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On pathwise uniqueness for stochastic differential equations driven by stable Lévy processes

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Abstract. We study a one-dimensional stochastic differential equation driven by a stable Lévy process of order α with drift and diffusion coefficients b, σ. When $\alpha \in (1,2)$, we investigate pathwise uniqueness for this equation. When $\alpha \in (0,1)$, we study another stochastic differential equation, which is equivalent in law, but for which pathwise uniqueness holds under much weaker conditions. We obtain various results, depending on whether $\alpha \in (0,1)$ or $\alpha \in (1,2)$ and on whether the driving stable process is symmetric or not. Our assumptions involve the regularity and monotonicity of b and σ.

Résumé. Nous étudions une équation différentielle stochastique de dimension 1 dirigée par un processus de Lévy stable. Lorsque $\alpha \in (1,2)$, nous examinons l'unicité trajectorielle pour cette équation. Quand $\alpha \in (0,1)$, nous étudions une autre équation, équivalente en loi, mais pour laquelle l'unicité trajectorielle s'avère vraie sous des hypothèses bien plus faibles. Nous obtenons des résultats variés, selon que $\alpha \in (0,1)$ ou $\alpha \in (1,2)$ et selon que le processus stable dirigeant l'équation est symétrique ou non. Nos hypothèses concernent la régularité et la monotonie des coefficients de dérive et de diffusion.

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1. Introduction and results

For a_-, a_+ in $[0,\infty)$ and $\alpha \in (0,2) \setminus \{1\}$, we consider the measure on $\mathbb{R}_*$:

$$\nu^\alpha_{a_-,a_+}(\mathrm{d}z) = |z|^{-\alpha-1}[a_-\mathbf{1}_{\{z<0\}} + a_+\mathbf{1}_{\{z>0\}}]\,\mathrm{d}z. \tag{1}$$

Let now $N(\mathrm{d}s\,\mathrm{d}z)$ be a Poisson measure on $[0,\infty) \times \mathbb{R}_*$ with intensity measure $\mathrm{d}s\,\nu^\alpha_{a_-,a_+}(\mathrm{d}z)$. Setting

$$\begin{cases} Z_t = \int_0^t \int_{\mathbb{R}_*} zN(\mathrm{d}s\,\mathrm{d}z) & \text{if } \alpha \in (0,1), \\ Z_t = \int_0^t \int_{\mathbb{R}_*} z\tilde{N}(\mathrm{d}s\,\mathrm{d}z) & \text{if } \alpha \in (1,2), \end{cases} \tag{2}$$

the process $(Z_t)_{t\ge 0}$ is a *stable process* of order α with parameters a_-, a_+, or a (α, a_-, a_+)-stable process in short. It is said to be *symmetric* if $a_- = a_+$. Here $\tilde{N}$ stands for the compensated Poisson measure, see Jacod and Shiryaev [9], Chapter II. Any one-dimensional stable process can be written as in (2), see Bertoin [4] and Sato [15] for many details on stable processes. We consider, for some measurable functions $\sigma, b : \mathbb{R} \mapsto \mathbb{R}$, the S.D.E.

$$X_t = x + \int_0^t \sigma(X_{s-})\,\mathrm{d}Z_s + \int_0^t b(X_s)\,\mathrm{d}s. \tag{3}$$

Our aim in this paper is to investigate pathwise uniqueness for this equation. Let us recall briefly the known results on this topic.

- Pathwise uniqueness classically holds when b, σ are both Lipschitz-continuous, see e.g. Ikeda and Watanabe [8], Chapter 4, and Protter [13], Chapter 5.
- When $\alpha \in (1,2)$, $a_+ = a_-$ and $b = 0$, Komatsu [10] has shown pathwise uniqueness if σ is Hölder-continuous with index $1/\alpha$, see also Bass [1].
- Bass, Burdzy and Chen [3] have proved that the above results are sharp: if $a_- = a_+$ and $b = 0$, for any $\beta < \min(1, 1/\alpha)$, one can find a function σ, Hölder-continuous with index β, bounded from above and from below, such that pathwise uniqueness fails for (3).
- When $\alpha \in (1,2)$ and $a_- = 0$, Li and Mytnik [12] have proved pathwise uniqueness if σ is non-decreasing and Hölder-continuous with index $1 - 1/\alpha$ and if b is the sum of a Lipschitz-continuous function and of a non-increasing function. This last result continues the work of Fu and Li [7].

We refer to the review paper of Bass [2] and to the works of Fu and Li [7] and Li and Mytnik [12] for many more details on the subject. In [7,12], much more general jumping S.D.E.s are considered. See also Situ [16] for a book on general S.D.E.s with jumps.

Our aims in this paper are the following. We will investigate pathwise uniqueness for (3) when $\alpha \in (1,2)$ without assuming $a_- = a_+$ or $a_- = 0$. When $\alpha \in (0,1)$, we will study another stochastic differential equation, which is equivalent in law, but for which pathwise uniqueness holds under much weaker conditions.

1.1. *Preliminaries*

When $\alpha \in (1,2)$, we will study the S.D.E. (3). When $\alpha \in (0,1)$, we will rather study the following equation: for $M(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u)$ a Poisson measure on $[0,\infty) \times \mathbb{R}_* \times \mathbb{R}_*$ with intensity measure $\mathrm{d}s\,\nu^{\alpha}_{a_-,a_+}(\mathrm{d}z)\,\mathrm{d}u$,

$$Y_t = x + \int_0^t \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} z[\mathbf{1}_{\{0<u<\gamma(Y_{s-})\}} - \mathbf{1}_{\{\gamma(Y_{s-})<u<0\}}] M(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u) + \int_0^t b(Y_s)\,\mathrm{d}s, \tag{4}$$

where $\gamma(x) = \mathrm{sign}(\sigma(x)) \cdot |\sigma(x)|^{\alpha}$. See Fu and Li [7], Eq. (5.6), for a similar equation. This equation is equivalent, in law, to (3). It has to be seen as another representation of (3).

Lemma 1. *Let $\alpha \in (0,1)$ and $a_-, a_+ \in [0,\infty)$.*

(i) *Let $(Y_t)_{t\geq 0}$ solve* (4). *There exists a (α, a_-, a_+)-stable process $(Z_t)_{t\geq 0}$ such that $(Y_t)_{t\geq 0}$ solves* (3).
(ii) *Let $(X_t)_{t\geq 0}$ solve* (3). *There exists, on an enlarged probability space, a Poisson measure M on $[0,\infty) \times \mathbb{R}_* \times \mathbb{R}_*$ with intensity measure $\mathrm{d}s\,\nu^{\alpha}_{a_-,a_+}(\mathrm{d}z)\,\mathrm{d}u$ such that $(X_t)_{t\geq 0}$ solves* (4).

Let us finally recall the following existence result.

Proposition 2. *Let $\alpha \in (0,2) \setminus \{1\}$ and $a_-, a_+ \in [0,\infty)$. Assume that σ, b have at most linear growth.*

(i) *If b, σ are continuous, there is weak existence for* (3).
(ii) *For any solution to* (3), *any $\beta \in (0,\alpha)$, any $T > 0$, $\mathbb{E}[\sup_{[0,T]} |X_t|^{\beta}] < \infty$.*

These results must be standard, but we found no precise reference. The weak existence is almost contained in Situ [16], Theorem 175.

1.2. *The case where $\alpha \in (1,2)$*

This section is devoted to the study of (3) when $\alpha \in (1,2)$. We first introduce some notation.

Lemma 3. *For $\alpha \in (1,2)$, set $a = \cos(\pi\alpha) \in (-1,1)$. Then for $c \in [0,1]$,*

$$\beta(\alpha, c) := \frac{1}{\pi} \arccos\left(\frac{c^2(1-a^2) - (1+ca)^2}{c^2(1-a^2) + (1+ca)^2}\right) \in [\alpha - 1, 1].$$

There holds $\beta(\alpha,0)=1$, $\beta(\alpha,1)=\alpha-1$ *and* $\beta(\alpha,c)\in(\alpha-1,1)$ *for* $c\in(0,1)$.

We may assume that $a_-\le a_+$ without loss of generality: if $a_->a_+$, write $\sigma(X_{s-})\,\mathrm{d}Z_s=\tilde\sigma(X_{s-})\,\mathrm{d}\tilde Z_s$, where $\tilde\sigma=-\sigma$ and $\tilde Z_t=-Z_t$ is a (α,a_+,a_-)-stable process.

Theorem 4. *Consider a stable process* $(Z_t)_{t\ge0}$ *of order* $\alpha\in(1,2)$ *with parameters* $0\le a_-\le a_+$. *Set* $\beta=\beta(\alpha,a_-/a_+)$ *as in Lemma 3. Assume that* σ,b *have at most linear growth and that for some constants* $\kappa_0,\kappa_1\in[0,\infty)$,

- σ *is Hölder-continuous with index* $(\alpha-\beta)/\alpha$ *(which lies in* $[1-1/\alpha,1/\alpha]$*)*,
- *for all* $x,y\in\mathbb{R}$, $\operatorname{sign}(x-y)(a_+-a_-)(\sigma(y)-\sigma(x))\le\kappa_1|x-y|$,
- *for all* $x,y\in\mathbb{R}$, $\operatorname{sign}(x-y)(b(x)-b(y))\le\kappa_0|x-y|$.

Consider two solutions $(X_t)_{t\ge0}$ *and* $(\tilde X_t)_{t\ge0}$ *to (3) started at* x *and* $\tilde x$.

