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ON PARTIAL BILATERAL AND IMPROPER PARTIAL BILATERAL GENERATING FUNCTIONS INVOLVING SOME SPECIAL FUNCTIONS

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Abstract

A group-theoretic method of obtaining more general class of generating functions from a given class of improper partial bilateral generating functions involving Hermite. Laguerre and Gegenbauer polynomials are discussed.

1 Introduction

The usual generating relation involving one special function may be called linear or unilateral generating relation. By the term usual (proper) bilateral generating function we mean a function $G(x, z, w)$ which can be expanded in powers of w in the following form

$$G(x, z, w) = \sum_{n=0}^{\infty} a_n f_n(x) g_n(z) w^n$$

where a_n is arbitrary that is independent of x and z and $f_n(x),\ g_n(z)$ are two different special functions.

In particular, when two special functions are same that is $f_n \equiv g_n$, we call the generating relation as bilinear generating relation.

Unlike the usual (proper) bilateral or bilinear generating relations [5], we shall introduce the concepts of usual (proper) partial bilateral generating relation and improper partial bilateral generating relation.

Definition 1.1. By the term usual (proper) partial bilateral generating relation for two classical polynomials, we mean the relation :

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$$G(x,z,w)=\sum_{n=0}^{\infty}a_nw^np_{m+n}^{(\alpha)}(x)q_{m+n}^{(\beta)}(z) \tag{1.1.1}$$

where the coefficients a_n's are quite arbitrary and $p_{m+n}^{(\alpha)}(x), q_{m+n}^{(\beta)}(z)$ are any two classical polynomials of order $(m+n)$ and of parameters α and β respectively.

Definition 1.2. By the term improper partial bilateral generating relation for two classical polynomials, we mean the relation :

$$G(x,z,w)=\sum_{n=0}^{\infty}a_nw^np_{m+n}^{(\alpha)}(x)q_{k+n}^{(\beta)}(z) \tag{1.2.1}$$

where the coefficients a_n's are quite arbitrary and $p_{m+n}^{(\alpha)}(x), q_{k+n}^{(\beta)}(z)$ are any two classical polynomials of order $(m+n),(k+n)$ and of parameters α,β respectively.

The object of this paper is to establish some general class of generating functions from a given class of improper partial bilateral generating functions.

2 Main Results

a) For improper partial bilateral generating functions :

Theorem 2.1 : *If there exist the following class of improper partial bilateral generating functions for the Hermite and Laguerre polynomials by means of the relation*

$$G(x,z,w)=\sum_{n=0}^{\infty}a_nw^nH_{m+n}(x)L_{k+n}^{(\alpha)}(z) \tag{2.1.1}$$

where a_n is arbitrary, then the following general class of generating functions hold :

$$\exp(2wx-w^2)(1-v)^{-(\alpha+k+1)}\exp\left(-\frac{\nu z}{1-\nu}\right)G\left(x-w,\frac{z}{1-\nu},\frac{w\nu}{1-\nu}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}a_nw^{n+s}v^{n+r}\frac{(k+n+1)_r}{s!\ r!}H_{m+n+s}(x)L_{k+n+r}^{(\alpha)}(z) \text{ where } |v|<1$$

Proof : Multiplying both sides of (2.1.1) by y^mt^k, we get

$$y^mt^kG(x,z,w)=\sum_{n=0}^{\infty}a_nw^n(H_{m+n}(x)y^m)(L_{k+n}^{(\alpha)}(z)t^k) \tag{2.1.2}$$

Now replacing 'w' by '$wvyt$' in (2.1.2) we get

$$y^m t^k G(x,z,wvyt) = \sum_{n=0}^{\infty} a_n (wv)^n \left(H_{m+n}(x) y^{m+n}\right) \left(L^{(\alpha)}{}_{k+n}(z) t^{k+n}\right) \tag{2.1.3}$$

We now choose the following two operators R_1 and R_2 of one-parameter groups ([1], [2]) namely

$$R_1 = 2xy - y\frac{\partial}{\partial x} \text{ and } R_2 = zt\frac{\partial}{\partial z} + t^2\frac{\partial}{\partial t} + (\alpha+1-z)t$$

so that

$$\begin{aligned} R_1\left[H_{m+n}(x)y^{m+n}\right] &= H_{m+n+1}(x)y^{m+n+1} \\ R_2\left[L^{(\alpha)}_{k+n}(z)t^{k+n}\right] &= (k+n+1)L^{(\alpha)}_{k+n+1}(z)t^{k+n+1} \end{aligned}$$

and ([3], [4])

