, $\mathbf{B}(\mathbf{p}) \in \mathbb{R}^{n\times m}$ and $\mathbf{C}(\mathbf{p}) \in \mathbb{R}^{p\times n}$ are parameter-dependent matrices with the vector of parameters $\mathbf{p} \in \mathbb{R}^d$. The vectors $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^p$ and $\mathbf{x}(t) \in \mathbb{R}^n$ denote the inputs, outputs and states of the system, respectively[1].

In the following, the system representation of (1) is referred to as $G(\mathbf{p})$. Assume that system matrices of high-order systems are computed for a set of k parameter vectors $\mathbf{p}_i$ with $i = 1...k$. The i-th local nonparametric system is then called G_i.

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[1]The time variable t is omitted in the following.

B. Projection-based model order reduction

Given is a high-dimensional system G_i which shall be approximated by a low-dimensional system of order $q \ll n$ using a Petrov-Galerkin projection. Conversant projection-based reduction methods are the Truncated Balanced Realization (TBR), Proper Orthogonal Decomposition (POD) or Krylov subspace methods. For detailed information on the reduction methods please refer to [1] and references therein.

The projection matrices $\mathbf{V}_i \in \mathbb{R}^{n\times q}$, $\mathbf{W}_i \in \mathbb{R}^{n\times q}$ span the right $\mathcal{V}_i$ and left subspace $\mathcal{W}_i$, respectively. Therefore, in the following the matrices $\mathbf{V}_i$ are referred to as right reduced order bases (ROBs) and $\mathbf{W}_i$ as left ROBs. They are calculated by any preferred projection-based reduction method. This leads to the following reduced order model:

$$G_{r,i} : \begin{cases} \mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i} = \mathbf{A}_{r,i}\mathbf{x}_{r,i} + \mathbf{B}_{r,i}\mathbf{u}, \\ \mathbf{y}_{r,i} = \mathbf{C}_{r,i}\mathbf{x}_{r,i}, \end{cases} \tag{2}$$

where

$$\begin{aligned} \mathbf{E}_{r,i} &= \mathbf{W}_i^T\mathbf{E}_i\mathbf{V}_i, \\ \mathbf{A}_{r,i} &= \mathbf{W}_i^T\mathbf{A}_i\mathbf{V}_i, \\ \mathbf{B}_{r,i} &= \mathbf{W}_i^T\mathbf{B}_i, \\ \mathbf{C}_{r,i} &= \mathbf{C}_i\mathbf{V}_i. \end{aligned} \tag{3}$$

III. GENERAL FRAMEWORK FOR PARAMETRIC MODEL ORDER REDUCTION BY MATRIX INTERPOLATION

In this section the pMOR approaches based on matrix interpolation from [5], [7] and [10], [11], [12] are arranged in a general framework. The reader can use the framework as a construction kit for pMOR by matrix interpolation including the following six steps:

A. Sampling of the parameter space.
B. Reduction of the local systems.
C. Adjustment of the right ROBs.
D. Adjustment of the left ROBs.
E. Choice of the interpolation manifold.
F. Choice of the interpolation method.

In the following the six steps and the degrees of freedom which arise therein are presented. Additionally, various alternatives and the respective advantages and difficulties are pointed out for the degrees of freedom so that the reader can determine a selection with respect to the considered system and the desired features.

A. Sampling of the parameter space

The parameter space is sampled in a suitable way for k vectors $\mathbf{p}_i$ and the high-order systems are computed at these parameter vectors. This results in a set of k high-dimensional LTI systems G_i with $i = 1...k$.

B. Reduction of the local systems

The local systems G_i are reduced to the same order $q \ll n$. In order to calculate proper subspaces $\mathcal{V}_i$ and $\mathcal{W}_i$ for every local model, the systems G_i are reduced individually. The basis vectors of the subspaces are normalized. Thereby, any projection-based reduction method can be applied which results in a set of reduced systems $G_{r,i}$.