(i) *For any* $t\ge0$, *there holds*

$$\mathbb{E}\big[|X_t-\tilde X_t|^\beta\big]\le|x-\tilde x|^\beta\mathrm{e}^{Ct},$$

where C *depends only on* $\kappa_0,\kappa_1,\alpha,a_-,a_+$. *Thus pathwise uniqueness holds for (3).*

(ii) *If furthermore* b *is constant and* $(a_+-a_-)\sigma$ *is non-decreasing, then* $\forall t\ge0$,

$$\mathbb{E}\big[|X_t-\tilde X_t|^\beta\big]=|x-\tilde x|^\beta.$$

Observe that the condition on b holds as soon as $b=b_1+b_2$, with b_1 non-increasing and b_2 Lipschitz-continuous. When $a_+=a_-$, we have $\beta=\alpha-1$ and thus we only assume that σ is Hölder-continuous with index $1/\alpha$, as Komatsu [10] or Bass [1]. But when $a_-<a_+$, there is automatically a compensation in the driving stable process, which introduces a sort of drift term. Our assumption on σ holds if $\sigma=\sigma_1+\sigma_2$, with σ_1 Lipschitz-continuous and σ_2 Hölder-continuous with index $(\alpha-\beta)/\alpha$ and non-decreasing. Observe that $(\alpha-\beta)/\alpha<1/\alpha$, so that if σ is non-decreasing, the assumption on σ is weaker if $a_-<a_+$ than if $a_-=a_+$. Finally, if $a_-=0$, then $\beta=1$, so that our assumption on σ holds if $\sigma=\sigma_1+\sigma_2$, with σ_1 Lipschitz-continuous and σ_2 Hölder-continuous with index $1-1/\alpha$ and non-decreasing. This last case is thus very similar to the result of Li and Mytnik [12].

As compared to [1,10,12], point (i) allows us to treat the case $a_-\ne a_+$ and $a_-\ne0$ and provides some simple stability estimates with respect to the initial datum. Point (ii) is a remarkable property. It was already discovered by Komatsu [10] when $a_-=a_+$ (and thus $\beta=\alpha-1$), although not explicitly stated. A similar remarkable identity holds in the Brownian case (with $\alpha=2$ and $\beta=\alpha-1=1$), see Le Gall [11], Theorem 1.3 and its proof.

As a by-product, our proof allows us to check the following statement. See [6], Theorems 4 and 5, for similar considerations about the stochastic heat equation.

Proposition 5. *Assume that* $\alpha\in(1,2)$ *and that* $a_-=a_+>0$. *Suppose that* σ,b *have at most linear growth, that* σ *is Hölder-continuous with index* $1/\alpha$ *and that* b *is non-increasing and continuous.*

(i) *If* (b,σ) *is injective, then (3) has at most one invariant distribution.*

(ii) *If there is a strictly increasing function* $\rho:\mathbb{R}_+\mapsto\mathbb{R}_+$ *such that*

$$\forall x,y\in\mathbb{R},\quad \mathbf{1}_{\{x\ne y\}}|x-y|^{\alpha-2}\big[\big|b(x)-b(y)\big|+\big|\sigma(x)-\sigma(y)\big|^\alpha\big]\ge\rho\big(|x-y|\big),$$

then for any pair of solutions $(X_t)_{t\ge0}$ *and* $(\tilde X_t)_{t\ge0}$ *to (3) started at* x *and* $\tilde x$ *(driven by the same stable process* $(Z_t)_{t\ge0}$*)*, $\lim_{t\to\infty}|X_t-\tilde X_t|=0$ *a.s.*

The basic example of application is the following: if $b(x)=-x$, then the conclusions of (i) and (ii) hold under the sole assumption that σ is Hölder-continuous with index $1/\alpha$. In particular, no positivity of σ is required at all. We only treat the case where $a_-=a_+$, because the other possible results are less interesting (although the proof is easily extended): some monotonicity conditions have to be imposed on the *true drift coefficient*, which involves b and σ.

1.3. *The case where* $\alpha \in (0, 1)$

Our goal is now to show that when $\alpha \in (0, 1)$, (4) is a *nice* representation of (3), in the sense that pathwise uniqueness holds for a larger class of functions σ, the Lipschitz condition being replaced by a weaker condition. First, we state a general result without monotonicity conditions on σ.

Theorem 6. *Let* $\alpha \in (0, 1)$ *and* $a_-, a_+ \in [0, \infty)$. *Consider a Poisson measure* M *on* $[0, \infty) \times \mathbb{R}_* \times \mathbb{R}_*$ *with intensity mesure* $\mathrm{d}s\, \nu^{\alpha}_{a_-,a_+}(\mathrm{d}z)\, \mathrm{d}u$. *Assume that* σ, b *have at most linear growth and that for some constant* $\kappa_0 \in [0, \infty)$,

- $\gamma(x) = \operatorname{sign}(\sigma(x)) \cdot |\sigma(x)|^{\alpha}$ *is Hölder-continuous with index* α,
- *for all* $x, y \in \mathbb{R}$, $\operatorname{sign}(x - y)(b(x) - b(y)) \le \kappa_0 |x - y|$.

Consider two solutions $(Y_t)_{t\ge 0}$ *and* $(\tilde{Y}_t)_{t\ge 0}$ *to* (4) *started at* x *and* $\tilde{x}$. *Then for any* $\beta \in (0, \alpha)$, *any* $t \ge 0$,

$$\mathbb{E}\big[|Y_t - \tilde{Y}_t|^{\beta}\big] \le |x - \tilde{x}|^{\beta} \mathrm{e}^{Ct},$$

where C *depends only on* $\alpha, a_-, a_+, \beta, \kappa_0$ *and on the Hölder constant of* γ. *Thus pathwise uniqueness holds for* (4).

Observe at once that if σ is bounded below by a positive constant and Hölder-continuous with index α, then γ is also Hölder-continuous with index α. But if σ vanishes, it has to be Lipschitz-continuous around its zeros. This is not only a technical condition as shown by Komatsu [10] or Bass [1], Remark 3.4: if $\alpha \in (0, 1)$, $x = 0$, $b = 0$, $a_- = a_+ = 1$ and $\sigma(x) = |x|^{\beta}$ (whence $\gamma(x) = |x|^{\beta\alpha}$) for some $\beta < 1$, then uniqueness in law fails for (3), whence it also fails for (4).

When $a_- = 0$ and σ is non-negative, (4) is a particular case of [7], Eq. (5.6). To apply [7], Theorem 5.6, one needs, roughly, the local Lipschitz-continuity of σ.

It might be surprising at first glance that in some cases, pathwise uniqueness holds for (4) but not for (3). This comes from the fact that, e.g. when starting from two initial positions x and $\tilde{x}$, (4) builds two different stable processes (coupled in a suitable way) to drive $(Y_t)_{t\ge 0}$ and $(\tilde{Y}_t)_{t\ge 0}$, while in (3), the same stable process drives $(X_t)_{t\ge 0}$ and $(\tilde{X}_t)_{t\ge 0}$. We see that the choice made in (4) is more efficient.

Let us now try to take advantage of some monotonicity considerations when $a_- \ne a_+$. This seems possible only if $\alpha \in (1/2, 1)$ and if a_-/a_+ is small enough.

Lemma 7. *For* $\alpha \in (1/2, 1)$, *set* $a = \cos(\pi\alpha) \in (-1, 0)$. *Then for* $c \in [0, -a)$,

$$\beta(\alpha, c) := \frac{1}{\pi} \arccos\left(\frac{1 - a^2 - (c + a)^2}{1 - a^2 + (c + a)^2}\right) \in (0, 2\alpha - 1].$$

There holds $\beta(\alpha, 0) = 2\alpha - 1$ *and, for any* $c \in (0, 1)$, $\lim_{\alpha\to 1-} \beta(\alpha, c) = 1$.

We only consider the case $a_- < a_+$ without loss of generality.

Theorem 8. *Assume that* $\alpha \in (1/2, 1)$, *that* $a_-/a_+ < |\cos(\pi\alpha)|$ *and set* $\beta := \beta(\alpha, a_-/a_+)$ *as in Lemma* 7. *Consider a Poisson measure* M *on* $[0, \infty) \times \mathbb{R}_* \times \mathbb{R}_*$ *with intensity measure* $\mathrm{d}s\, \nu^{\alpha}_{a_-,a_+}(\mathrm{d}z)\, \mathrm{d}u$. *Assume that* σ, b *have at most linear growth and that for some constants* $\kappa_0, \kappa_1 \in [0, \infty)$,

- $\gamma(x) = \operatorname{sign}(\sigma(x)) \cdot |\sigma(x)|^{\alpha}$ *is Hölder-continuous with index* $\alpha - \beta$,
- *for all* $x, y \in \mathbb{R}$, $\operatorname{sign}(x - y)(\gamma(x) - \gamma(y)) \le \kappa_1 |x - y|^{\alpha}$,
- *for all* $x, y \in \mathbb{R}$, $\operatorname{sign}(x - y)(b(x) - b(y)) \le \kappa_0 |x - y|$.

Consider two solutions $(Y_t)_{t\ge 0}$ *and* $(\tilde{Y}_t)_{t\ge 0}$ *to* (4) *started at* x *and* $\tilde{x}$.