$$\begin{aligned} \exp(wR_1)f(x,y) &= \exp(2wxy - w^2y^2)f(x-wy,y) \\ \exp(vR_2)f(z,t) &= (1-vt)^{-\alpha-1}\exp\left(-\frac{vzt}{1-vt}\right) f\left(\frac{z}{1-vt}, \frac{t}{1-vt}\right). \end{aligned}$$

where $|vt| < 1$

We now operate both sides of (2.1.3) by $\exp(wR_1)\exp(vR_2)$ and as a result of it, the relation (2.1.3) becomes

$$\begin{aligned} &\exp(2wxy - w^2y^2)(1-vt)^{-\alpha-1}\exp\left(-\frac{vzt}{1-vt}\right) y^m \left(\frac{t}{1-vt}\right)^k \times \\ &\qquad \times G\left(x-wy, \frac{z}{1-vt}, \frac{wvyt}{1-vt}\right) \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n (wv)^n \left(\frac{(wR_1)^s}{s!}\left(H_{m+n}(x)y^{m+n}\right)\right) \times \\ &\quad \times \left(\tfrac{(vR_2)^r}{r!}\left(L^{(\alpha)}_{k+n}(z)t^{k+n}\right)\right) \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n \frac{w^{n+s}v^{n+r}}{s!\, r!}(k+n+1)_r \left(H_{m+n+s}(x)y^{m+n+s}\right. \times \\ &\quad \times (L^{(\alpha)}_{k+n+r}(z)t^{k+n+r}) \end{aligned}$$

Now putting $y = t = 1$ in the above relation, we get

$$\exp(2wx - w^2)(1-v)^{-(\alpha+k+1)}\exp\left(-\frac{vz}{1-v}\right) G\left(x-w,\ \frac{z}{1-v}, \frac{wv}{1-v}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n \frac{w^{n+s}v^{n+r}}{s!\ r!}(k+n+1)_r\ H_{m+n+s}(x)L_{k+n+r}^{(\alpha)}(z),$$

where

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) L_{k+n}^{(\alpha)}(z) \text{ and } |v| < 1.$$

Theorem 2.2 *If there exist the following class of improper partial bilateral generating functions for Hermite and Gegenbauer polynomials by means of the relation*

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) C_{k+n}^{(\alpha)}(z) \tag{2.2.1}$$

where a_n is arbitrary, then the following general class of generating functions hold :

$$\exp(2wx-w^2)(1-2vz+v^2)^{-\alpha-\frac{k}{2}} G\left(x-w,\ \frac{z-v}{\sqrt{1-2vz+v^2}}, \frac{wv}{\sqrt{1-2vz+v^2}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n \frac{w^{n+s}v^{n+r}}{s!\ r!}(k+n+1)_r\ H_{m+n+s}(x)C_{k+n+r}^{(\alpha)}(z),$$

where $|2vz - v^2| < 1$.

Proof : Multiplying both sides of (2.2.1) by $y^m t^k$, we get :-

$$y^m t^k G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n (H_{m+n}(x) y^m)(C_{k+n}^{(\alpha)}(z) t^k) \tag{2.2.2}$$

Now replacing 'w' by '$wvyt$' in (2.2.2), we get

$$y^m t^k G(x,z,wvyt) = \sum_{n=0}^{\infty} a_n (wv)^n (H_{m+n}(x) y^{m+n})(C_{k+n}^{(\alpha)}(z) t^{k+n}) \tag{2.2.3}$$

We now choose the following two operators R_1 and R_2 of one-parameter groups ([1], [2]) namely

$$R_1 = 2xy - y\frac{\partial}{\partial x} \text{ and } R_2 = (z^2-1)t\frac{\partial}{\partial z} + zt^2\frac{\partial}{\partial t} + (2\alpha+k)zt$$

so that

$$\begin{aligned} R_1[H_{m+n}(x)y^{m+n}] &= H_{m+n+1}(x)y^{m+n+1} \\ R_2[C_{k+n}^{(\alpha)}(z)t^{k+n}] &= (k+n+1)C_{k+n+1}^{(\alpha)}(z)t^{k+n+1} \end{aligned}$$

and ([3], [4])

$$\begin{aligned} \exp(wR_1)f(x,y) &= \exp(2wxy - w^2y^2)f(x-wy,y) \\ \exp(vR_2)f(z,t) &= (1-2vzt+v^2t^2)^{-\alpha} f\left(\tfrac{z-vt}{\sqrt{1-2vzt+v^2t^2}}, \tfrac{t}{\sqrt{1-2vzt+v^2t^2}}\right). \end{aligned}$$

where $|2vzt - v^2t^2| < 1$

We now operate both sides of (2.2.3) by $\exp(wR_1)\exp(vR_2)$ and as a result of it, the relation (2.2.3) becomes