C. Adjustment of the right ROBs

The states of the reduced order systems $G_{r,i}$ lie in different right subspaces spanned by $\mathbf{V}_i$. For a meaningful interpolation of the system matrices, the states of the reduced systems $G_{r,i}$ have to be described in a set of generalized coordinates. This is done by the state transformations $\mathbf{x}_{r,i} = \mathbf{T}_i\tilde{\mathbf{x}}_{r,i}$ with $\mathbf{T}_i \in \mathbb{R}^{q\times q}$, which leave the input-output behavior of the local systems unchanged and lead to the following systems:

$$\begin{aligned} \mathbf{E}_{r,i}\mathbf{T}_i\dot{\tilde{\mathbf{x}}}_{r,i} &= \mathbf{A}_{r,i}\mathbf{T}_i\tilde{\mathbf{x}}_{r,i} + \mathbf{B}_{r,i}\mathbf{u}, \\ \mathbf{y}_{r,i} &= \mathbf{C}_{r,i}\mathbf{T}_i\tilde{\mathbf{x}}_{r,i}. \end{aligned} \tag{4}$$

The state transformations are supposed to introduce a set of generalized coordinates with respect to a reference subspace spanned by the columns of $\mathbf{R}_V \in \mathbb{R}^{n\times q}$.

1) Reference subspace: The reference subspace should comprise the directions which describe the most important dynamics of all local models. The states of $G_{r,i}$ lie in the right subspaces spanned by the columns of $\mathbf{V}_i$. Hence, there are a few possibilities for the calculation of $\mathbf{R}_V$:

- Fixed [11]:
  The simplest way of choosing $\mathbf{R}_V$ is to take the right ROB $\mathbf{V}_{i_0}$ of one of the reduced systems i_0:

  $$\mathbf{R}_V = \mathbf{V}_{i_0}. \tag{5}$$

  This choice does not need any additional calculation and is determined only once. Therefore, it can be used for real-time applications. However, it cannot be stated in advance which subspace $\mathcal{V}_{i_0}$ approximates the most important dynamics of all systems best.

- Non-weighted SVD [7]:
  This approach sums up all right ROBs $\mathbf{V}_i$ and takes the q most important directions by calculating the economy version of the Singular Value Decomposition (SVD). The first q columns of $\mathbf{U}$ are then chosen as $\mathbf{R}_V$:

  $$\begin{aligned} \mathbf{U}\mathbf{\Sigma}\mathbf{N}^T &= \mathrm{svd}([\mathbf{V}_1...\mathbf{V}_k], \text{'econ'}) \\ &\Rightarrow \mathbf{R}_V = \mathbf{U}(:, 1:q). \end{aligned} \tag{6}$$

  This approach does not need knowledge of the system dynamics in advance as the most important directions of the subspaces $\mathcal{V}_i$ are considered. The calculation is done only once. Therefore, this approach is suited for real-time applications.

- Weighted SVD [7]:
  This approach resembles the non-weighted SVD, but weights ω_i for the right ROBs $\mathbf{V}_i$ are introduced:

  $$\begin{aligned} \mathbf{U}\mathbf{\Sigma}\mathbf{N}^T &= \mathrm{svd}([\omega_1(\mathbf{p})\mathbf{V}_1...\omega_k(\mathbf{p})\mathbf{V}_k], \text{'econ'}) \\ &\Rightarrow \mathbf{R}_V = \mathbf{U}(:, 1:q). \end{aligned} \tag{7}$$

  If the subspaces are very different, subspaces near the interpolation point can be favored in order to capture the most important dynamics near every local system. As the weights are parameter-dependent, $\mathbf{R}_V$ has to be calculated for every new interpolation parameter. Hence, this approach is not suited for real-time applications.

To sum up, the choice of $\mathbf{R}_V$ depends on available a priori knowledge of the system dynamics, the need for real-time capability and the difference of the right subspaces.

2) Generalized coordinates: The way of computing the matrices $\mathbf{T}_i$ which ensures that the systems $G_{r,i}$ are described in generalized coordinates with respect to a reference subspace shall be motivated by the following proposition:

Proposition 1: The right subspace of a reduced order system $G_{r,i}$ can be expressed by a variety of bases.