(i) *Then for any* $t \ge 0$,

$$\mathbb{E}\big[|Y_t - \tilde{Y}_t|^{\beta}\big] \le |x - \tilde{x}|^{\beta} e^{Ct},$$

where C *depends only on* $a_-, a_+, \alpha, \kappa_0, \kappa_1$. *Thus pathwise uniqueness holds for* (4).

(ii) *If furthermore b is constant and γ is non-increasing*, *then* $\forall t \geq 0$,

$$\mathbb{E}\big[|Y_t - \tilde{Y}_t|^\beta\big] = |x - \tilde{x}|^\beta.$$

This last property is of course remarkable. If $a_- = 0$, the above result holds when $\gamma = \gamma_1 + \gamma_2$, with γ_1 Hölder-continuous with index α and γ_2 non-increasing and Hölder-continuous with index $1-\alpha$, which is very small when α is close to 1. More generally, when $a_- < a_+$ and if α is very close to 1, one has to assume only very few regularity on γ, provided it is non-increasing.

1.4. *Comments*

First observe that when $a_- < a_+$, the favorable monotonicity of σ is not the same if $\alpha \in (0,1)$ and if $\alpha \in (1,2)$. This is due to the fact that when $\alpha \in (1,2)$, the main problem is due to the compensation (which appears negatively in the equation).

Let us summarize roughly our results. Denote by $H(\delta)$ the set of Hölder-continuous functions with index δ and by $H^\downarrow(\delta)$ (resp. $H^\uparrow(\delta)$) its subset of non-increasing (resp. non-decreasing) functions. Recall that when σ is bounded below by a positive constant, the regularity of $\gamma(x) = \text{sign}(\sigma(x)) \cdot |\sigma(x)|^\alpha$ is the same as that of σ. We have pathwise uniqueness for (4) (if $\alpha \in (0,1)$) and (3) (if $\alpha \in (1,2)$) if $b = b_1 + b_2$ has at most linear growth, with $b_1 \in H(1)$ and b_2 non-increasing and if $\sigma = \sigma_1 + \sigma_2$ (or $\gamma = \gamma_1 + \gamma_2$) satisfies (we set $\beta(\alpha, c) = 0$ if $\alpha \in (0, 1/2]$ or if $c \geq -\cos(\pi\alpha)$):

	$\alpha \in (0,1)$	$\alpha \in (1,2)$
$a_- = a_+$	$\gamma \in H(\alpha)$	$\sigma \in H(1/\alpha)$
$a_- < a_+$	$\gamma_1 \in H(\alpha)$, $\gamma_2 \in H^\downarrow(\alpha - \beta(\alpha, a_-/a_+))$	$\sigma_1 \in H(1)$, $\sigma_2 \in H^\uparrow(1 - \beta(\alpha, a_-/a_+)/\alpha)$
$a_- = 0$	$\gamma_1 \in H(\alpha)$, $\gamma_2 \in H^\downarrow(1-\alpha)$	$\sigma_1 \in H(1)$, $\sigma_2 \in H^\uparrow(1 - 1/\alpha)$

See Fig. 1 for an illustration. Thus the situation is quite intricate. When $a_- = a_+$ and σ is bounded from below, we have to assume that $\sigma \in H(\min\{\alpha, 1/\alpha\})$. It seems quite strange that the required regularity of σ is low when α is small, maximal when $\alpha = 1$ and small again when α is near 2. A more tricky representation of (3) might allow one to obtain some better results.

When $a_- = 0$ and σ is bounded from below and monotonic, we have to assume that $\sigma \in H(\alpha)$ (if $\alpha \in (0, 1/2]$), $\sigma \in H^\downarrow(1-\alpha)$ (if $\alpha \in (1/2, 1)$) and $\sigma \in H^\uparrow(1 - 1/\alpha)$ (if $\alpha \in (1,2)$). Thus few regularity is needed when α is near 0 or 1 and higher regularity is needed when α is near 1/2 and 2.

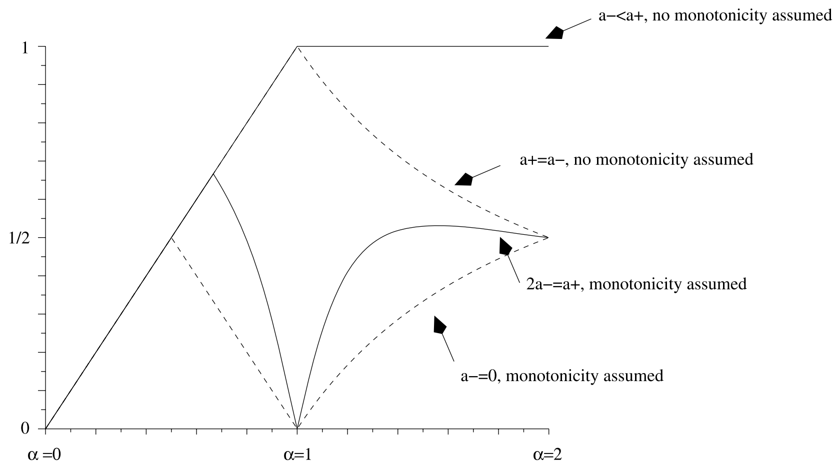

Fig. 1. Index of Hölder regularity of γ (if $\alpha \in (0,1)$) or σ (if $\alpha \in (1,2)$) required for pathwise uniqueness as a function of α. The four curves coincide on $[0, 1/2]$.

Theorem 4 is not so good when $a_- < a_+$, because we have to assume the Lipschitz-continuity of the decreasing part of σ. On the contrary, Theorem 6 works quite well for any value of a_-, a_+.

Theorems 4 and 8 really rely on specific properties of stable processes. Theorem 6, of which the proof is much simpler, may be easily extended to other jumping S.D.E.s with finite variations. For example, some of the Lipschitz assumptions of [5] can be consequently weakened.

1.5. *Plan of the paper*

In the next section, we prove Lemmas 3 and 7 and show that some integrals vanish. These integrals are those that appear when we use the Itô formula to compute $|X_t - \tilde{X}_t|^\beta$ for two solutions to (3) or (4). Section 3 shows how to approximate these integrals. We prove Theorem 4 and Proposition 5 in Section 4. Section 5 is devoted to the proofs of Theorems 6 and 8. We finally check Proposition 2 and Lemma 1 in Section 6.

2. Computation of some integrals

This technical section contains the main tools of the paper. We introduce

$$\text{for } \alpha \in (0,1) \text{ and } \beta \in (0,\alpha), \quad I^{\alpha,\beta}_{a_-,a_+} = \int_{\mathbb{R}_*} \left[|1-x|^\beta - 1\right] \nu^\alpha_{a_-,a_+}(\mathrm{d}x), \tag{5}$$

$$\text{for } \alpha \in (1,2) \text{ and } \beta \in (0,\alpha), \quad \tilde{I}^{\alpha,\beta}_{a_-,a_+} = \int_{\mathbb{R}_*} \left[|1+x|^\beta - 1 - \beta x\right] \nu^\alpha_{a_-,a_+}(\mathrm{d}x). \tag{6}$$

Observe that all the above integrals converge absolutely. The aim of this section is to prove Lemmas 3 and 7, as well as the following identities.

Lemma 9. (i) *Let* $\alpha \in (1/2, 1)$ *and* $0 \le a_- \le a_+$ *such that* $a_-/a_+ < -\cos(\pi\alpha)$. *Set* $\beta = \beta(\alpha, a_-/a_+) \in (0, 2\alpha - 1]$ *as in Lemma 7. There holds* $I^{\alpha,\beta}_{a_-,a_+} = 0$.

(ii) *Let* $\alpha \in (1,2)$ *and* $0 \le a_- \le a_+$. *Set* $\beta = \beta(\alpha, a_-/a_+) \in [\alpha - 1, 1]$ *as in Lemma 3. There holds* $\tilde{I}^{\alpha,\beta}_{a_-,a_+} = 0$.

Proof. We start with point (i). Observe that $\beta \le 2\alpha - 1 < \alpha$, so that the integral is convergent. We write $I^{\alpha,\beta}_{a_-,a_+} = a_- A_1 + a_+ A_2 + a_+ A_3$, where

$$A_1 = \int_0^\infty \left[(1+x)^\beta - 1\right] x^{-\alpha-1}\,\mathrm{d}x,$$

$$A_2 = \int_0^1 \left[(1-x)^\beta - 1\right] x^{-\alpha-1}\,\mathrm{d}x,$$

$$A_3 = \int_1^\infty \left[(x-1)^\beta - 1\right] x^{-\alpha-1}\,\mathrm{d}x.$$

Using an integration by parts and then putting $u = 1/(1+x)$, one can check that

$$A_1 = \frac{\beta}{\alpha}\int_0^\infty (1+x)^{\beta-1} x^{-\alpha}\,\mathrm{d}x = \frac{\beta}{\alpha}\int_0^1 u^{\alpha-\beta-1}(1-u)^{-\alpha}\,\mathrm{d}u = \frac{\beta\Gamma(\alpha-\beta)\Gamma(1-\alpha)}{\alpha\Gamma(1-\beta)},$$

where Γ is the Euler function. Next, an integration by parts implies that

$$A_2 = \frac{1}{\alpha} - \frac{\beta}{\alpha}\int_0^1 (1-x)^{\beta-1} x^{-\alpha}\,\mathrm{d}x = \frac{1}{\alpha} - \frac{\beta\Gamma(\beta)\Gamma(1-\alpha)}{\alpha\Gamma(1-(\alpha-\beta))}.$$

