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On Partitioning and Subtractive Subsemimodules of Semimodules over Semirings

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Abstract. In this paper, we introduce a partitioning subsemimodule of a semimodule over a semiring which is useful to develop the quotient structure of semimodule. Indeed we prove : 1) The quotient semimodule $M/N_{(Q)}$ is essentially independent of choice of Q. 2) If $f : M \to M^{'}$ is a maximal R-semimodule homomorphism, then $M/kerf_{(Q)} \cong M^{'}$. 3) Every partitioning subsemimodule is subtractive. 4) Let N be a Q-subsemimodule of an R-semimodule M. Then A is a subtractive subsemimodule of M with $N \subseteq A$ if and only if $A/N_{(Q\cap A)} = \{q + N : q \in Q \cap A\}$ is a subtractive subsemimodule of $M/N_{(Q)}$.

1. Introduction

For the definitions of monoid and semiring we refer [5]. All semirings in this paper are commutative with identity element. $\mathbb{Z}_0^+$ will denote the set of all non-negative integers. An element a of a monoid $(M, *)$ is called idempotent if $a*a = a$. An ideal I of a semiring R is called a subtractive ideal (k-ideal) if a, $a+b \in I$, $b \in R$, then $b \in I$. An ideal I of a semiring R is called a partitioning ideal (=Q-ideal) if there exist a subset Q of R such that:

1) $R = \cup\{q + I : q \in Q\}$.
2) If q_1, $q_2 \in Q$, then $(q_1 + I) \cap (q_2 + I) \neq \emptyset \Leftrightarrow q_1 = q_2$.

Definition 1.1. Let R be a semiring. A left R-semimodule is a commutative monoid $(M, +)$ with additive identity 0_M for which we have a function $R\times M \to M$, defined by $(r, x) \mapsto rx$ called scalar multiplication, which satisfies the following conditions for all elements r and $r^{'}$ of R and all elements x and y of M:

1) $(rr^{'})x = r(r^{'}x)$;
2) $r(x + y) = rx + ry$;
3) $(r + r^{'})x = rx + r^{'}x$;
4) $1_Rx = x$;

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5) $r0_M = 0_M = 0_R x$.

A nonempty subset N of a left R-semimodule M is called subsemimodule of M if N is closed under addition and closed under scalar multiplication.

Throughout this paper by an R-semimodule we mean a left semimodule over a semiring R. Every semiring R is $(\mathbb{Z}_0^+, +, \cdot)$-semimodule ([5], P.151).

Definition 1.2. If R is a semiring and M and $M^{'}$ are R-semimodules, then a function f from M to $M^{'}$is an R-semimodule homomorphism if and only if the following conditions are satisfied:

1) $f(x+y) = f(x) + f(y)$ for all $x, y \in M$;
2) $f(rx) = rf(x)$ for all $x \in M$ and $r \in R$.

An R-semimodule homomorphism f from an R-semimodule M to an R-semimodule $M^{'}$ is called isomorphism if f is one to one and onto.

The following lemma which is similar to lemma([6], Lemma 1) is easy to prove.

Lemma 1.3. *Let N be a subsemimodule of an R-semimodule M and x, $y \in M$ such that $x + N \subseteq y + N$. Then $x + z + N \subseteq y + z + N$ and $rx + N \subseteq ry + N$ for all $z \in M$, $r \in R$.*

2. Partitioning Subsemimodules

In this section we extend some definitions and results of Allen [1], [2] and Atani [3] to semimodules over semirings.

Definition 2.1. A subsemimodule N of an R-semimodule M will be called a partitioning subsemimodule (= Q-subsemimodule) if there exists a subset Q of M such that

1) $M = \cup\{q + N : q \in Q\}$.
2) if q_1, $q_2 \in Q$, then $(q_1 + N) \cap (q_2 + N) \neq \emptyset \Leftrightarrow q_1 = q_2$.

Clearly, every semiring is semimodule over itself. Hence every partitioning ideal of a semiring R is partitioning subsemimodule of an R-semimodule R.

Lemma 2.2. *Let N be a partitioning subsemimodule of an R-semimodule M. If $x \in M$, then there exists a unique $q \in Q$ such that $x + N \subseteq q + N$. Hence $x = q + a$ for some $a \in N$.*

Proof. Trivial. □

Now we extend a result of P. J. Allen ([2], Lemma 36) for semirings to semimodules over semirings.