With this, the state transformations with matrices $\mathbf{T}_i$ can also be interpreted as changes of basis of the right subspaces of the reduced models. The new right ROBs are:

$$\tilde{\mathbf{V}}_i = \mathbf{V}_i \mathbf{T}_i. \tag{8}$$

The general idea, which was formulated in [11], is that the corresponding vectors of $\tilde{\mathbf{V}}_i$ and $\mathbf{R}_V$ are in good correlation by a proper choice of $\mathbf{T}_i$. The correlation between two vectors can be evaluated by the Modal Assurance Criterion (MAC) [14]. The maximal value of the MAC is 1, which corresponds to the best correlation, and the minimal value is 0. As the bases are normalized, the MAC between the j-th vector of $\tilde{\mathbf{V}}_i$ and the l-th vector of $\mathbf{R}_V$ is:

$$\mathrm{MAC}\big(\tilde{\mathbf{V}}_i(:,j), \mathbf{R}_V(:,l)\big) = \left|\tilde{\mathbf{V}}_i(:,j)^T \mathbf{R}_V(:,l)\right|^2. \tag{9}$$

Hence, the diagonal elements of the product $\tilde{\mathbf{V}}_i^T \mathbf{R}_V$ contain the square roots of the MACs between the corresponding vectors of $\tilde{\mathbf{V}}_i$ and $\mathbf{R}_V$ and the off-diagonal elements comprise the square roots of the MACs between the non-corresponding vectors. Therefore, the diagonal elements of $\tilde{\mathbf{V}}_i^T \mathbf{R}_V$ shall be maximal and the off-diagonal elements shall be minimal. In the following, two approaches for the calculation of the matrices $\mathbf{T}_i$ which fulfill the MACs in a strong and weak manner are described.

- Strong fulfillment of the MACs:
  The MACs between the corresponding vectors of $\tilde{\mathbf{V}}_i$ and $\mathbf{R}_V$ are explicitly set to the maximal value 1 and the MACs of the non-corresponding vectors are set to the minimal value 0, which leads to:

  $$\mathbf{T}_i^T \mathbf{V}_i^T \mathbf{R}_V \stackrel{!}{=} \mathbf{I}, \tag{10}$$

  which is equivalent to:

  $$\mathbf{T}_i = (\mathbf{R}_V^T \mathbf{V}_i)^{-1}. \tag{11}$$

  This result is identical to the choice of $\mathbf{T}_i$ in [7], where the derivation is based on a projection-based view.
  The matrices $\mathbf{T}_i \in \mathbb{R}^{q \times q}$ are real and describe permutations, rotations and distortions of the bases $\mathbf{V}_i$ in the corresponding right subspace $\mathcal{V}_i$. Therefore, this method is independent of the representation of $\mathbf{V}_i$, i.e. the bases $\mathbf{V}_i$ do not have to be orthogonal. Additionally, the MACs between the corresponding vectors are always maximal and the MACs between the non-corresponding vectors are minimal and hence optimal. However, the matrices $\mathbf{T}_i$ can be close to singular.

- Weak fulfillment of the MACs [11]:
  As a weaker fulfillment of the MACs, the sum of the diagonal elements of the product $\tilde{\mathbf{V}}_i^T \mathbf{R}_V$ shall be maximized. This results in the following maximization problems, where tr denotes the trace of a matrix:

  $$\mathbf{T}_i = \arg\max_{\mathbf{T}_i} \operatorname{tr}\left(\mathbf{T}_i^T \mathbf{V}_i^T \mathbf{R}_V\right). \tag{12}$$

  This optimization problem is known as the orthogonal Procrustes problem. An analytical solution based on the Singular Value Decomposition (SVD) is given in [13] which uses orthogonal matrices[2] $\mathbf{T}_i \in O(q)$. After performing the SVD of $\mathbf{V}_i^T \mathbf{R}_V = \mathbf{U}_{V,i} \mathbf{\Sigma}_{V,i} \mathbf{Z}_{V,i}^T$, one gets for the transformation matrices $\mathbf{T}_i$:

  $$\begin{aligned} \mathbf{T}_i &= \arg\max_{\mathbf{T}_i \in O(q)} \operatorname{tr}\left(\mathbf{T}_i^T \mathbf{U}_{V,i} \mathbf{\Sigma}_{V,i} \mathbf{Z}_{V,i}^T\right) \\ &= \arg\max_{\mathbf{T}_i \in O(q)} \operatorname{tr}\left(\mathbf{Z}_{V,i}^T \mathbf{T}_i^T \mathbf{U}_{V,i} \mathbf{\Sigma}_{V,i}\right), \end{aligned} \tag{13}$$

  which can be solved by the choice:

  $$\mathbf{T}_i = \mathbf{U}_{V,i} \mathbf{Z}_{V,i}^T. \tag{14}$$

  The matrices $\mathbf{T}_i$ describe permutations and rotations of the bases $\mathbf{V}_i$ in the corresponding right subspace $\mathcal{V}_i$. As distortions cannot be captured, the matrix interpolation typically gets less accurate if $\mathbf{V}_i$ are not orthogonal. Additionally, the MACs are generally not optimal. An advantage is that the matrices $\mathbf{T}_i$ are always orthogonal and hence well-defined.