Finally, setting $x = 1/u$, we get

$$A_3 = \int_1^\infty (x-1)^\beta x^{-\alpha-1}\,\mathrm{d}x - \frac{1}{\alpha} = \int_0^1 (1-u)^\beta u^{\alpha-\beta-1}\,\mathrm{d}u - \frac{1}{\alpha} = \frac{\Gamma(\beta+1)\Gamma(\alpha-\beta)}{\Gamma(\alpha+1)} - \frac{1}{\alpha}.$$

We thus find, recalling that $\Gamma(a+1)=a\Gamma(a)$,

$$I^{\alpha,\beta}_{a_-,a_+}=\frac{\beta}{\alpha}\left[a_-\frac{\Gamma(\alpha-\beta)\Gamma(1-\alpha)}{\Gamma(1-\beta)}-a_+\frac{\Gamma(\beta)\Gamma(1-\alpha)}{\Gamma(1-(\alpha-\beta))}+a_+\frac{\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\right].$$

Using now Euler's reflection formula $\Gamma(x)\Gamma(1-x)=\pi/\sin(\pi x)$ for $x\in(0,1)$,

$$\begin{aligned}I^{\alpha,\beta}_{a_-,a_+}&=\frac{\beta\Gamma(\beta)\Gamma(\alpha-\beta)}{\alpha\Gamma(\alpha)}\left[a_-\frac{\sin(\pi\beta)}{\sin(\pi\alpha)}-a_+\frac{\sin(\pi(\alpha-\beta))}{\sin(\pi\alpha)}+a_+\right]\\&=\frac{a_+\beta\Gamma(\beta)\Gamma(\alpha-\beta)}{\alpha\Gamma(\alpha)\sin(\pi\alpha)}\big[c\sin(\pi\beta)-\sin\big(\pi(\alpha-\beta)\big)+\sin(\pi\alpha)\big],\end{aligned}$$

where we have set $c=a_-/a_+$. We have chosen $\beta=\beta(\alpha,c)$ in such a way that $c\sin(\pi\beta)-\sin(\pi(\alpha-\beta))+\sin(\pi\alpha)=0$, whence $I^{\alpha,\beta}_{a_-,a_+}=0$ as desired. Indeed, recall that $\cos(\pi\beta)=b$, where $b=(1-a^2-(c+a)^2)/(1-a^2+(c+a)^2)$, with $a=\cos(\pi\alpha)$. Since $\beta,\alpha\in(0,1)$, we have $\sin(\pi\alpha)=\sqrt{1-a^2}$ and $\sin(\pi\beta)=\sqrt{1-b^2}$, whence

$$\begin{aligned}&c\sin(\pi\beta)-\sin\big(\pi(\alpha-\beta)\big)+\sin(\pi\alpha)\\&\quad=c\sin(\pi\beta)-\sin(\pi\alpha)\cos(\pi\beta)+\sin(\pi\beta)\cos(\pi\alpha)+\sin(\pi\alpha)\\&\quad=(a+c)\sqrt{1-b^2}+(1-b)\sqrt{1-a^2}.\end{aligned}$$

Recall that $a+c<0<1-b$, since $c=a_-/a_+<-\cos(\pi\alpha)=-a$. We thus need to check that $(a+c)^2(1-b^2)=(1-b)^2(1-a^2)$, i.e. $(a+c)^2(1+b)=(1-b)(1-a^2)$. This is easily verified.

We now prove (ii). We write $\tilde{I}^{\alpha,\beta}_{a_-,a_+}=a_+B_1+a_-B_2+a_-B_3$, where

$$\begin{aligned}B_1&=\int_0^\infty\big[(1+x)^\beta-1-\beta x\big]x^{-\alpha-1}\,\mathrm{d}x,\\B_2&=\int_1^\infty\big[(x-1)^\beta-1+\beta x\big]x^{-\alpha-1}\,\mathrm{d}x,\\B_3&=\int_0^1\big[(1-x)^\beta-1+\beta x\big]x^{-\alpha-1}\,\mathrm{d}x.\end{aligned}$$

Using two integrations by parts and then putting $u=1/(1+x)$, one can prove that, if $\alpha-1\le\beta<1$,

$$\begin{aligned}B_1&=\frac{\beta(\beta-1)}{\alpha(\alpha-1)}\int_0^\infty(1+x)^{\beta-2}x^{1-\alpha}\,\mathrm{d}x\\&=\frac{\beta(\beta-1)}{\alpha(\alpha-1)}\int_0^1u^{\alpha-\beta-1}(1-u)^{1-\alpha}\,\mathrm{d}u\\&=-\frac{\beta(1-\beta)\Gamma(2-\alpha)\Gamma(\alpha-\beta)}{\alpha(\alpha-1)\Gamma(2-\beta)}.\end{aligned}$$

Since now $\alpha\in(1,2)$ and $\beta\in(0,1)$,

$$\begin{aligned}\Gamma(2-\alpha)&=\frac{\pi}{\Gamma(\alpha-1)\sin(\pi(\alpha-1))}=\frac{\pi(\alpha-1)}{\Gamma(\alpha)\sin(\pi(\alpha-1))},\\\Gamma(2-\beta)&=(1-\beta)\Gamma(1-\beta)=\frac{\pi(1-\beta)}{\Gamma(\beta)\sin(\pi\beta)}.\end{aligned}\tag{7}$$

Hence

$$B_1=-\frac{\beta\Gamma(\beta)\Gamma(\alpha-\beta)\sin(\pi\beta)}{\alpha\Gamma(\alpha)\sin(\pi(\alpha-1))}.$$

This formula remains valid if $\beta = 1$, since then $B_1 = 0$ and $\sin(\pi\beta) = 0$. Next we use one integration by parts and we put $u = 1/x$ to get

$$\begin{aligned} B_2 &= \frac{\beta-1}{\alpha} + \frac{\beta}{\alpha}\int_1^\infty \big[1+(x-1)^{\beta-1}\big]x^{-\alpha}\,\mathrm{d}x \\ &= \frac{\beta-1}{\alpha} + \frac{\beta}{\alpha(\alpha-1)} + \frac{\beta}{\alpha}\int_0^1 u^{\alpha-\beta-1}(1-u)^{\beta-1}\,\mathrm{d}u \\ &= -\frac{1}{\alpha} + \frac{\beta}{\alpha-1} + \frac{\beta\Gamma(\beta)\Gamma(\alpha-\beta)}{\alpha\Gamma(\alpha)}. \end{aligned}$$

Finally, an integration by parts shows that

$$B_3 = \frac{1-\beta}{\alpha} - \frac{\beta}{\alpha}\int_0^1 \big[(1-x)^{\beta-1}-1\big]x^{-\alpha}\,\mathrm{d}x = \frac{1-\beta}{\alpha} - \frac{\beta}{\alpha}[G_1+G_2],$$

where

$$G_1 = \int_0^1 \big[(1-x)^{\beta-1}-(1-x)^{\beta}\big]x^{-\alpha}\,\mathrm{d}x = \int_0^1 (1-x)^{\beta-1}x^{1-\alpha}\,\mathrm{d}x = \frac{\Gamma(\beta)\Gamma(2-\alpha)}{\Gamma(\beta+2-\alpha)}$$

and, using an integration by parts,

$$\begin{aligned} G_2 &= \int_0^1 \big[(1-x)^{\beta}-1\big]x^{-\alpha}\,\mathrm{d}x \\ &= \frac{1}{\alpha-1} - \frac{\beta}{\alpha-1}\int_0^1 (1-x)^{\beta-1}x^{1-\alpha}\,\mathrm{d}x \\ &= \frac{1}{\alpha-1} - \frac{\beta\Gamma(\beta)\Gamma(2-\alpha)}{(\alpha-1)\Gamma(\beta+2-\alpha)}. \end{aligned}$$

Thus

$$\begin{aligned} B_3 &= \frac{1}{\alpha} - \frac{\beta}{\alpha-1} + \frac{\beta(\beta+1-\alpha)\Gamma(\beta)\Gamma(2-\alpha)}{\alpha(\alpha-1)\Gamma(\beta+2-\alpha)} \\ &= \frac{1}{\alpha} - \frac{\beta}{\alpha-1} + \frac{\beta\Gamma(\beta)\Gamma(\alpha-\beta)\sin(\pi(\alpha-\beta))}{\alpha\Gamma(\alpha)\sin(\pi(\alpha-1))}. \end{aligned}$$

We used (7) and that

$$\frac{\Gamma(\beta+2-\alpha)}{\beta+1-\alpha} = \Gamma(\beta+1-\alpha) = \frac{\pi}{\Gamma(\alpha-\beta)\sin(\pi(\alpha-\beta))}.$$

This last equality uses that $\beta - \alpha + 1 \in (0,1)$, but one easily checks that the expression of B_3 remains valid if $\beta = \alpha - 1$, because then $1 + \beta - \alpha = \sin(\pi(\alpha-\beta)) = 0$. We finally find that

$$\tilde{I}^{\alpha,\beta}_{a_-,a_+} = \frac{\beta\Gamma(\beta)\Gamma(\alpha-\beta)}{\alpha\Gamma(\alpha)}\left[-a_+\frac{\sin(\pi\beta)}{\sin(\pi(\alpha-1))} + a_- + a_-\frac{\sin(\pi(\alpha-\beta))}{\sin(\pi(\alpha-1))}\right].$$