Lemma 2.3. *If N is a partitioning subsemimodule of an R-semimodule M, then* *there exists a unique* $q_0 \in Q$ *such that* $N = q_0 + N$.

Proof. Since N is a partitioning subsemimodule, by Lemma 2.2, there exists a unique $q_0 \in Q$ such that $0 = q_0 + a_0$ for some $a_0 \in N$. If $b \in N$, then by Lemma 2.2, there exists a unique $q \in Q$ such that $b = q + a$ for some $a \in N$. Therefore, $q + a = b = b + 0 = b + q_0 + a_0 \in q_0 + N$. Hence $N \subseteq q_0 + N$. Again by Lemma 2.2, there exists a unique $q^{'} \in Q$ such that $q_0 + q_0 = q^{'} + c$ for some $c \in N$. Now $q_0 = q_0 + 0 = q_0 + q_0 + a_0 = q^{'} + c + a_0 \in q^{'} + N$. Also $q_0 \in q_0 + N$. Hence $(q^{'} + N) \cap (q_0 + N) \neq \emptyset$ and so $q_0 = q^{'}$. Thus, $q_0 + N = q^{'} + c + a_0 + N = q_0 + c + a_0 + N = c + q_0 + a_0 + N = c + N \subseteq N$. Now $N = q_0 + N$ where $q_0 \in Q$ is a unique element. □

Let N be a partitioning subsemimodule of an R-semimodule M. Then $M/N_{(Q)} = \{q + N : q \in Q\}$ forms an R-semimodule under the following addition "$\oplus$"and scalar multiplication "$\odot$", $(q_1 + N) \oplus (q_2 + N) = q_3 + N$ where $q_3 \in Q$ is a unique element such that $q_1 + q_2 + N \subseteq q_3 + N$ and $r \odot (q_1 + N) = q_4 + N$ where $q_4 \in Q$ is a unique element such that $rq_1 + N \subseteq q_4 + N$. This R-semimodule $M/N_{(Q)}$ will be called a quotient semimodule of M by N and denoted by $(M/N_{(Q)}, \oplus, \odot)$ or just $M/N_{(Q)}$. By Lemma 2.3, there exists a unique $q_0 \in Q$ such that $q_0 + N = N$. This $q_0 + N$ is the zero element of $M/N_{(Q)}$. If N is a subsemimodule of an R-semimodule M, then it is possible that N can be considered to be a partitioning subsemimodule with respet to many different subsets Q of M. However, the next theorem proves that the structure $(M/N_{(Q)}, \oplus, \odot)$ is essentially independent of Q.

Theorem 2.4. *If* N *is a partitioning subsemimodule with respect to two subsets* Q_1 *and* Q_2 *of an* R*-semimodule* M*, then* $M/N_{(Q_1)} \cong M/N_{(Q_2)}$.

Proof. Define $f : M/N_{(Q_1)} \to M/N_{(Q_2)}$ by $f(q_1 + N) = q_2 + N$ where $q_2 \in Q_2$ is a unique such that $q_1 + N \subseteq q_2 + N$. Clearly, f is well defined.
1) Let $q_1 + N$, $q_1^{'} + N \in M/N_{(Q_1)}$ and $r \in R$. Therefore,

(i) $$f\big((q_1 + N) \oplus (q_1{}^{'} + N)\big) = f(q_1{}^{''} + N) = q_2 + N$$

where $q_1{}^{''} \in Q_1$ is a unique such that $q_1 + q_1{}^{'} + N \subseteq q_1{}^{''} + N$ and $q_2 \in Q_2$ is a unique such that $q_1{}^{''} + N \subseteq q_2 + N$. Also

(ii) $$f(q_1 + N) \oplus f(q_1{}^{'} + N) = (q_2{}^{'} + N) \oplus (q_2{}^{''} + N) = q_2{}^{'''} + N$$

where $q_2{}^{'}, q_2{}^{''} \in Q_2$ are unique such that $q_1 + N \subseteq q_2{}^{'} + N$ and $q_1{}^{'} + N \subseteq q_2{}^{''} + N$ and $q_2{}^{'''} \in Q_2$ is a unique such that $q_2{}^{'} + q_2{}^{''} + N \subseteq q_2{}^{'''} + N$. Now

