Commun. Korean Math. Soc. **27** (2012), No. 3, pp. 477–482
http://dx.doi.org/10.4134/CKMS.2012.27.3.477

ON PARTITION CONGRUENCES FOR OVERCUBIC PARTITION PAIRS

Byungchan Kim

Abstract. In this note, we investigate partition congruences for a certain type of partition function, which is named as the overcubic partition pairs in light of the literature. Let $\overline{cp}(n)$ be the number of overcubic partition pairs. Then we will prove the following congruences:

$$\overline{cp}(8n+7) \equiv 0 \pmod{64} \quad \text{and} \quad \overline{cp}(9m+3) \equiv 0 \pmod{3}.$$

1. Introduction

In a paper [3], H.-C. Chan initiated the study of the cubic partitions by showing a close relation between a certain type of partition function and Ramanujan's cubic continued fraction. Cubic partition function $c(n)$ is defined by

$$\sum_{n=0}^{\infty} c(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}.$$

Here and in the sequel, we will use the following standard q-series notation:

$$(a;q)_\infty := \prod_{n=1}^{\infty} (1-aq^{n-1}), \ |q|<1.$$

Motivated by his works ([3, 4, 5]), many partition congruences for analogous partition functions have been investigated. In particular, the author studied its overpartition analog [7] in which the overcubic partition function $\overline{c}(n)$ was defined by

$$\sum_{n=0}^{\infty} \overline{c}(n)q^n = \frac{(-q;q)_\infty (-q^2;q^2)_\infty}{(q;q)_\infty (q^2;q^2)_\infty}. \tag{1.1}$$

Received April 26, 2011.
2010 *Mathematics Subject Classification.* 11P83.
Key words and phrases. partition, cubic partition, overcubic partition pair.

More recently, H. Zhao and Z. Zhong [13] investigated congruences for the following partition function:

$$\sum_{n=0}^{\infty} cp(n)q^n = \frac{1}{(q;q)_\infty^2 (q^2;q^2)_\infty^2}. \tag{1.2}$$

Since $cp(n)$ counts a pair of cubic partitions, we will call $cp(n)$ the number of cubic partition pairs. We can interpret $cp(n)$ as the number of 4-color partitions of n with colors r, y, o, and b subject to the restriction that the colors o and b appear only in even parts.

In this article, we will investigate congruence properties of the following partition function $\overline{cp}(n)$, which is named as the number of overcubic partition pairs in light of (1.1) and (1.2),

$$\sum_{n=0}^{\infty} \overline{cp}(n)q^n = \frac{(-q;q)_\infty^2 (-q^2;q^2)_\infty^2}{(q;q)_\infty^2 (q^2;q^2)_\infty^2}. \tag{1.3}$$

We can interpret $\overline{cp}(n)$ as the number of 4-color partitions of n with colors r, y, o, and b subject to the restriction that the colors o and b appear only in even parts in which we may overline the first occurrence parts.

First, we will investigate congruences for $\overline{cp}(n)$ modulo 64 by using arithmetic properties of quadratic forms.

Theorem 1. *For all nonnegative integers n,*

$$\overline{cp}(8n+7) \equiv 0 \pmod{64}.$$

Next, we find the following congruence for $\overline{cp}(n)$ via the aid of modular forms.

Theorem 2. *For all nonnegative integers n,*

$$\overline{cp}(9n+3) \equiv 0 \pmod{3}.$$

Combining the above two congruences, we can obtain the following congruences.

Corollary 3. *For all nonnegative integer n,*

$$\overline{cp}(72n+39) \equiv 0 \pmod{192}.$$

This paper is organized as follows. In Section 2, we review a 2-adic generating function of the overpartition function and basic properties of modular forms. In Section 3, we prove our results.

2. Preliminaries

Let $\overline{P}(q)$ be

$$\overline{P}(q) = \frac{(-q;q)_\infty}{(q;q)_\infty},$$

and $\varphi(q)$ be

$$\varphi(q)=\sum_{n=1}^{\infty} q^{n^2}.$$

Then, the coefficients of $\varphi(q)^k=\sum_{n\geq 1} c_k(n)q^n$ are the number of representations of $n=n_1^2+\cdots+n_k^2$ where each n_i is a positive integer. From Mahlburg's paper [10], we know that

$$\overline{P}(q)=1+\sum_{k=1}^{\infty}(-2)^k\varphi(-q)^k. \tag{2.1}$$

By reducing (2.1) modulo 64, we arrive at

(2.2)