Set $c = a_-/a_+$ and recall that $b := \cos(\pi\beta) = (c^2(1-a^2) - (1+ca)^2)/(c^2(1-a^2) + (1+ca)^2)$, for $a = \cos(\pi\alpha) \in (-1,1)$. It remains to check that $\sin(\pi\beta) = c\sin(\pi(\alpha-1)) + c\sin(\pi(\alpha-\beta))$, i.e. $\sin(\pi\beta) = -c\sin(\pi\alpha) + c\sin(\pi\alpha)\cos(\pi\beta) - c\sin(\pi\beta)\cos(\pi\alpha)$. Observe that since $\alpha \in (1,2)$, $\sin(\pi\alpha) = -\sqrt{1-a^2}$, while since $\beta \in (0,1]$, $\sin(\pi\beta) = \sqrt{1-b^2}$. We need to verify that $\sqrt{1-b^2} = c\sqrt{1-a^2} - cb\sqrt{1-a^2} - ca\sqrt{1-b^2}$, i.e. that $(1+ac)\sqrt{1-b^2} = c(1-b)\sqrt{1-a^2}$, i.e. that $(1+ac)^2(1+b) = c^2(1-a^2)(1-b)$. This is easily done. □

We now give the

Proof of Lemma 3. Recall that $\alpha \in (1,2)$, that $a = \cos(\pi\alpha) \in (-1,1)$, that $0 \le c \le 1$ and that $\beta(\alpha,c) = \pi^{-1}\arccos b$, where $b := (c^2(1-a^2) - (1+ca)^2)/(c^2(1-a^2) + (1+ca)^2)$.

First, we have $\beta(\alpha,0) = \pi^{-1}\arccos(-1) = 1$ and $\beta(\alpha,1) = \pi^{-1}\arccos(-a) = \pi^{-1}\arccos(-\cos(\pi\alpha)) = \alpha - 1$.

To check that $\beta(\alpha,c) \in (\alpha-1,1)$ if $c \in (0,1)$, it suffices to prove that $b \in (-1,-a)$ (because $-1 = \cos(\pi)$ and $-a = \cos(\pi(\alpha-1))$). First, $b > -1$ is obvious if $c > 0$. Next, we have to check that $c^2(1-a^2) - (1+ca)^2 < -a(c^2(1-a^2) + (1+ca)^2)$, i.e. that $c^2(1+a)^2 < (1+ca)^2$, which holds true because $c < 1$ and $|a| < 1$. □

We conclude this section with the

Proof of Lemma 7. Recall that $\alpha \in (1/2,1)$, that $0 \le c < -a = -\cos(\pi\alpha) < 1$ and that $\beta(\alpha,c) = \pi^{-1}\arccos b$, where $b = (1-a^2-(c+a)^2)/(1-a^2+(c+a)^2)$. First, $\beta(\alpha,0) = \pi^{-1}\arccos(1-2a^2) = \pi^{-1}\arccos(1-2\cos^2(\pi\alpha)) = 2\alpha - 1$.

To prove that $\beta(\alpha,c) \in (0,2\alpha-1]$, it suffices to check that $b \in [1-2a^2,1)$. First, $b < 1$ is obvious. Next, $b \ge 1-2a^2$ because $(1-a^2)-(c+a)^2 \ge (1-2a^2)[(1-a^2)+(c+a)^2]$, since $2a^2(1-a^2) \ge 2(c+a)^2(1-a^2)$. Indeed, $a^2 > (c+a)^2$, since $0 \le c < -a$.

Finally, for $c < 1$ fixed, $\lim_{\alpha\to1-} a = -1$, whence $\lim_{\alpha\to1-} b = -1$. Thus $\lim_{\alpha\to1-}\beta(\alpha,c) = \pi^{-1} \times \arccos(-1) = 1$. □

3. Approximation lemmas

To prove our main results, we will apply Itô's formula to compute $|X_t - \tilde{X}_t|^\beta$, for $(X_t)_{t\ge0}$ and $(\tilde{X}_t)_{t\ge0}$ two solutions to (3) or (4), with some suitable value of $\beta \in (0,\alpha)$. This is not licit, since the function $|x|^\beta$ is not of class C^2. The two lemmas below will allow us to overcome this difficulty.

Lemma 10. *Let $0 < \beta < \alpha < 1$ and $a_-, a_+ \in [0,\infty)$. For $\eta > 0$, set $\phi_\eta(x) = (\eta^2 + x^2)^{\beta/2}$. For $\Delta \in \mathbb{R}_*$,*

$$J^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta) := \int_{\mathbb{R}_*} \big[\phi_\eta(\Delta - z) - \phi_\eta(\Delta)\big]\nu^\alpha_{a_-,a_+}(\mathrm{d}z) \to |\Delta|^{\beta-\alpha}\big[\mathbf{1}_{\Delta>0} I^{\alpha,\beta}_{a_-,a_+} + \mathbf{1}_{\Delta<0} I^{\alpha,\beta}_{a_+,a_-}\big]$$

as $\eta \to 0$, recall (5). Furthermore, for all $\eta > 0$, all $\Delta \in \mathbb{R}_*$,

$$\big|J^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta)\big| \le K^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta) := \int_{\mathbb{R}_*} \big|\phi_\eta(\Delta - z) - \phi_\eta(\Delta)\big|\nu^\alpha_{a_-,a_+}(\mathrm{d}z) \le C|\Delta|^{\beta-\alpha},$$

where C depends only on α, a_-, a_+, β.

Proof. We fix $\Delta \in \mathbb{R}_*$ and we observe that for all $\eta > 0$,

$$\big|\phi_\eta(\Delta - z) - \phi_\eta(\Delta)\big| \le C\min\big\{|z|^\beta, |\Delta|^{\beta-1}|z|\big\}. \tag{8}$$

This is easily deduced from the facts that $|\phi_\eta(x+y) - \phi_\eta(x)| \le |y|^\beta$ and $|\phi'_\eta(x)| \le \beta|x|^{\beta-1}$. Separate the cases $|z| \le |\Delta|/2$ and $|z| \ge |\Delta|/2$. But now

$$\int_{\mathbb{R}_*} \min\big\{|z|^\beta, |\Delta|^{\beta-1}|z|\big\}|z|^{-\alpha-1}\,\mathrm{d}z \le \int_{|z|\ge|\Delta|} |z|^{\beta-\alpha-1}\,\mathrm{d}z + \int_{|z|\le|\Delta|} |\Delta|^{\beta-1}|z|^{-\alpha}\,\mathrm{d}z$$
$$= C|\Delta|^{\beta-\alpha}.$$

We immediately deduce that $|K^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta)| \le C|\Delta|^{\beta-\alpha}$. And by Lebesgue's dominated convergence theorem, since $\lim_{\eta\to 0}\phi_\eta(x) = |x|^\beta$ for all $x \in \mathbb{R}$,

$$\begin{aligned}\lim_{\eta\to 0} J^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta) &= \int_{\mathbb{R}_*} \big[|\Delta - z|^\beta - |\Delta|^\beta\big]\nu_{a_-,a_+}(\mathrm{d}z)\\ &= |\Delta|^\beta \int_{\mathbb{R}_*} \big[|1 - z/\Delta|^\beta - 1\big]\nu_{a_-,a_+}(\mathrm{d}z)\\ &= \mathbf{1}_{\{\Delta>0\}}|\Delta|^\beta \int_{\mathbb{R}_*} \big[\big|1 - z/|\Delta|\big|^\beta - 1\big]\nu_{a_-,a_+}(\mathrm{d}z)\\ &\quad + \mathbf{1}_{\{\Delta<0\}}|\Delta|^\beta \int_{\mathbb{R}_*} \big[\big|1 - z/|\Delta|\big|^\beta - 1\big]\nu_{a_+,a_-}(\mathrm{d}z).\end{aligned}$$

In the last inequality and when $\Delta < 0$, we have used the substitution $x = -z$, which leads to $\nu^\alpha_{a_-,a_+}(\mathrm{d}z) = \nu^\alpha_{a_+,a_-}(\mathrm{d}x)$. Using finally the substitution $x = z/|\Delta|$, for which $\nu^\alpha_{a_-,a_+}(\mathrm{d}z) = |\Delta|^{-\alpha}\nu^\alpha_{a_-,a_+}(\mathrm{d}x)$, we get

$$\begin{aligned}\lim_{\eta\to 0} J^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta) &= \mathbf{1}_{\{\Delta>0\}}|\Delta|^{\beta-\alpha} \int_{\mathbb{R}_*} \big[|1 - x|^\beta - 1\big]\nu_{a_-,a_+}(\mathrm{d}x)\\ &\quad + \mathbf{1}_{\{\Delta<0\}}|\Delta|^{\beta-\alpha} \int_{\mathbb{R}_*} \big[|1 - x|^\beta - 1\big]\nu_{a_+,a_-}(\mathrm{d}x)\end{aligned}$$

as desired. □

Lemma 11. *Let* $0 < \beta \le 1 < \alpha < 2$ *and* a_-, a_+ *in* $[0,\infty)$. *For* $\eta > 0$ *and* $x \in \mathbb{R}$, *set* $\phi_\eta(x) = (\eta^2 + x^2)^{\beta/2}$. *Define, for* $\Delta, \delta \in \mathbb{R}$,

$$\begin{aligned}\tilde{J}^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta) &:= \int_{\mathbb{R}_*} \big\{\phi_\eta(\Delta + \delta z) - \phi_\eta(\Delta) - \delta z \phi'_\eta(\Delta)\big\}\nu^\alpha_{a_-,a_+}(\mathrm{d}z),\\ \tilde{K}^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta) &:= \int_{|\delta z|\le|\Delta|} \big(\phi_\eta(\Delta + \delta z) - \phi_\eta(\Delta) - |\Delta + \delta z|^\beta + |\Delta|^\beta\big)^2 \nu^\alpha_{a_-,a_+}(\mathrm{d}z),\\ \tilde{L}^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta) &:= \int_{|\delta z|>|\Delta|} \big|\phi_\eta(\Delta + \delta z) - \phi_\eta(\Delta) - |\Delta + \delta z|^\beta + |\Delta|^\beta\big| \nu^\alpha_{a_-,a_+}(\mathrm{d}z).\end{aligned}$$