(iii) $$q_1 + q_1{}^{'} \in q_1 + q_1{}^{'} + N \subseteq q_1{}^{''} + N \subseteq q_2 + N.$$

Also by Lemma 1.3,

$$\begin{aligned} q_1 + N \subseteq q_2{}^{'} + N and \quad q_1^{'} + N \subseteq q_2^{''} + N \Rightarrow q_1 + q_1{}^{'} + N &\subseteq q_2{}^{'} + q_1{}^{'} + N \\ &\subseteq q_2{}^{'} + q_2{}^{''} + N \\ &\subseteq q_2{}^{'''} + N. \end{aligned}$$

Therefore,

$$\text{(iv)} \qquad q_1 + q_1' \in q_1 + q_1' + N \subseteq q_2''' + N.$$

From (iii) and (iv), $q_2 = q_2'''$. Hence by (i) and (ii), $f\big((q_1 + N) \oplus (q_1' + N)\big) = f(q_1 + N) \oplus f(q_1' + N)$. Similarly, it can be shown that $f\big(r \odot (q_1 + N)\big) = r \odot f(q_1 + N)$.
2) Let $q_2 + N \in M/N_{(Q_2)}$. Since $q_2 \in M$, there exists a unique $q_1 \in Q_1$ such that $q_2 + N \subseteq q_1 + N$. But then there exists a unique $q_2' \in Q_2$ such that $q_1 + N \subseteq q_2' + N$. Now $q_2 = q_2'$ implies $q_2 + N = q_2' + N$ and hence $f(q_1 + N) = q_2 + N$. So f is onto.
3) Suppose that $f(q_1 + N) = f(q_1' + N) = q_2 + N$ say, where $q_2 \in Q_2$ is a unique such that $q_1 + N \subseteq q_2 + N$ and $q_1' + N \subseteq q_2 + N$. Choose $t_1 \in Q_1$ such that $q_2 + N \subseteq t_1 + N$. But then $q_1 = t_1 = q_1'$. So $q_1 + N = q_1' + N$. Thus $f : M/N_{(Q_1)} \to M/N_{(Q_2)}$ is an isomorphism. □

Theorem 2.5. *If N is a partitioning subsemimodule with respect to two subsets Q_1 and Q_2 of an R-semimodule M, then $M/N_{(Q_1)}$ and $M/N_{(Q_2)}$ are equal as sets.*

Proof. Let $q_1 + N \in M/N_{(Q_1)}$. Then $q_1 \in Q_1 \subseteq M$ and hence by Lemma 2.2, there exists a unique $q_2 \in Q_2$ such that $q_1 + N \subseteq q_2 + N$. Again there exists a unique $q_3 \in Q_1$ such that $q_2 + N \subseteq q_3 + N$. Now $q_1 + N = q_3 + N = q_2 + N \in M/N_{(Q_2)}$. So $M/N_{(Q_1)} \subseteq M/N_{(Q_2)}$. Similarly, $M/N_{(Q_2)} \subseteq M/N_{(Q_1)}$. □

Example 2.6. The monoid $M = (\mathbb{Z}_6, +_6)$ is semimodule over $(\mathbb{Z}_0^+, +, \cdot)$, ([5], P.151). Then clearly $N = \{\overline{0}, \overline{2}, \overline{4}\}$ is a partitioning subsemimodule of M with respect to three sets $Q_1 = \{\overline{0}, \overline{1}\}$, $Q_2 = \{\overline{0}, \overline{3}\}$, $Q_3 = \{\overline{0}, \overline{5}\}$ where $M/N_{(Q_1)} = \{\overline{0} + N, \overline{1} + N\} = \{\{\overline{0}, \overline{2}, \overline{4}\}, \{\overline{1}, \overline{3}, \overline{5}\}\}$, $M/N_{(Q_2)} = \{\overline{0} + N, \overline{3} + N\} = \{\{\overline{0}, \overline{2}, \overline{4}\}, \{\overline{1}, \overline{3}, \overline{5}\}\}$ and $M/N_{(Q_3)} = \{\overline{0} + N, \overline{5} + N\} = \{\{\overline{0}, \overline{2}, \overline{4}\}, \{\overline{1}, \overline{3}, \overline{5}\}\}$. Here $M/N_{(Q_1)}$, $M/N_{(Q_2)}$ and $M/N_{(Q_3)}$ are equal as sets. But $M/N_{(Q_1)}$, $M/N_{(Q_2)}$ and $M/N_{(Q_3)}$ considered as $(\mathbb{Z}_0^+, +, \cdot)$-semimodules are not equal because $\overline{1} + N \in M/N_{(Q_1)}$ but $\overline{1} + N \notin M/N_{(Q_2)}$ and $\overline{1} + N \notin M/N_{(Q_3)}$ as $\overline{1} \notin Q_2$ and $\overline{1} \notin Q_3$.

