On parametrical expressibility in the free void-generated diagonalizable algebra*

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Let $\mathfrak{A}$ be any universal algebra. We say formula A is explicitly expressible on algebra $\mathfrak{A}$ via system of formulas Σ, if A can be obtained on $\mathfrak{A}$ of variables and formulas of Σ by means of superpositions. We say a system of formulas Σ is complete in $\mathfrak{A}$, if any formula is expressible via Σ. We say a system Σ is precomplete as to expressibility on $\mathfrak{A}$ if Σ is not complete on $\mathfrak{A}$, but for any formula F, which is not expressible via Σ on $\mathfrak{A}$, then the system $\Sigma \cup \{F\}$ is complete as to expressibility on $\mathfrak{A}$. It is known [1, 2] that there are only five precomplete as to explicite expressibility classes of boolean functions, that there are only finitely many precomplete classes of functions in any general k-valued logic [3], that there are only 12 precomplete classes of pseudo-boolean functions [4, 5], and other similar results. Note that preudo-boolean functions cannot be defined by finite tables.

At the same time we can consider other tools to get new functions of a given system of functions. We say formula A is parametrical expressible on algebra $\mathfrak{A}$ via system of formulas Σ if there exist numbers l and m, variables $\pi, \pi_1, \ldots, \pi_l$, not occuring in A, formulas $B_1, C_1, \ldots, B_m, C_m$, which are explicitly expressible on $\mathfrak{A}$ via Σ, and formulas $D_1 \ldots, D_l$ such that next relations are valid on $\mathfrak{A}$:

$$(A = \pi) \Longrightarrow (\bigwedge_{i=1}^{m})(B_i = C_i)[\pi_1/D_1] \ldots [\pi_l/D_l],$$

$$(\bigwedge_{i=1}^{m})(B_i = C_i) \Longrightarrow (A = \pi).$$

In the case of k-valued logics this definition was given by A.V. Kuznetsov as was reported in [6], and he also had proved that a two-element set has only 25 parametrically closed classes of functions, A.F. Danil'čenko [7] subsequently proved that a three-element set has only finitely many parametrically closed classes of functions, S. Burris and R. Willard [8] proved recently that a k-element set also has only finitely many parametrically closed classes of functions. The main result of this paper is that there are systems of functions, namely of functions of diagonalizable algebra, such that contains infinitely many parametrically closed classes of functions.

A diagonalizable algebra $\mathfrak{D}$ is a boolean algebra $\mathfrak{A} = (A; \&, \vee, \supset, \neg)$ with an additional operator Δ satisfying the following identies:

$$\Delta(\alpha \supset \beta) \leq \Delta\alpha \supset \Delta\beta,$$

$$\Delta\alpha \leq \Delta\Delta\alpha,$$

$$\Delta(\Delta\alpha \supset \alpha) = \Delta\alpha,$$

$$\Delta 1 = 1,$$

where 1 is the unit of $\mathfrak{A}$.

We consider the diagonalizable algebra $\mathfrak{M}^* = (M; \&, \vee, \supset, \neg, \Delta)$ of all infinite binary sequences of the type $\alpha = (\mu_1, \mu_2, \dots), \mu_i \in \{0, 1\}, i = 1, 2, \dots$. The boolean operations $\&, \vee, \supset, \neg$ over elements of M are defined component componentwise, and the operation Δ over element α we define by the equality $\Delta\alpha = (1, \nu_1, \nu_2, \dots)$, where $\nu_i = \mu_1 \& \dots \& \mu_2$. We consider then the subalgebra generated $\mathfrak{M}^*$ of the algebra $\mathfrak{M}$ which is generated by its zero element $(0, 0, \dots)$.

We prove next

THEOREM. There are infinitely many precomplete with respect to parametrical expressibility classes of functions in the free diagonalizable algebra $\mathfrak{M}^*$.

The theorem is based on the example of an infinite family of parametrically precomplete classes of formulas presented in the following.

EXAMPLE. The classes K_1, $K_2, \dots$ of formulas, which preserve on algebra $\mathfrak{M}^*$ respectively the relations $x = \neg\Delta 0$, $x = \neg\Delta^2 0, \dots,$ constitute a numerable collection of parametrically precomplete in $\mathfrak{M}^*$ classes of formulas.

It is known [6] that these classes are closed with respect to parametrical expressibility.

It is easy to check that functions $\neg\Delta^i 0$, $p\&q$, $p \vee q$ belong to the class K_i, and $\neg p$, Δp does not belong to K_i, $i = 1, 2, \dots$. So these classes are not complete as to parametrical expressibility.

Let us show that they are distimct two by two. It is clear that the function $\neg\Delta^j 0 \notin K_i$, $i \neq j$.

Let us prove that these classes are parametrically precomplete. Suppose we have an arbitrary function $F_i(p_1, \dots, p_n) \notin K_i$. It means that

$F_i(\neg\Delta^i 0, \ldots, \neg\Delta^i 0) \neq \neg\Delta^i 0$. Let us denote by $\Box p$ the function $p\&\Delta p$, and by ∇p the function $\Box\neg\Box\neg\Box p$.

Let us consider in the following two functions, denoted respectively by $F_\neg$ and F_Δ:

$$((\nabla\neg(p \sim q)\&((\neg p \sim q) \sim F_i(\neg\Delta^i 0, \ldots, \neg\Delta^i 0))) \vee (\nabla(p \sim q)\&\neg\Delta^i 0),$$

$$(\nabla q\&((\Delta p \sim q) \sim F_i(\neg\Delta^i 0, \ldots, \neg\Delta^i 0))) \vee (\neg\nabla q\&\neg\Delta^i 0).$$

It is clear that functions $F_\neg$ and F_Δ are from class K_i. Then it is not so difficult to check that next relations are valid on $\mathfrak{M}^*$:

$$(\neg p = q) \Longleftrightarrow (F_\neg(p, q) = F_i(\neg\Delta^i 0, \ldots, \neg\Delta^i 0)),$$

$$(\Delta p = q) \Longleftrightarrow (F_\Delta(p, q) = F_i(\neg\Delta^i 0, \ldots, \neg\Delta^i 0)).$$

The last thing together with previous facts means that classes $K_1, K_2, \ldots$ are parametrically precomplete on $\mathfrak{M}^*$.

The theorem is proved.

Remark. The theorem was announced for the first time at the CAIM-1999 organized by ROMAI (Romanian Society of Applied and Industrial Mathematics) at Pitești, Romania.

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