For any $\Delta \in \mathbb{R}_*$, *any* $\delta \in \mathbb{R}$,

$$\begin{aligned}&\lim_{\eta\to 0} \tilde{J}^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta) = |\Delta|^{\beta-\alpha}|\delta|^\alpha\big[\mathbf{1}_{\{\delta\Delta>0\}}\tilde{I}^{\alpha,\beta}_{a_-,a_+} + \mathbf{1}_{\{\delta\Delta<0\}}\tilde{I}^{\alpha,\beta}_{a_+,a_-}\big],\\ &\lim_{\eta\to 0} \tilde{K}^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta) = \lim_{\eta\to 0} \tilde{L}^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta) = 0.\end{aligned}$$

Furthermore, we can find a constant C, *depending only on* α, a_-, a_+, β, *such that for all* $\eta > 0$, *all* $\Delta \in \mathbb{R}_*$, *all* $\delta \in \mathbb{R}$,

$$\begin{aligned}&\big|\tilde{J}^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta)\big| \le C|\Delta|^{\beta-\alpha}|\delta|^\alpha,\\ &\tilde{K}^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta) \le C|\Delta|^{2\beta-\alpha}|\delta|^\alpha,\\ &\tilde{L}^{\alpha,\beta}_{a_-,a_+}(\Delta,\delta) \le C|\Delta|^{\beta-\alpha}|\delta|^\alpha.\end{aligned}$$

Proof. We first observe that there is a constant C such that for all $\eta > 0$,

$$\begin{aligned}&\big|\phi_\eta(\Delta + \delta z) - \phi_\eta(\Delta)\big| \le C\min\big\{|\Delta|^{\beta-1}|\delta z|, |\delta z|^\beta\big\},\\ &\big|\phi_\eta(\Delta + \delta z) - \phi_\eta(\Delta) - \delta z\phi'_\eta(\Delta)\big| \le C\min\big\{|\Delta|^{\beta-2}(\delta z)^2, |\Delta|^{\beta-1}|\delta z|\big\}.\end{aligned}$$

This is easily deduced from the facts that $|\phi_\eta(x+y)-\phi_\eta(x)|\le|y|^\beta$, $|\phi'_\eta(x)|\le\beta|x|^{\beta-1}$, $|\phi''_\eta(x)|\le C|x|^{\beta-2}$ and $\beta-1\le 0$. Separate the cases $|\delta z|\le|\Delta|/2$ and $|\delta z|\ge|\Delta|/2$. Similarly,

$$\big||\Delta+\delta z|^\beta-|\Delta|^\beta\big|\le C\min\big\{|\Delta|^{\beta-1}|\delta z|,|\delta z|^\beta\big\}.$$

Next we observe that

$$\begin{aligned}&\int_{\mathbb{R}_*}\min\big\{|\Delta|^{\beta-2}(\delta z)^2,|\Delta|^{\beta-1}|\delta z|\big\}|z|^{-\alpha-1}\,\mathrm{d}z\\&\quad\le|\Delta|^{\beta-2}|\delta|^2\int_{\{|z|\le|\Delta|/|\delta|\}}|z|^{1-\alpha}\,\mathrm{d}z+|\Delta|^{\beta-1}|\delta|\int_{\{|z|\ge|\Delta|/|\delta|\}}|z|^{-\alpha}\,\mathrm{d}z=C|\Delta|^{\beta-\alpha}|\delta|^\alpha\end{aligned}$$

from which we immediately deduce that $|\tilde J^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta)|\le C|\Delta|^{\beta-\alpha}|\delta|^\alpha$ and that we can apply Lebesgue's dominated convergence theorem:

$$\begin{aligned}\lim_{\eta\to0}\tilde J^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta)&=\int_{\mathbb{R}_*}\big\{|\Delta+\delta z|^\beta-|\Delta|^\beta-\beta\delta z\cdot\operatorname{sign}(\Delta)|\Delta|^{\beta-1}\big\}\nu^\alpha_{a_-,a_+}(\mathrm{d}z)\\&=|\Delta|^\beta\int_{\mathbb{R}_*}\big\{\big|1+(\delta/\Delta)z\big|^\beta-1-\beta(\delta/\Delta)z\big\}\nu^\alpha_{a_-,a_+}(\mathrm{d}z)\\&=\mathbf{1}_{\{\delta\Delta>0\}}|\Delta|^\beta\int_{\mathbb{R}_*}\big\{\big|1+|\delta/\Delta|z\big|^\beta-1-\beta|\delta/\Delta|z\big\}\nu^\alpha_{a_-,a_+}(\mathrm{d}z)\\&\quad+\mathbf{1}_{\{\delta\Delta<0\}}|\Delta|^\beta\int_{\mathbb{R}_*}\big\{\big|1+|\delta/\Delta|z\big|^\beta-1-\beta|\delta/\Delta|z\big\}\nu^\alpha_{a_+,a_-}(\mathrm{d}z)\\&=|\Delta|^{\beta-\alpha}|\delta|^\alpha\big[\mathbf{1}_{\{\delta\Delta>0\}}\tilde I^{\alpha,\beta}_{a_-,a_+}+\mathbf{1}_{\{\delta\Delta<0\}}\tilde I^{\alpha,\beta}_{a_+,a_-}\big].\end{aligned}$$

We finally have put $x=|\delta/\Delta|z$, for which $\nu^\alpha_{a_-,a_+}(\mathrm{d}z)=(|\delta|/|\Delta|)^\alpha\nu^\alpha_{a_-,a_+}(\mathrm{d}x)$.

To study $\tilde K^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta)$, we note that

$$\begin{aligned}&\int_{|\delta z|\le|\Delta|}\min\big\{|\Delta|^{\beta-1}|\delta z|,|\delta z|^\beta\big\}^2|z|^{-\alpha-1}\,\mathrm{d}z\\&\quad\le|\Delta|^{2\beta-2}|\delta|^2\int_{|\delta z|\le|\Delta|}|z|^{1-\alpha}\,\mathrm{d}z=C|\Delta|^{2\beta-\alpha}|\delta|^\alpha.\end{aligned}$$

This implies that $\tilde K^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta)\le C|\Delta|^{2\beta-\alpha}|\delta|^\alpha$ and that we can apply the dominated convergence theorem to get $\lim_{\eta\to0}\tilde K^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta)=0$.

Finally,

$$\begin{aligned}&\int_{|\delta z|>|\Delta|}\min\big\{|\Delta|^{\beta-1}|\delta z|,|\delta z|^\beta\big\}|z|^{-\alpha-1}\,\mathrm{d}z\\&\quad\le|\delta|^\beta\int_{|\delta z|>|\Delta|}|z|^{\beta-\alpha-1}\,\mathrm{d}z=C|\Delta|^{\beta-\alpha}|\delta|^\alpha\end{aligned}$$

implies that $\tilde L^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta)\le C|\Delta|^{\beta-\alpha}|\delta|^\alpha$ and that $\lim_{\eta\to0}\tilde L^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta,\delta)=0$. □

4. The case with infinite variation

We now have all the weapons in hand to study pathwise uniqueness when $\alpha\in(1,2)$. We first prove that we can apply Itô's formula with the function $|x|^\beta$.

Lemma 12. *Let $\alpha \in (1,2)$, $0 \le a_- \le a_+$ and $\beta \in (0,1]$. Assume that σ, b have at most linear growth and that for some constant $\kappa_0 \ge 0$, for some $\beta \in (0,1]$,*

- *for all $x, y \in \mathbb{R}$, $\operatorname{sign}(x-y)(b(x)-b(y)) \le \kappa_0|x-y|$,*
- *σ is Hölder-continuous with index $(\alpha-\beta)/\alpha$.*

Consider two solutions $(X_t)_{t\ge 0}$ and $(\tilde{X}_t)_{t\ge 0}$ to (3) started at x and $\tilde{x}$, driven by the same (α, a_-, a_+)-stable process $(Z_t)_{t\ge 0}$ defined by (2). Put $\Delta_t = X_t - \tilde{X}_t$ and $\delta_t = \sigma(X_t) - \sigma(\tilde{X}_t)$. Then a.s., for all $t \ge 0$,

$$|\Delta_t|^\beta = |x-\tilde{x}|^\beta + \beta \int_0^t \mathbf{1}_{\{\Delta_s \neq 0\}} |\Delta_s|^{\beta-1} \operatorname{sign}(\Delta_s)\big[b(X_s) - b(\tilde{X}_s)\big]\,\mathrm{d}s$$
$$+ \int_0^t \mathbf{1}_{\{\Delta_s \neq 0\}} |\Delta_s|^{\beta-\alpha} |\delta_s|^\alpha \big(\mathbf{1}_{\{\delta_s \Delta_s > 0\}} \tilde{I}^{\alpha,\beta}_{a_-,a_+} + \mathbf{1}_{\{\delta_s \Delta_s < 0\}} \tilde{I}^{\alpha,\beta}_{a_+,a_-}\big)\,\mathrm{d}s + M_t,$$

where $\tilde{I}^{\alpha,\beta}_{a_-,a_+}$ was defined in (6) and where $(M_t)_{t\ge 0}$ is the L^1-martingale given by

$$M_t = \int_0^t \int_{\mathbb{R}_*} \big[|\Delta_{s-} + \delta_{s-} z|^\beta - |\Delta_{s-}|^\beta\big] \tilde{N}(\mathrm{d}s\,\mathrm{d}z).$$