Definition 2.7. An onto R-semimodule homomorphism $f : M \to M'$ will be called maximal if for each $a \in M'$ there exists a unique $q_a \in f^{-1}(\{a\})$ such that $x + kerf \subseteq q_a + kerf$, for each $x \in f^{-1}(\{a\})$ where $kerf = \{x \in M : f(x) = 0_{M'}\}$.

Clearly every R-module homomorphism is a maximal R-semimodule homomorphism.
P. J. Allen ([1], Lemma 14, Lemma 15 and Theorem 16) has proved the results for semirings. However, we extend the following Lemma 2.8, Lemma 2.11 and Theorem 2.12 for semimodules over semirings.

Lemma 2.8. *If $f : M \to M'$ is a maximal R-semimodule homomorphism, then $kerf$ is a partitioning subsemimodule of M.*

Proof. Since f is maximal, for each $a \in M'$ there exists a unique $q_a \in f^{-1}(\{a\})$

such that $x + kerf \subseteq q_a + kerf$ for all $x \in f^{-1}(\{a\})$. Take $Q = \{q_a : a \in M^{'}\}$. Clearly $\cup\{q_a + kerf : q_a \in Q\} \subseteq M$. On the other hand, if $m \in M$, then $f(m) \in M^{'}$. Now $m \in f^{-1}(\{f(m)\})$ implies $m \in m + kerf \subseteq q_{f(m)} + kerf$. Hence $M \subseteq \cup \{q_a + kerf : q_a \in Q\}$. Now for q_a, $q_b \in Q$, suppose that $(q_a + kerf) \cap (q_b + kerf) \neq \emptyset$. Let $q_a + k = q_b + k^{'}$ for some k, $k^{'} \in kerf$. Now $a = f(q_a) + f(k) = f(q_a + k) = f(q_b + k^{'}) = f(q_b) + f(k^{'}) = b$. Hence $q_a = q_b$. Thus, $kerf$ is a partitioning subsemimodule of M. □

The converse of Lemma 2.8 is not true.

Example 2.9. Let $M = (\mathbb{Z}_0{}^{+}, \max)$, $M^{'} = (\{0, 1\}, \max)$ and $R = (\mathbb{Z}_0^{+}, +, \cdot)$. Then M, $M^{'}$ are R-semimodules. Define $f : M \to M^{'}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 5 \\ 1 & \text{if } x > 5 \end{cases}$$

Clearly, f is an onto R-semimodule homomorphism. Also $kerf = \{0, 1, 2, 3, 4, 5\}$ is a partitioning subsemimodule of M with $Q = \{0, 6, 7, \dots\}$. For $1 \in M^{'}$ there cannot exists any $q_1 \in f^{-1}(\{1\})$ such that $x + kerf \subseteq q_1 + kerf$ for all $x \in f^{-1}(\{1\})$. So f is not a maximal R-semimodule homomorphism.

Example 2.10. Let $M = (\mathbb{Z}_0^{+}, +)$, $M^{'} = (\mathbb{Z}_6, +_6)$ and $R = (\mathbb{Z}_0^{+}, +, \cdot)$. Then M, $M^{'}$ are R-semimodules. Define $f : M \to M^{'}$ by $f(x) = \overline{r}$ where $x \equiv r \pmod 6$, $0 \leq r \leq 5$. Clearly, f is onto R-semimodule homomorphism. Also $kerf = \{0, 6, 12, 18, \dots\}$. For any $\overline{a} \in M^{'}$ there exists a unique $q_a = a \in f^{-1}(\{\overline{a}\})$ such that $x + kerf \subseteq q_a + kerf$ for all $x \in f^{-1}(\{\overline{a}\})$. Hence f is a maximal R-semimodule homomorphism.