$$\begin{aligned}
\overline{CP}(q) =& \sum_{n=0}^{\infty}\overline{cp}(n)q^n\\
=&\ \overline{P}(q)^2\overline{P}(q^2)^2\\
=&\ (1-2\varphi(-q)+2^2\varphi(-q)^2-\cdots)^2(1-2\varphi(-q^2)+2^2\varphi(-q^2)^2-\cdots)^2\\
=&\ 1-4\varphi(-q)-4\varphi(-q^2)+12\varphi(-q)^2+12\varphi(-q^2)^2+16\varphi(-q)\varphi(-q^2)\\
&+16\varphi(-q)^4+16\varphi(-q^2)^4-48\varphi(-q)^2\varphi(-q^2)-48\varphi(-q)\varphi(-q^2)^2\\
&-48\varphi(-q)^2\varphi(-q^2)^2+144\varphi(-q)^2\varphi(-q^2)^2-32\varphi(-q)^3\\
&-32\varphi(-q^2)^3 \pmod{64}.
\end{aligned}$$

Now we give some basic properties of modular functions. For more details on this subject, consult [12]. Define

$$\Gamma=SL_2(\mathbb{Z}),\ \Gamma_0(N):=\left\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma : c\equiv 0 \pmod{N}\right\}.$$

For a meromorphic function f on the complex upper half plane $\mathcal{H}$, define the slash operator by

$$f|\begin{pmatrix} a & b\\ c & d\end{pmatrix}:=f\left(\frac{az+b}{cz+d}\right).$$

We say a meromorphic function f is a modular function on $\Gamma_0(N)$ if f is invariant under the slash operator. Let $\mathcal{M}_0(\Gamma_0(N))$ denote the vector space of modular functions on $\Gamma_0(N)$. For a prime p, we define the U_p-operator as follows. If $f(q)$ has a Fourier expansion $f(q)=\sum a(n)q^n$, then

$$U_pf(z):=\sum a(pn)q^n.$$

It is well known that $U_pf(z)\in\mathcal{M}_0(\Gamma_0(Np))$ provided $f(z)\in\mathcal{M}_0(\Gamma_0(Np^2))$. Let $\eta(z)=q^{1/24}(q;q)_\infty$ be Dedekind eta function, where $q=\exp(2\pi iz)$ and z is in the upper half complex plane $\mathcal{H}$. For a fixed N and integers r_i's, a function of the form

$$f(z) := \prod_{\substack{n|N \\ n>0}} \eta(nz)^{r_n} \tag{2.3}$$

is called an η-quotient. By the famous theorem of Newman [11], we can determine when an η-quotient becomes a modular function. We call the orbits of $\mathbb{Q} \cup \{i\infty\} \pmod{\Gamma_0(N)}$ the cusps of $\Gamma_0(N)$. Moreover, by the work of Ligozat [9], we can calculate the order of the η-quotient f at the cusps c/d of $\Gamma_0(N)$ provided $f \in \mathcal{M}_0(\Gamma_0(N))$. Recall that if $p|N$ and $f \in \mathcal{M}_0(\Gamma_0(pN))$, then $U_p f \in \mathcal{M}_0(\Gamma_0(N))$. The following theorem gives bounds on the order of $U_p f$ at cusps of $\Gamma_0(N)$ in terms of the order of f at cusps of $\Gamma_0(pN)$.

Theorem 4 (Theorem 4 of [6])**.** *Let p be a prime and $\pi(n)$ be the highest power of p dividing n. Suppose that $f \in \mathcal{M}_0(\Gamma_0(pN))$, where $p|N$ and $\alpha = c/d$ is a cusp of $\Gamma_0(N)$. Then,*

$$\mathrm{ord}_\alpha U_p f \geq \begin{cases} \frac{1}{p}\mathrm{ord}_{\alpha/p} f, & \text{if } \pi(d) \geq \frac{1}{2}\pi(N), \\ \mathrm{ord}_{\alpha/p} f, & \text{if } 0 < \pi(d) < \frac{1}{2}\pi(N), \\ \min_{0 \leq \beta \leq p-1} \mathrm{ord}_{(\alpha+\beta)/p} f, & \text{if } \pi(d) = 0. \end{cases}$$

3. Proofs of theorems

We now start the proof of Theorem 1.