$$\exp(2wxy - w^2y^2)(1-2vzt+v^2t^2)^{-\alpha} y^m \left(\frac{t}{\sqrt{1-2vzt+v^2t^2}}\right)^k$$

$$G\left(x-wy, \frac{z-vt}{\sqrt{1-2vzt+v^2t^2}}, \frac{wvyt}{\sqrt{1-2zvt+v^2t^2}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n (wv)^n \left(\frac{(wR_1)^s}{s!} H_{m+n}(x) y^{m+n}\right)\left(\frac{(vR_2)^r}{r!} C_{k+n}^{(\alpha)}(z) t^{k+n}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n \frac{w^{n+s}v^{n+r}}{s!\ r!}(k+n+1)_r \left(H_{m+n+s}(x) y^{m+n+s}\right) \times$$

$$\times \left(C_{k+n+r}^{(\alpha)}(z) t^{k+n+r}\right)$$

Now putting $y = t = 1$ in the above relation, we get

$$\exp(2wx - w^2)(1-2vz+v^2)^{-\alpha-\frac{1}{2}} G\left(x-w, \frac{z-v}{\sqrt{1-2vz+v^2}}, \frac{wv}{\sqrt{1-2vz+v^2}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n \frac{w^{n+s}v^{n+r}}{s!\ r!}(k+n+1)_r H_{m+n+s}(x) C_{k+n+r}^{(\alpha)}(z),$$

where

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) C_{k+n}^{(\alpha)}(z) \text{ and } |2vz - v^2| < 1.$$

Theorem 2.3. *If there exist the following class of improper partial bilateral generating functions for Leguerre and Gegenbauer polynomials by means of the relation*

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n L_{m+n}^{(\alpha)}(x) C_{k+n}^{(\beta)}(z), \qquad (2.3.1)$$

where a_n is arbitrary, then the following general class of generating functions hold :

$$(1-w)^{-\alpha-m-1}(1-2vz+v^2)^{-\beta-\frac{k}{2}} \exp\left(-\frac{wx}{1-w}\right) \times$$

$$\times G\left(\frac{x}{1-w}, \frac{z-v}{\sqrt{1-2vz+v^2}}, \frac{wv}{(1-w)\sqrt{1-2vz+v^2}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n w^{n+s} v^{n+r} \frac{(m+n+1)_s}{s!} \frac{(m+n+1)_r}{r!} L_{m+n+s}^{(\alpha)}(x) C_{k+n+r}^{(\beta)}(z),$$

where $|2vz - v^2| < 1$

Proof : Multiplying both sides of (2.3.1) by $y^m t^k$, we get

$$y^m t^k G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n \left(L_{m+n}^{(\alpha)}(x) y^m\right)\left(C_{k+n}^{(\beta)}(z) t^k\right) \quad (2.3.2)$$

Now replacing 'w' by '$wvyt$' in (2.3.2), we get

$$y^m t^k G(x,z,wvyt) = \sum_{n=0}^{\infty} a_n (wv)^n \left(L_{m+n}^{(\alpha)}(x) y^{m+n}\right)\left(C_{k+n}^{(\beta)}(z) t^{k+n}\right) \quad (2.3.3)$$

We now choose the following two operators R_1 and R_2 of one-parameter groups ([1], [2]) namely

$$R_1 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (\alpha+1-x)y \text{ and } R_2 = (z^2-1)t\frac{\partial}{\partial z} + zt^2\frac{\partial}{\partial t} + (2\beta+k)zt.$$

so that

$$R_1\left[L_{m+n}^{(\alpha)}(x) y^{m+n}\right] = (m+n+1) L_{m+n+1}^{(\alpha)}(x) y^{m+n+1}$$

$$R_2\left[C_{k+n}^{(\beta)}(z) t^{k+n}\right] = (k+n+1) C_{k+n+1}^{(\beta)}(z) t^{k+n+1}$$

and ([3], [4])

$$\exp(wR_1) f(x,y) = (1-wy)^{-\alpha-1} \exp\left(-\frac{wxy}{1-wy}\right) f\left(\frac{x}{1-wy}, \frac{y}{1-wy}\right)$$

$$\exp(vR_2) f(z,t) = (1-2vzt+v^2t^2)^{-\beta} f\left(\frac{z-vt}{\sqrt{1-2vzt+v^2t^2}}, \frac{t}{\sqrt{1-2vzt+v^2t^2}}\right).$$

where $|2vzt - v^2t^2| < 1$

We now operate both sides of (2.3.3) by $\exp(wR_1)\exp(vR_2)$ and as a result of it, the relation (2.3.3) reduces to