Lemma 2.11. *Let M, $M^{'}$, f and Q be as stated in Lemma 2.8. Let q_a, q_b, $q_c \in Q$ and $r \in R$, then*

(i) *If $q_a + q_b + kerf \subseteq q_c + kerf$, then $a + b = c$.*

(ii) *If $rq_a + kerf \subseteq rq_c + kerf$, then $ra = rc$.*

Proof. (i) Since $q_a + q_b \in q_a + q_b + kerf \subseteq q_c + kerf$, $q_a + q_b = q_c + k$ for some $k \in kerf$. Now $a + b = f(q_a) + f(q_b) = f(q_a + q_b) = f(q_c + k) = f(q_c) + f(k) = c$. (ii) Since $rq_a \in rq_a + kerf \subseteq rq_c + kerf$, $rq_a = rq_c + k^{'}$ for some $k^{'} \in kerf$. Now $ra = rf(q_a) = f(rq_a) = f(rq_c + k^{'}) = f(rq_c) + f(k^{'}) = rf(q_c) = rc$ □

Theorem 2.12. *If $f : M \to M^{'}$ is a maximal R-semimodule homomorphism, then $M/kerf_{(Q)} \cong M^{'}$ where Q is as stated in Lemma 2.8.*

Proof. By Lemma 2.8, $kerf$ is a partitioning subsemimodule of M. Define $\overline{f} : M/kerf_{(Q)} \to M^{'}$ by $\overline{f}(q_a + kerf) = f(q_a) = a$ for each $q_a \in Q$. If $q_a + kerf$, $q_b + kerf \in M/kerf_{(Q)}$, then $\overline{f}(q_a + kerf) = \overline{f}(q_b + kerf) \Leftrightarrow a = b \Leftrightarrow q_a + kerf = q_b + kerf$. Hence $\overline{f}$ is well defined and one-one. Since f is maximal, $\overline{f}$ is onto. For $q_a + kerf$, $q_b + kerf \in M/kerf_{(Q)}$, $r \in R$, consider (i) $\overline{f}\big((q_a + kerf) \oplus (q_b + kerf)\big)$

$= \overline{f}(q_c + kerf) = c$ where q_c is a unique element in Q such that $q_a + q_b + kerf \subseteq q_c + kerf$. By Lemma 2.11, $a + b = c$. Now $\overline{f}(q_a + kerf) + \overline{f}(q_b + kerf) = a + b = c = \overline{f}\big((q_a + kerf) \oplus (q_b + kerf)\big)$. (ii) $\overline{f}(r \odot (q_a + kerf)) = \overline{f}(q_d + kerf) = d$ where q_d is a unique element in Q such that $rq_a + kerf \subseteq q_d + kerf$. By Lemma 2.11, $ra = d$. Therefore, $r\overline{f}(q_a + kerf) = ra = d = \overline{f}\big(r \odot (q_a + kerf)\big)$. Hence $\overline{f}$ is an R-semimodule isomorphism. Thus, $M/kerf_{(Q)} \cong M^{'}$. □

3. Subtractive subsemimodules

In this section we extend some results of S. E. Atani [4] to semimodules over semirings.

Definition 3.1. A subsemimodule N of an R-semimodule M is called a subtractive subsemimodule ($= k$-subsemimodule) if x, $x + y \in N$, $y \in M$ then $y \in N$.