Note that both alternatives are identical if $\mathbf{V}_i$ and $\mathbf{R}_V$ span the same subspace. In that case the solutions are simply $\mathbf{T}_i = \mathbf{V}_i^T \mathbf{R}_V$.

D. Adjustment of the left ROBs

Besides the adaption of the right ROBs, the left ROBs have to be adjusted with respect to a reference subspace spanned by $\mathbf{R}_W$ for a meaningful interpolation of the system matrices of the reduced systems $G_{r,i}$. For the adjustment of the left ROBs the concept of duality between the left and right subspace is exploited.

Duality between systems is a well-known notion in control theory [15]. For any system $G_{r,i}$ there is a dual or adjoint system $G_{r,i}^T$ with state vector $\boldsymbol{\xi}_{r,i}$ which is:

$$G_{r,i}^T : \begin{cases} \mathbf{E}_{r,i}^T \dot{\boldsymbol{\xi}}_{r,i} = \mathbf{A}_{r,i}^T \boldsymbol{\xi}_{r,i} + \mathbf{C}_{r,i}^T \mathbf{y}_{r,i}, \\ \mathbf{u} = \mathbf{B}_{r,i}^T \boldsymbol{\xi}_{r,i}. \end{cases} \tag{15}$$

The right subspaces of the dual reduced systems are spanned by the columns of $\mathbf{W}_i$. Therefore, the adjustment of the left ROBs of the systems $G_{r,i}$ is equivalent to the adaption of the right ROBs of the adjoint systems $G_{r,i}^T$, which is done in analogy to the explications for the right ROBs of the systems $G_{r,i}$ in section III-C.

[2] $O(q)$ denotes the orthogonal group of square matrices of order q.

1) Generalized coordinates of the dual systems: The states of the dual systems $G_{r,i}^T$ lie in different subspaces spanned by $\mathbf{W}_i$. A set of generalized coordinates for the dual reduced systems is introduced by the state transformations $\boldsymbol{\xi}_{r,i} = \mathbf{M}_i \tilde{\boldsymbol{\xi}}_{r,i}$.

Considering Proposition 1, the state transformations with matrices $\mathbf{M}_i$ can again be interpreted as changes of basis of the right subspaces of the adjoint reduced models. The new bases of their right subspaces are:

$$\tilde{\mathbf{W}}_i = \mathbf{W}_i \mathbf{M}_i.$$

In order to calculate proper matrices $\mathbf{M}_i$, the notion of duality is exploited. In analogy to section III-C the concept of maximizing the MACs between the corresponding vectors of $\tilde{\mathbf{W}}_i$ and $\mathbf{R}_W$ and minimizing the MACs between the non-corresponding vectors is employed. There are again two possible approaches depending on the fulfillment of the MACs in a strong or weak manner. Due to duality the solutions are given below. For the respective advantages and disadvantages the reader is referred to section III-C:

- Strong fulfillment of the MACs:
  As the MACs between the corresponding vectors of $\tilde{\mathbf{W}}_i$ and $\mathbf{R}_W$ are set to the maximal value 1 and the MACs of the non-corresponding vectors are set to the minimal value 0, one can again set $\mathbf{M}_i^T \mathbf{W}_i^T \mathbf{R}_W \overset{!}{=} \mathbf{I}$ and get:
  $$\mathbf{M}_i = (\mathbf{R}_W^T \mathbf{W}_i)^{-1}. \tag{16}$$
  This result is confirmed by [7], where this choice of $\mathbf{M}_i$ was explicitly deduced by a projection-based approach and without the notion of duality.
- Weak fulfillment of the MACs:
  After performing the SVD of $\mathbf{W}_i^T \mathbf{R}_W = \mathbf{U}_{W,i} \boldsymbol{\Sigma}_{W,i} \mathbf{Z}_{W,i}^T$, one gets for the transformation matrices $\mathbf{M}_i$:
  $$\begin{aligned} \mathbf{M}_i &= \arg \max_{\mathbf{M}_i \in O(q)} \operatorname{tr}\left(\mathbf{M}_i^T \mathbf{W}_i^T \mathbf{R}_W\right) \\ &= \mathbf{U}_{W,i} \mathbf{Z}_{W,i}^T. \end{aligned} \tag{17}$$