$$(1-wy)^{-\alpha-1}(1-2vzt+v^2t^2)^{-\beta}\times$$

$$\times \exp\left(-\frac{wxy}{1-wy}\right)\left(\frac{y}{1-wy}\right)^m \left(\frac{t}{\sqrt{1-2vzt+v^2t^2}}\right)^k \times$$

$$\times G\left(\frac{x}{1-wy}, \frac{z-vt}{\sqrt{1-2vzt+v^2t^2}}, \frac{wvyt}{(1-wy)\sqrt{1-2vzt+v^2t^2}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n (wv)^n \left(\frac{(wR_1)^s}{s!} L_{m+n}^{(\alpha)}(x) y^{m+n}\right)\left(\frac{(vR_2)^r}{r!} C_{k+n}^{(\beta)}(z) t^{k+n}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n w^{n+s} v^{n+r} \frac{(m+n+1)_s}{s!}\frac{(k+n+1)_r}{r!} \times$$

$$\times L_{m+n+s}^{(\alpha)}(x) y^{m+n+s} C_{k+n+r}^{(\beta)} z) t^{k+n+r}$$

Now putting $y = t = 1$ in the above relation, we get

$$(1-w)^{-\alpha-m-1}(1-2vz+v^2)^{-\beta-\frac{k}{2}} \exp\left(-\frac{wx}{1-w}\right) \times$$

$$\times G\left(\frac{x}{1-w}, \frac{z-v}{\sqrt{1-2vz+v^2}}, \frac{wv}{(1-w)\sqrt{1-2vz+v^2}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n w^{n+s} v^{n+r} \frac{(m+n+1)_s}{s!}\frac{(k+n+1)_r}{r!} L_{m+n+s}^{(\alpha)}(x) C_{k+n+r}^{(\beta)}(z),$$

where

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n L_{m+n}^{(\alpha)}(x) C_{k+n}^{(\beta)}(z) \text{ and } |2vz - v^2| < 1.$$

Particular Cases : It may be of interest to point out that for $k = m$, the above Theorems 2.1, 2.2 & 2.3 become nice general class of generating functions from the given class of usual (proper) partial bilateral generating functions, which need not be derived independently. We state those resutls in the following form :

b) For proper partial bilateral generating functions

Theorem 2'.1 *If there exist the following class of (proper) partial bilateral generating functions for Hermite and Leguerre polynomials by means of the relation*

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) L_{m+n}^{(\alpha)}(z), \tag{2'.1.1}$$

where a_n is arbitrary, then the following general class of generating functions hold :

$$\exp(2wx - w^2)(1-v)^{-(\alpha+m+1)} \exp\left(-\frac{vz}{1-v}\right) G\left(x-w, \frac{z}{1-v}, \frac{wv}{1-v}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n w^{n+s} \frac{(m+n+1)_r}{s!\ r!} H_{m+n+s}(x) L_{m+n+r}^{(\alpha)}(z), \text{ where } |v|<1.$$

Theorem 2'.2 *If there exist the following class of (proper) partial bilateral generating function for Hermite and Gengenbauer polynomials by means of the relation*

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) C_{m+n}^{(\alpha)}(z), \qquad (2'2.1)$$

where a_n is arbitrary, then the following general class of generating functions hold :

$$\exp(2wx-w^2)(1-2vz+v^2)^{-\alpha-\frac{m}{2}} G\left(x-w, \frac{z-v}{\sqrt{1-2vz+v^2}}, \frac{wv}{\sqrt{1-2vz+v^2}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n w^{n+s} v^{n+r} \frac{(m+n+1)_r}{s!\ r!} H_{m+n+s}(x) C_{m+n+r}^{(\alpha)}(z),$$

where $|2vz-v^2|<1$.

Theorem 2'.3 *If there exist the following class of (proper) partial bilateral generating functions for Laguerre and Gegenbauer polynomials by means of the relation*

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n L_{m+n}^{(\alpha)}(x) C_{m+n}^{(\beta)}(z), \qquad (2'.3.1)$$

where a_n is arbitrary, then the following general class of generating functions hold :

$$(1-w)^{-\alpha-m-1}(1-2vz+v^2)^{-\beta-\frac{m}{2}} \exp\left(-\frac{wx}{1-w}\right) \times$$

$$\times G\left(\frac{x}{1-w}, \frac{z-v}{\sqrt{1-2vz+v^2}}, \frac{wv}{(1-w)\sqrt{1-2vz+v^2}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n w^{n+s} v^{n+r} \frac{(m+n+1)_s}{s!} \frac{(m+n+1)_r}{r!} L_{m+n+s}^{(\alpha)}(x), C_{m+n+r}^{(\beta)}(z),$$

where $|2vz-v^2|<1$.

Remark : In a similar manner some new results can also be derived for (proper) bilinear as well as improper partial bilinear generating functions.

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