Proof of Theorem 1. For a quadratic form $Q = \sum_{i=1}^k a_i x_i^2$, let $R(n, Q)$ be the number of representations of n by $n = a_1 n_1^2 + a_2 n_2^2 + \cdots + a_k n_k^2$, where a_i's and n_i's are positive integers. By (2.2), it suffices to show the following congruences:

$$\begin{aligned} R(8n+7, x_1^2) &\equiv R(8n+7, 2x_1^2) \\ &\equiv R(8n+7, x_1^2 + x_2^2) \\ &\equiv R(8n+7, 2x_1^2 + 2x_2^2) \equiv 0 \pmod{16}, \\ R(8n+7, x_1^2 + 2x_2^2) &\equiv R(8n+7, x_1^2 + x_2^2 + 2x_3^2) \\ &\equiv R(8n+7, x_1^2 + 2x_2^2 + 2x_3^2) \\ &\equiv R(8n+7, x_1^2 + x_2^2 + x_3^2 + x_4^2) \\ &\equiv R(8n+7, 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2) \\ &\equiv R(8n+7, x_1^2 + x_2^2 + 2x_3^2 + 2x_4^2) \equiv 0 \pmod{4}, \\ R(8n+7, x_1^2 + x_2^2 + x_3^2) &\equiv R(8n+7, 2x_1^2 + 2x_2^2 + 2x_3^2) \equiv 0 \pmod{2}. \end{aligned}$$

Note that $x^2 \equiv 0,\ 1,\ \text{or}\ \ 4 \pmod 8$. By using this, we can see that there is no integer solution for $8n + 7 = x_1^2 + x_2^2 + 2x_3^2$ and we also see that if $8n+7 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, then at least one of x_i is different from others. By using a similar argument, we can verify the above congruences hold. $\square$

Remark. By proceeding as in [8], one can prove that $\{n \in \mathbb{N} : \overline{cp}(n) \equiv 0 \pmod{2^k}\}$ has the arithmetic density 1 for any fixed positive integer k. As we focus on explicit congruences in this article, we will omit the proof of this fact.

Now we turn to the proof of Theorem 2.

Proof of Theorem 2. Let define $G(z)$ as

$$G(z) = \frac{\eta^2(4z)\eta^4(9z)\eta^2(18z)}{\eta^4(z)\eta^2(2z)\eta^2(36z)}.$$

Then, by applying Newman's theorem, we see that $G(z) \in \mathcal{M}(\Gamma_0(36))$. By employing U_3 operator, we see that $U_3G(z) \in \mathcal{M}(\Gamma_0(12))$ and arrive at

$$U_3G(z) = \left(\sum_{n=0}^{\infty} \overline{cp}(3n)q^n\right) \frac{(q^3;q^3)_\infty^4 (q^9;q^9)_\infty^2}{(q^{12};q^{12})_\infty^2}.$$

By applying Theorem 4, we can find lower bounds for the orders of $U_3G(z)$ at the cusps as follows:

d	1	2	3	4	6	12
$\mathrm{ord}_{c/d} U_3G(z) \geq$	-6	-6	2	-6	1	0

After multiplying $\Delta(z)^2 = \eta^{48}(z)$ to $U_3G(z)$, we observe that $U_3G(z)\Delta(z)^2$ is in $\mathcal{M}_{24}(\Gamma_0(12))$ and $U_3G(z)\Delta(z)^2$ is a holomorphic modular form and so is $U_3(U_3G(z)\Delta(z)^2)$. Moreover,

$$\begin{aligned} U_3(U_3G(z)\Delta(z)^2) &\equiv U_3(U_3G(z)\eta^{16}(3z)) \pmod 3 \\ &\equiv U_3\left(\sum_{n=0}^{\infty} \overline{cp}(3n)q^{n+2}\right) \frac{(q;q)_\infty^{20}(q^3;q^3)_\infty^2}{(q^4;q^4)_\infty^2} \pmod 3 \\ &\equiv \left(\sum_{n=1}^{\infty} \overline{cp}(9n-6)q^n\right) \frac{(q;q)_\infty^{20}(q^3;q^3)_\infty^2}{(q^4;q^4)_\infty^2} \pmod 3. \end{aligned}$$

Since the dimension of holomorphic modular form of weight 24 with level 12 is 49, by Sturm's theorem, we can verify that $\overline{cp}(9n-6) \equiv 0 \pmod 3$ by checking up to $n = 50$. □

Remark. There is no single eta-quotient in $\mathcal{M}_0(\Gamma_0(12))$ which coincides with $U_3G(z)$.

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School of Liberal Arts
Seoul National University of Science and Technology
Seoul 139-743, Korea
E-mail address: bkim4@seoultech.ac.kr