Theorem 3.2. *Every partitioning subsemimodule N of an R-semimodule M is subtractive.*

Proof. Since N is a partitioning subsemimodule, by Lemma 2.3, $N = q_0 + N$ for some $q_0 \in Q$. Let x, $x + y \in N$ where $y \in M$. Therefore $x = q_0 + \alpha$, $x + y = q_0 + \beta$ for some α, $\beta \in N$. By Lemma 2.2, there exists a unique $q^{'} \in Q$ such that $y = q^{'} + \lambda$ for some $\lambda \in N$. Now $y + x = q^{'} + \lambda + x \in q^{'} + N$ and $x + y \in N = q_0 + N$. Hence $(q_0 + N) \cap (q^{'} + N) \neq \emptyset$ and so $q_0 = q^{'}$. Thus, $y = q^{'} + \lambda \in q^{'} + N = q_0 + N = N$. □

If $(M, +)$ is an idempotent commutative monoid, then M is $(\mathbb{Z}_0^+, +, \cdot)$-semimodule with scalar multiplication defined by $rm = 0$ if $r = 0$ and $rm = m$ if $r > 0$ for all $r \in \mathbb{Z}_0^+$ and $m \in M$ ([5] , P.151). In a semiring $R = (\mathbb{Z}_0^+, \gcd, lcm)$, the ideal $2\mathbb{Z}_0^+ = \{0, 2, 4, 6, \dots\}$ of R is subtractive but not partitioning ([7]).

The converse of Theorem 3.2 is not true.

Example 3.3. Let $M = (\mathbb{Z}_0^+ , \gcd)$ and $R = (\mathbb{Z}_0^+ ,+ , \cdot)$. Clearly, M is a commutative monoid in which every element is idempotent. Hence M is an R-semimodule in which $N = \{0, 2, 4, 6, \dots\}$ is a subtractive subsemimodule of M but not a partitioning subsemimodule.

S. E. Atani ([4], Lemma 2.1, Proposition 2.2 and Theorem 2.3) has proved the results for semirings. However, we extend the following Lemma 3.4, Theorem 3.5 and Theorem 3.6 for semimodules over semirings.

Lemma 3.4. *Let N be a Q-subsemimodule of an R-semimodule M. If A is a subtractive subsemimodule of M such that $N \subseteq A$, then N is a $Q \cap A$-subsemimodule of A.*

Proof. It is sufficient to show that $A = \cup\{q + N : q \in Q \cap A\}$. Clearly, $\cup\{q + N : q \in Q \cap A\} \subseteq A$. On the other hand, let $x \in A$. Since N is a Q-subsemimodule, by Lemma 2.2, $x = q + a$ for some $q \in Q$, $a \in N \subseteq A$. Then $q \in Q \cap A$, since A is a subtractive subsemimodule. So we have an equality. □

Theorem 3.5. *Let N be a Q-subsemimodule, A be a subtractive subsemimodule of an R-semimodule M with $N \subseteq A$. Then $A/N_{(Q\cap A)} = \{q + N : q \in Q \cap A\}$ is a subtractive subsemimodule of $M/N_{(Q)}$.*

Proof. By Lemma 2.3, let $q_0 \in Q$ be unique such that $q_0 + N$ is the zero element of $M/N_{(Q)}$. First, we show that $q_0 + N \in A/N_{(Q\cap A)}$. Let $a + N \in A/N_{(Q\cap A)} \subseteq M/N_{(Q)}$ where $a \in Q \cap A$.Then $(a + N) \oplus (q_0 + N) = a + N$ where $a \in Q$ is a unique such that $a+q_0+N \subseteq a+N$. So $a+q_0 = a+b$ for some $b \in N \subseteq A$. Since A is a subtractive subsemimodule, $q_0 \in A$. Thus, $q_0 + N \in A/N_{(Q\cap A)}$. Next suppose that $q_1 + N$, $q_2 + N \in A/N_{(Q\cap A)}$ where $q_1, q_2 \in Q\cap A$. Then $(q_1 + N) \oplus (q_2 + N) = q_3 + N$ where $q_3 \in Q$ is a unique such that $q_1 + q_2 + N \subseteq q_3 + N$. So $q_1 + q_2 = q_3 + c$ for some $c \in N \subseteq A$. Hence $q_3 \in Q \cap A$, since A is a subtractive subsemimodule. Now $(q_1+N) \oplus (q_2+N) = q_3+N \in A/N_{(Q\cap A)}$. Now let $r \in R$, $q+N \in A/N_{(Q\cap A)}$. Then $r \odot (q+N) = q_4+N$ where $q_4 \in Q$ is a unique such that $rq+N \subseteq q_4+N$. So $rq = q_4+d$ for some $d \in N \subseteq A$. Hence $q_4 \in Q \cap A$, since A is a subtractive subsemimodule. Thus, $r \odot (q + N) = q_4 + N \in A/N_{(Q\cap A)}$. Thus $A/N_{(Q\cap A)}$ is a subsemimodule of $M/N_{(Q)}$. Finally, assume that $x+N, (x+N) \oplus (y+N) = z+N \in A/N_{(Q\cap A)}$ where $x, z \in Q\cap A, y \in Q$ and $x+y+N \subseteq z+N$. Then $x+y = z+e$ for some $e \in N \subseteq A$. Now $y \in Q \cap A$, since A is a subtractive subsemimodule. Thus $y + N \in A/N_{(Q\cap A)}$ as needed. □