The changes of basis of the left subspaces of the reduced systems $G_{r,i}$ by the matrices $\mathbf{M}_i$ lead - based on equations (4) - to the following systems $\tilde{G}_{r,i}$:

$$\tilde{G}_{r,i} : \begin{cases} \tilde{\mathbf{E}}_{r,i} \dot{\tilde{\mathbf{x}}}_{r,i} = \tilde{\mathbf{A}}_{r,i} \tilde{\mathbf{x}}_{r,i} + \tilde{\mathbf{B}}_{r,i} \mathbf{u}, \\ \mathbf{y}_{r,i} = \tilde{\mathbf{C}}_{r,i} \tilde{\mathbf{x}}_{r,i}, \end{cases} \tag{18}$$

where

$$\begin{aligned} \tilde{\mathbf{E}}_{r,i} &= \mathbf{M}_i^T \mathbf{E}_{r,i} \mathbf{T}_i, \\ \tilde{\mathbf{A}}_{r,i} &= \mathbf{M}_i^T \mathbf{A}_{r,i} \mathbf{T}_i, \\ \tilde{\mathbf{B}}_{r,i} &= \mathbf{M}_i^T \mathbf{B}_{r,i}, \\ \tilde{\mathbf{C}}_{r,i} &= \mathbf{C}_{r,i} \mathbf{T}_i. \end{aligned} \tag{19}$$

Therefore, the adjustment of the left ROBs of the reduced systems $G_{r,i}$ is obtained by multiplying the system matrices (4) from the left with the matrices $\mathbf{M}_i^T$. The multiplication leaves the transfer behavior of the local systems unchanged.

To sum up, the matrices $\mathbf{T}_i$ linearly combine the columns and the matrices $\mathbf{M}_i^T$ linearly combine the rows of the system matrices (19) in such a way that the correct coefficients of the matrices are interpolated.

Note that the adjustment of the left ROBs is even necessary for original systems G_i which are not in descriptor form, i.e. $\mathbf{E}_i = \mathbf{I}$, as the reduced order systems $G_{r,i}$ are in descriptor form with $\mathbf{E}_{r,i} = \mathbf{W}_i^T \mathbf{V}_i$.

2) Reference subspace for the dual systems: Due to duality of the left and right subspaces, the reference subspace spanned by $\mathbf{R}_W$ can be calculated in the same way as $\mathbf{R}_V$ just by replacing $\mathbf{V}_i$ by $\mathbf{W}_i$, see section III-C.1. But other choices are also possible. The special choices of the existing approaches in the literature are presented below:

- The systems $\tilde{G}_{r,i}$ have compatible state vectors with regard to $\mathbf{R}_V$ and are described in the basis $\mathbf{R}_W$ [7]. Therefore, it was suggested to set the reference subspace for adjusting the left ROBs to the reference subspace for adjusting the right ROBs, i.e. $\mathbf{R}_W = \mathbf{R}_V$.
- In [11], [12] the adjustment of the left subspaces is not mentioned. Instead, the systems (4) are multiplied from the left with $\mathbf{E}_{r,i}^{-1}$. However, this is equivalent to the following choice of $\mathbf{M}_i$:
  $$\mathbf{M}_i = \mathbf{E}_{r,i}^{-T} = (\mathbf{V}_i^T \mathbf{E}_i^T \mathbf{W}_i)^{-1}. \tag{20}$$
  Therefore, the multiplication with $\mathbf{E}_{r,i}^{-1}$ implicitly adopts the approach from (16) by choosing for every system $\tilde{G}_{r,i}$ its own reference subspace spanned by the columns of $\mathbf{R}_W = \mathbf{E}_i \mathbf{V}_i$ instead of a common subspace for all systems. Furthermore, the systems are multiplied from the left with $\mathbf{T}_i^{-1}$ in order to get reduced systems $\tilde{G}_{r,i}$ in non-descriptor form, i.e. $\tilde{\mathbf{E}}_{r,i} = \mathbf{I}$. Hence, this approach requires less storage space as the matrices $\tilde{\mathbf{E}}_{r,i}$ are the identity matrix and do not need to be stored for the interpolation process.