Proof. For $\eta > 0$, consider $\phi_\eta(x) = (\eta^2 + x^2)^{\beta/2}$ as in Lemma 11. Applying the Itô formula, see e.g. Jacod and Shiryaev [9], Theorem 4.57, p. 56, we get, recalling (2),

$$\phi_\eta(\Delta_t) = \phi_\eta(x-\tilde{x}) + \int_0^t \int_{\mathbb{R}_*} \big[\phi_\eta(\Delta_{s-} + \delta_{s-} z) - \phi_\eta(\Delta_{s-})\big] \tilde{N}(\mathrm{d}s\,\mathrm{d}z)$$
$$+ \int_0^t \int_{\mathbb{R}_*} \big[\phi_\eta(\Delta_{s-} + \delta_{s-} z) - \phi_\eta(\Delta_{s-}) - \delta_{s-} z \phi'_\eta(\Delta_{s-})\big] \nu^\alpha_{a_-,a_+}(\mathrm{d}z)\,\mathrm{d}s$$
$$+ \int_0^t \phi'_\eta(\Delta_{s-})\big[b(X_s) - b(\tilde{X}_s)\big]\,\mathrm{d}s$$
$$=: \phi_\eta(x-\tilde{x}) + M^\eta_t + \int_0^t \tilde{J}^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta_s, \delta_s)\,\mathrm{d}s + \int_0^t A^\eta_s\,\mathrm{d}s,$$

where $\tilde{J}^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta, \delta)$ was defined in Lemma 11. First, we clearly have a.s.

$$\lim_{\eta\to 0} \phi_\eta(\Delta_t) = |\Delta_t|^\beta \quad \text{and} \quad \lim_{\eta\to 0} \phi_\eta(x-\tilde{x}) = |x-\tilde{x}|^\beta.$$

Next, we observe that $\tilde{J}^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta_t, \delta_t) = \tilde{J}^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta_t, \delta_t)\mathbf{1}_{\{\Delta_t \neq 0\}}$, since $\Delta_t = 0$ implies that $\delta_t = 0$. Since σ is Hölder-continuous with index $(\alpha-\beta)/\alpha$ by assumption, we deduce that $|\Delta_t|^{\beta-\alpha}|\delta_t|^\alpha$ is uniformly bounded. Thus, using Lemma 11 and Lebesgue's dominated convergence theorem, we get a.s.

$$\lim_{\eta\to 0} \int_0^t \tilde{J}^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta_s, \delta_s)\,\mathrm{d}s$$
$$= \int_0^t \mathbf{1}_{\{\Delta_s \neq 0\}} |\Delta_s|^{\beta-\alpha} |\delta_s|^\alpha \big(\mathbf{1}_{\{\delta_s \Delta_s > 0\}} \tilde{I}^{\alpha,\beta}_{a_-,a_+} + \mathbf{1}_{\{\delta_s \Delta_s < 0\}} \tilde{I}^{\alpha,\beta}_{a_+,a_-}\big)\,\mathrm{d}s.$$

Since $\phi'_\eta(x) = \beta x(\eta^2 + x^2)^{(\beta-2)/2}$, we may write $A^\eta_t = A^{\eta,+}_t - A^{\eta,-}_t$, where

$$A^{\eta,+}_t = \beta|\Delta_t|\big(\Delta_t^2 + \eta^2\big)^{(\beta-2)/2}\big(\operatorname{sign}(\Delta_t)[b(X_t) - b(\tilde{X}_t)]\big)_+,$$
$$A^{\eta,-}_t = \beta|\Delta_t|\big(\Delta_t^2 + \eta^2\big)^{(\beta-2)/2}\big(\operatorname{sign}(\Delta_t)[b(X_t) - b(\tilde{X}_t)]\big)_-.$$

First, $\lim_{\eta\to 0}\int_0^t A_s^{\eta,-}\,\mathrm{d}s = \beta\int_0^t \mathbf{1}_{\{\Delta_s\neq 0\}}|\Delta_s|^{\beta-1}(\mathrm{sign}(\Delta_s)[b(X_s)-b(\tilde X_s)])_-\,\mathrm{d}s$ by the monotone convergence theorem. Next, our assumption on b implies that $A_t^{\eta,+}\le \beta\kappa_0|\Delta_t|^\beta$. Hence we can apply the dominated convergence theorem to compute $\lim_{\eta\to 0}\int_0^t A_s^{\eta,+}\,\mathrm{d}s$ and we finally get a.s.,

$$\lim_{\eta\to 0}\int_0^t A_s^\eta\,\mathrm{d}s = \beta\int_0^t \mathbf{1}_{\{\Delta_s\neq 0\}}|\Delta_s|^{\beta-1}\,\mathrm{sign}(\Delta_s)\big[b(X_s)-b(\tilde X_s)\big]\,\mathrm{d}s$$

as desired. It only remains to prove that M_t^η tends to M_t in L^1. We write $M_t = M_t^1 + M_t^2$ and $M_t^\eta = M_t^{\eta,1}+M_t^{\eta,2}$, where

$$M_t^1 = \int_0^t\int_{\mathbb{R}_*}\mathbf{1}_{\{|\delta_{s-}z|\le|\Delta_{s-}|\}}\big[|\Delta_{s-}+\delta_{s-}z|^\beta - |\Delta_{s-}|^\beta\big]\tilde N(\mathrm{d}s\,\mathrm{d}z),$$
$$M_t^2 = \int_0^t\int_{\mathbb{R}_*}\mathbf{1}_{\{|\delta_{s-}z|>|\Delta_{s-}|\}}\big[|\Delta_{s-}+\delta_{s-}z|^\beta - |\Delta_{s-}|^\beta\big]\tilde N(\mathrm{d}s\,\mathrm{d}z),$$
$$M_t^{1,\eta} = \int_0^t\int_{\mathbb{R}_*}\mathbf{1}_{\{|\delta_{s-}z|\le|\Delta_{s-}|\}}\big[\phi_\eta(\Delta_{s-}+\delta_{s-}z) - \phi_\eta(\Delta_{s-})\big]\tilde N(\mathrm{d}s\,\mathrm{d}z),$$
$$M_t^{2,\eta} = \int_0^t\int_{\mathbb{R}_*}\mathbf{1}_{\{|\delta_{s-}z|>|\Delta_{s-}|\}}\big[\phi_\eta(\Delta_{s-}+\delta_{s-}z) - \phi_\eta(\Delta_{s-})\big]\tilde N(\mathrm{d}s\,\mathrm{d}z).$$

Using Lemma 11 and Lebesgue's dominated convergence theorem, there holds

$$\lim_{\eta\to 0}\mathbb{E}\big[\big|M_t^1 - M_t^{1,\eta}\big|^2\big] = \lim_{\eta\to 0}\int_0^t \mathbb{E}\big[\tilde K^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta_s,\delta_s)\big]\,\mathrm{d}s = 0,$$

since $\tilde K^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta_s,\delta_s)\le C|\Delta_s|^{2\beta-\alpha}|\delta_s|^\alpha \le C|\Delta_s|^\beta$ and since $\mathbb{E}[\sup_{[0,t]}|\Delta_s|^\beta]<\infty$ by Proposition 2(ii). Similarly, writing $\tilde N(\mathrm{d}s\,\mathrm{d}z) = N(\mathrm{d}s\,\mathrm{d}z) - \nu^\alpha_{a_-,a_+}(\mathrm{d}z)\,\mathrm{d}s$,

$$\lim_{\eta\to 0}\mathbb{E}\big[\big|M_t^2 - M_t^{2,\eta}\big|\big] \le 2\lim_{\eta\to 0}\int_0^t \mathbb{E}\big[\tilde L^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta_s,\delta_s)\big]\,\mathrm{d}s = 0,$$

because $\tilde L^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta_s,\delta_s)\le C|\Delta_t|^{\beta-\alpha}|\delta_t|^\alpha \le C$. This ends the proof. □

Proof of Theorem 4. We thus consider $\alpha\in(1,2)$, $0\le a_-\le a_+$ and two solutions $(X_t)_{t\ge 0}$ and $(\tilde X_t)_{t\ge 0}$ to (3). We put $\Delta_t = X_t - \tilde X_t$ and $\delta_t = \sigma(X_t)-\sigma(\tilde X_t)$. We set $\beta = \beta(\alpha, a_-/a_+)\in[\alpha-1,1]$ as in Lemma 3 and we use Lemma 12. With our choice for β, there holds $\tilde I^{\alpha,\beta}_{a_-,a_+} = 0$ by Lemma 9. We thus find

$$|\Delta_t|^\beta = |x-\tilde x|^\beta + \beta\int_0^t \mathbf{1}_{\{\Delta_s\neq 0\}}|\Delta_s|^{\beta-1}\,\mathrm{sign}(\Delta_s)\big[b(X_s)-b(\tilde X_s)\big]\,\mathrm{d}s$$
$$+ C\mathbf{1}_{\{a_-\neq a_+\}}\int_0^t \mathbf{1}_{\{\Delta_s\delta_s<0\}}|\Delta_s|^{\beta-\alpha}|\delta_s|^\alpha\,\mathrm{d}s + M_t,$$

where $C = I^{\alpha,\beta}_{a_+,a_-}$ and where $(M_t)_{t\ge 0}$ is a L^1-martingale.