Theorem 3.6. *Let N be a Q-subsemimodule of an R-semimodule M and L be a subtractive subsemimodule of $M/N_{(Q)}$. Then $L = P/N_{(Q\cap P)}$ for some subtractive subsemimodule P of M with $N \subseteq P$.*

Proof. By Lemma 2.2, let $q_0 + N = N$ be the zero element of $M/N_{(Q)}$ where $q_0 \in Q$. Denote $P = \{x \in M : \textit{there exists a unique } q \in Q \textit{ such that } x + N \subseteq q + N \in L\}$. We show that P is a subtractive subsemimodule of M with $N \subseteq P$ and $L = P/N_{(Q\cap P)}$.
(1) Let $x \in N$. Then $x + N \subseteq N = q_0 + N \in L$, so $x \in P$. Thus, $N \subseteq P$.
(2) Let $x, y \in P$. Then there are unique elements $q_1, q_2 \in Q$ such that $x + N \subseteq q_1 + N \in L$ and $y + N \subseteq q_2 + N \in L$. Now $(q_1 + N) \oplus (q_2 + N) = q_3 + N \in L$ where $q_3 \in Q$ is a unique such that $q_1 + q_2 + N \subseteq q_3 + N$. By Lemma 1.3, $x + N \subseteq q_1 + N$ and $y + N \subseteq q_2 + N$ implies $x + y + N \subseteq q_1 + y + N \subseteq q_1 + q_2 + N \subseteq q_3 + N \in L$. Hence $x + y \in P$. Similarly, if $r \in R, x \in P$ then $rx \in P$. Thus, P is a subsemimodule of M.
(3) Let $x, x + y \in P$ where $y \in M$. Then there are unique elements $q_1, q_2, q_3 \in Q$ such that $x+N \subseteq q_1+N \in L$, $x+y+N \subseteq q_2+N \in L$, $y+N \subseteq q_3+N \in M/N_{(Q)}$. Since $q_1 + N$, $q_3 + N \in M/N_{(Q)}$, there exists a unique element $q_4 \in Q$ such that $(q_1 + N) \oplus (q_3 + N) = q_4 + N$ where $q_1 + q_3 + N \subseteq q_4 + N$. By Lemma 1.3, $x + N \subseteq q_1 + N$ and $y + N \subseteq q_3 + N$ implies $x + y + N \subseteq q_1 + y + N \subseteq q_1 + q_3 + N \subseteq q_4 + N$. Hence $x + y \in (q_2 + N) \cap (q_4 + N)$. So $q_4 + N = q_2 + N \in L$. Since L is a subtractive subsemimodule, $q_3 + N \in L$. Now $y + N \subseteq q_3 + N \in L$. So $y \in P$. Hence P is a subtractive subsemimodule of M.

(4) By Lemma 3.4, N is a $Q \cap P$-subsemimodule of P. If $q + N \in L$ where $q \in Q$ then $q \in P$. So $q \in Q \cap P$, hence $q + N \in P/N_{(Q \cap P)}$. Thus, $L \subseteq P/N_{(Q \cap P)}$. On the other hand, if $q + N \in P/N_{(Q \cap P)}$, then $q \in Q \cap P \subseteq P$. So $q + N \subseteq q^{'} + N \in L$ for some unique $q^{'} \in Q$. Therefore, $q + N = q^{'} + N \in L$. Thus, $P/N_{(Q \cap P)} \subseteq L$ □

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