E. Choice of the interpolation manifold

The system matrices of $\tilde{G}_{r,i}$ are elements of a Riemannian manifold $\mathcal{M}$. Therefore, the interpolated matrices should as well be elements of the respective manifold. The concept of matrix manifolds was introduced for pMOR by matrix interpolation in [9], [10], [11], [12]. For detailed information on the application of matrix manifolds for pMOR please refer to these references.

Let $\mathbf{X}_{r,i} \in \mathcal{M}$ be an element of a matrix manifold, where $\mathbf{X}_{r,i}$ stands for a precomputed system matrix from (19), e.g. $\tilde{\mathbf{E}}_{r,i}$. After choosing a reference system i_0, the tangent space is constructed in $\mathbf{X}_{r,i_0}$. The tangent space is a linear space and exists as Riemannian manifolds are differentiable manifolds. The matrices $\mathbf{X}_{r,i}$ of the other local systems are mapped into the tangent space by the so called logarithmic mapping. The interpolation of the matrices takes place in the tangent space and the interpolated matrix is mapped back to the original space by the exponential mapping resulting in the matrix $\mathbf{X}_{r,\mathrm{int}}$.

The interpolation formulae for real and regular matrix manifolds are given below with weighting functions $\omega_i(\mathbf{p})$:

- Interpolation on manifold of real matrices:

$$\mathbf{X}_{r,\text{int}} = \sum_{i=1}^{k} \omega_i(\mathbf{p})(\mathbf{X}_{r,i} - \mathbf{X}_{r,i_0}) + \mathbf{X}_{r,i_0}. \quad (21)$$

  This is in a way the generalization of the scalar weighted arithmetic mean for matrix interpolation using the Euclidean metric.
- Interpolation on manifold of regular matrices:

$$\mathbf{X}_{r,\text{int}} = \exp^{\sum_{i=1}^{k} \omega_i(\mathbf{p}) \log(\mathbf{X}_{r,i}\mathbf{X}_{r,i_0}^{-1})} \mathbf{X}_{r,i_0}. \quad (22)$$

  This can in a way be seen as the generalization of the scalar weighted geometric mean for matrix interpolation using Euclidean metric in the logarithmic space. This interpretation was pointed out for the case of the interpolation on the manifold of symmetric positive definite matrices, where it was shown that the determinant of the interpolated matrix is the scalar geometric mean of the determinants of the considered matrices, see e.g. [16].

In this paper and in [5], [7] and [10], [11], [12] the matrix manifold for $\tilde{\mathbf{B}}_{r,i}$ is the manifold of real matrices $\mathbb{R}^{q\times m}$ and for $\tilde{\mathbf{C}}_{r,i}$ the manifold of real matrices $\mathbb{R}^{p\times q}$.

For regular matrices $\tilde{\mathbf{E}}_{r,i}$ and $\tilde{\mathbf{A}}_{r,i}$ the manifold of real or regular matrices are possible. In [11] always the manifold of regular matrices is used. The interpolation method in [7] was presented without the notion of matrix manifolds. In the context of interpolating matrices on manifolds, this approach always employs the manifold of real matrices $\mathbb{R}^{q\times q}$ under the assumption $\sum_{i=1}^{k} \omega_i(\mathbf{p}) = 1$. In [12] it is noted that the appropriate manifold is problem dependent. Besides, an a priori heuristic criterion based on linear regression is proposed for finding the most appropriate manifold. This criterion can be used for spline interpolation. We want to point out that the choice of the interpolation manifold is equivalent in the scalar case to the choice of the proper mean, either arithmetic or geometric mean.

Note that interpolating on the manifold of regular matrices is not always possible. If $\mathbf{X}_{r,i}$ is not in the neighborhood of $\mathbf{X}_{r,i_0}$ and hence the matrix $\mathbf{X}_{r,i}\mathbf{X}_{r,i_0}^{-1}$ has negative eigenvalues, then $\log(\mathbf{X}_{r,i}\mathbf{X}_{r,i_0}^{-1})$ may not be unique and delivers imaginary matrices. This follows from the theorem below:

Theorem 1 ([17]): A matrix has a unique and real logarithm if and only if it is non-singular and each Jordan block belonging to a negative eigenvalue occurs an even number of times.

As a side note we want to present a group of matrices the interpolation on the manifold of regular matrices can always be applied to. Consider reduced systems whose matrices $\tilde{\mathbf{A}}_{r,i}$ have a symmetric negative definite part, i.e. $\tilde{\mathbf{A}}_{r,i} + \tilde{\mathbf{A}}_{r,i}^T < \mathbf{0}$. Such matrices arise in so called contractive systems [8].

It follows from Theorem 1 that matrices with complex and positive eigenvalues have a unique and real logarithm. Then the following corollary holds:

Corollary 1: If $\tilde{\mathbf{A}}_{r,i} + \tilde{\mathbf{A}}_{r,i}^T < \mathbf{0}$, then the product $\tilde{\mathbf{A}}_{r,i}\tilde{\mathbf{A}}_{r,i_0}^{-1}$ only has complex and positive eigenvalues and according to Theorem 1, has a unique and real logarithm.

Proof: It shall be investigated if the real eigenvalues λ_j to the real eigenvectors $\mathbf{v}_j$ of $\tilde{\mathbf{A}}_{r,i}\tilde{\mathbf{A}}_{r,i_0}^{-1}$ can be negative.

$$(\tilde{\mathbf{A}}_{r,i}\tilde{\mathbf{A}}_{r,i_0}^{-1})\mathbf{v}_j = \lambda_j\mathbf{v}_j \Rightarrow \tilde{\mathbf{A}}_{r,i_0}^{-1}\mathbf{v}_j = \lambda_j\tilde{\mathbf{A}}_{r,i}^{-1}\mathbf{v}_j$$
$$\Rightarrow \underbrace{\mathbf{v}_j^T\tilde{\mathbf{A}}_{r,i_0}^{-1}\mathbf{v}_j}_{<0} = \lambda_j \underbrace{\mathbf{v}_j^T\tilde{\mathbf{A}}_{r,i}^{-1}\mathbf{v}_j}_{<0} \Rightarrow \lambda_j > 0 \;\forall\; j = 1...q.$$

The eigenvalues of $\tilde{\mathbf{A}}_{r,i}\tilde{\mathbf{A}}_{r,i_0}^{-1}$ can only be complex or positive. Therefore, the logarithm is unique and real. ■

F. Choice of the interpolation method

The interpolation formulae for the system matrices (21) and (22) require a proper choice of the weighting functions $\omega_i(\mathbf{p})$. Alternatively, the matrices can be interpolated element-wise in the chosen manifolds by any preferred interpolation method like linear or spline interpolation.

It is noted that the matrices (19) are precomputed and stored if the reference subspaces are calculated by the fixed order non-weighted SVD version. Then the interpolation is done online and the method is real-time capable.

IV. INTEGRATION OF EXISTING METHODS

The two approaches of pMOR by matrix interpolation [5], [7] and [10], [11], [12] were developed independently and presented as different methods. In fact, they are special cases of the same general framework proposed in this paper and differ in the selection of the degrees of freedom.

TABLE I
INTEGRATION OF EXISTING METHODS INTO THE FRAMEWORK.

	Panzer et al. [7]	Amsallem et al. [11]
Ref. subspace $\mathbf{R}_V$	non-/weighted SVD $*$	fixed
Gen. coordinates	$\mathbf{T}_i = (\mathbf{R}_V^T\mathbf{V}_i)^{-1} *$	$\mathbf{T}_i = \mathbf{U}_{V,i}\mathbf{Z}_{V,i}^T$
Ref. subspace $\mathbf{R}_W$	$\mathbf{R}_W = \mathbf{R}_V *$	$\mathbf{R}_W = \mathbf{E}_i\mathbf{V}_i$
Gen. co. dual sys.	$\mathbf{M}_i = (\mathbf{R}_W^T\mathbf{W}_i)^{-1} *$	$\mathbf{M}_i = (\mathbf{R}_W^T\mathbf{W}_i)^{-1}\mathbf{T}_i$
Interpol. manifold	real matrices	real/regular matrices $*$
Interpol. method	linear	spline $*$

Besides the two choices of the degrees of freedom, there are many other possible selections. As the most accurate selection is problem dependent, a restriction to the two special choices shown in the left and right column in Table I is not useful. Besides, the degrees of freedom can be chosen with respect to desired features like real-time ability, storage requirements or structure preservation.

V. NUMERICAL EXAMPLE

The considered example is a FE model which was generated in [6]. It describes the motion of a 3D cantilever Timoshenko beam. The parameter of the system is the length L of the beam which varies between $L = 0.4$m and $L = 2.4$m. The model input is a vertical force, which is applied at the tip of the beam, and the model output is the vertical displacement at this point.

The model has 200 nodes along the beam, each having six degrees of freedom: three translational displacements and three rotational degrees of freedom. Formulating the

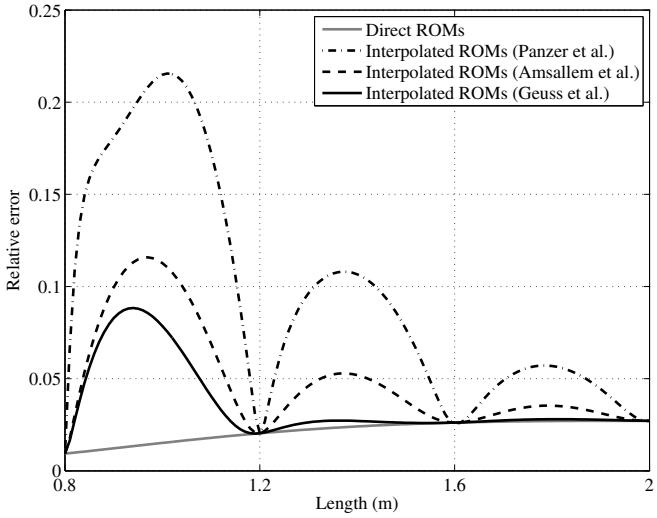

Fig. 1. Relative error in $\mathcal{H}_2$-norm of different interpolated ROMs at various lengths L for the cantilever beam.

resulting second-order system as a first-order system gives a model with $n = 2400$ degrees of freedom. Six local models uniformly distributed for lengths from $L = 0.4$m to $L = 2.4$m are computed. The local models are reduced using a two-sided Krylov subspace method with the reduced order $q = 10$ and an expansion point $s_0 = 0$.

In Fig. 1 the relative error in $\mathcal{H}_2$-norm for directly reduced order models (ROMs) and interpolated reduced order models is shown for parameters in the interval between $L = 0.8$m and $L = 2$m. In this example the degrees of freedom for the general framework are chosen as a combination of the two existing methods. The selection, with the non-weighted SVD for the calculation of $\mathbf{R}_V$, is marked in Table I with $(*)$. This selection is compared to the latest methods from Panzer et al. [7] and Amsallem et al. [11]. The selection of the degrees of freedom for Panzer et al. is given in the left column of Table I with the non-weighted SVD for the calculation of $\mathbf{R}_V$. For Amsallem et al. the degrees of freedom are chosen according to the right column of Table I.

At the parameter values $L = 0.8$m, $L = 1.2$m, $L = 1.6$m and $L = 2$m all considered methods share the relative error of the locally reduced models and between them an additional error due to the interpolation occurs. In this example the proposed selection of the degrees of freedom delivers more accurate results than the two existing approaches. However, the best selection of the degrees of freedom is problem dependent and a priori statements about the best choice are in general not possible.

VI. CONCLUSIONS

In this paper a general framework for pMOR by matrix interpolation was proposed and the degrees of freedom including respective advantages and difficulties were pointed out. For this framework the concept of duality was introduced and the analogy to the scalar case of arithmetic and geometric interpolation was highlighted.

It was shown that the existing approaches of pMOR by matrix interpolation determined a selection of the degrees of freedom of the proposed general framework. However, the general procedure offers the user a wider range of possibilities in order to apply this framework for a considered problem with respect to desired features like accuracy, real-time ability, storage requirements or structure preservation.

As pMOR by matrix interpolation makes few assumptions, it is in general not possible to make a priori statements on the best choice of the degrees of freedom concerning accuracy. Therefore, the following work has to concentrate on developing error bounds for special types of systems.

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