Step 1. We prove point (i). Due to our assumption on b, $\mathrm{sign}(\Delta_s)[b(X_s)-b(\tilde X_s)]\le\kappa_0|\Delta_s|$. If $a_- = a_+$, we thus get, taking expectations, $\mathbb{E}[|\Delta_t|^\beta]\le |x-\tilde x|^\beta + \beta\kappa_0\int_0^t\mathbb{E}[|\Delta_s|^\beta]\,\mathrm{d}s$ and we conclude with the Gronwall Lemma. If $a_- < a_+$, our assumption on σ guarantees us that if $\delta_s\Delta_s<0$, then $|\delta_s|\le\kappa_1|\Delta_s|/(a_+-a_-)$, whence $|\Delta_s|^{\beta-\alpha}|\delta_s|^\alpha\mathbf{1}_{\{\delta_s\Delta_s<0\}}\le C|\Delta_s|^\beta$. Hence taking expectations, we get $\mathbb{E}[|\Delta_t|^\beta]\le|x-\tilde x|^\beta + C\int_0^t\mathbb{E}[|\Delta_s|^\beta]\,\mathrm{d}s$: we also conclude with the Gronwall Lemma.

Step 2. We check point (ii). Assuming that $(a_+ - a_-)\sigma$ is non-decreasing, we deduce that either $a_- = a_+$ or for all $s \geq 0$, a.s., $\delta_s \Delta_s \geq 0$. If furthermore b is constant, we thus get $|\Delta_t|^\beta = |x - \tilde{x}|^\beta + M_t$, whence $\mathbb{E}[|\Delta_t|^\beta] = |x - \tilde{x}|^\beta$. □

We now study the large time behavior of solutions when $a_- = a_+$.

Proof of Proposition 5. We thus assume that $a_- = a_+ > 0$, that $\alpha \in (1, 2)$, that b is non-increasing and continuous and that σ is Hölder-continuous with index $1/\alpha$.

Step 1. Consider any pair of solutions $(X_t)_{t\geq 0}$, $(\tilde{X}_t)_{t\geq 0}$ to (3) driven by the same stable process and set, as usual, $\Delta_t = X_t - \tilde{X}_t$, $\delta_t = \sigma(X_t) - \sigma(\tilde{X}_t)$. Lemma 12 with $\beta = \beta(\alpha, 1) = \alpha - 1$ implies, since $I^{\alpha,\beta}_{a_-,a_+} = I^{\alpha,\beta}_{a_+,a_-} = 0$ by Lemma 9, that

$$|\Delta_t|^{\alpha-1} = |x - \tilde{x}|^{\alpha-1} + M_t + (\alpha - 1)\int_0^t \operatorname{sign}(\Delta_s)\mathbf{1}_{\{\Delta_s\neq 0\}}|\Delta_s|^{\alpha-2}\big[b(X_s) - b(\tilde{X}_s)\big]\,\mathrm{d}s,$$

where $M_t = \int_0^t\int_{\mathbb{R}_*}(|\Delta_{s-} + \delta_{s-}z|^{\alpha-1} - |\Delta_{s-}|^{\alpha-1})\tilde{N}(\mathrm{d}s\,\mathrm{d}z)$. Using that b is non-increasing, we deduce that

$$|\Delta_t|^{\alpha-1} + (\alpha - 1)\int_0^t \mathbf{1}_{\{\Delta_s\neq 0\}}|\Delta_s|^{\alpha-2}\big|b(X_s) - b(\tilde{X}_s)\big|\,\mathrm{d}s = |x - \tilde{x}|^{\alpha-1} + M_t =: U_t. \tag{9}$$

Consequently, U_t is a non-negative martingale. Thus it a.s. converges as $t \to \infty$, as well as its bracket:

$$\int_0^\infty\int_{\mathbb{R}_*}\big[|\Delta_s + \delta_s z|^{\alpha-1} - |\Delta_s|^{\alpha-1}\big]^2\nu^\alpha_{a_-,a_+}(\mathrm{d}z)\,\mathrm{d}s < \infty.$$

But if $\Delta_s \neq 0$, since $a_- = a_+$, setting $c = a_+\int_{\mathbb{R}_*}[|1 + x|^{\alpha-1} - 1]^2|x|^{-\alpha-1}\,\mathrm{d}x > 0$,

$$\begin{aligned}\int_{\mathbb{R}_*}\big[|\Delta_s + \delta_s z|^{\alpha-1} - |\Delta_s|^{\alpha-1}\big]^2\nu^\alpha_{a_-,a_+}(\mathrm{d}z) &= a_+|\Delta_s|^{2\alpha-2}\int_{\mathbb{R}_*}\big[|1 + \delta_s z/\Delta_s|^{\alpha-1} - 1\big]^2|z|^{-\alpha-1}\,\mathrm{d}z\\ &= c|\Delta_s|^{2\alpha-2}\big(|\delta_s/\Delta_s|\big)^\alpha = c|\Delta_s|^{\alpha-2}|\delta_s|^\alpha,\end{aligned}$$

whence

$$\int_0^\infty \mathbf{1}_{\{\Delta_s\neq 0\}}|\Delta_s|^{\alpha-2}|\delta_s|^\alpha\,\mathrm{d}s < \infty \quad \text{a.s.} \tag{10}$$

We also have $\sup_{[0,\infty)} U_t < \infty$, whence, due to (9),

$$\int_0^\infty \mathbf{1}_{\{\Delta_s\neq 0\}}|\Delta_s|^{\alpha-2}\big|b(X_s) - b(\tilde{X}_s)\big|\,\mathrm{d}s < \infty \quad \text{a.s.} \tag{11}$$

Finally, Doob's L^1 inequality (see e.g. Revuz and Yor [14], Theorem 1.7, p. 54) implies, since U_t is a non-negative L^1 càdlàg martingale, that for any $a > 0$,

$$\Pr\Big[\sup_{[0,\infty)} U_t \geq a\Big] \leq a^{-1}\sup_{[0,\infty)}\mathbb{E}[U_t] = a^{-1}|x - \tilde{x}|^{\alpha-1}.$$

But $\sup_{[0,\infty)}|\Delta_t|^{\alpha-1} \leq \sup_{[0,\infty)} U_t$ by (9). Hence for any $\beta \in (0, \alpha - 1)$, for any $c > 0$,

$$\begin{aligned}\mathbb{E}\Big[\sup_{[0,\infty)}|\Delta_t|^\beta\Big] &= \int_0^\infty \Pr\Big[\sup_{[0,\infty)}|\Delta_t|^{\alpha-1} \geq a^{(\alpha-1)/\beta}\Big]\,\mathrm{d}a\\ &\leq c + \int_c^\infty a^{-(\alpha-1)/\beta}|x - \tilde{x}|^{\alpha-1}\,\mathrm{d}a\\ &= c + \frac{\beta}{\alpha - 1 - \beta}|x - \tilde{x}|^{\alpha-1}c^{1-(\alpha-1)/\beta}.\end{aligned}$$

Choose $c = |x - \tilde{x}|^\beta$: for some constant C depending only on α, β,

$$\mathbb{E}\Big[\sup_{[0,\infty)} |\Delta_t|^\beta\Big] \le C|x - \tilde{x}|^\beta. \tag{12}$$

Step 2. We now prove point (i). Consider two invariant distributions Q and $\tilde{Q}$ for (3). Let $X_0 \sim Q$ and $\tilde{X}_0 \sim \tilde{Q}$ be two random variables independent of $(Z_t)_{t\ge 0}$. Consider the associated solutions $(X_t)_{t\ge 0}$ and $(\tilde{X}_t)_{t\ge 0}$ to (3) starting from X_0 and $\tilde{X}_0$ (pathwise existence holds for (3): we have checked pathwise uniqueness in Theorem 4 and weak existence in Proposition 2). Then we have $X_t \sim Q$ and $\tilde{X}_t \sim \tilde{Q}$ for all $t \ge 0$. From (10) and (11), we have a.s.

$$\int_0^\infty \Gamma(X_t, \tilde{X}_t)\,\mathrm{d}t < \infty,$$

where

$$\begin{aligned}\Gamma(x, y) &:= \mathbf{1}_{\{x\ne y\}}\big(1 + |x - y|\big)^{\alpha-2}\big[\big|b(x) - b(y)\big| + \big|\sigma(x) - \sigma(y)\big|^\alpha\big]\\ &\le \mathbf{1}_{\{x\ne y\}}|x - y|^{\alpha-2}\big[\big|b(x) - b(y)\big| + \big|\sigma(x) - \sigma(y)\big|^\alpha\big].\end{aligned}$$

We easily deduce, see e.g. [6], Lemma 10, that there is a deterministic sequence $(t_n)_{n\ge 1}$ increasing to infinity such that $\Gamma(X_{t_n}, \tilde{X}_{t_n})$ goes to 0 in probability. Since (σ, b) is injective by assumption, we have $\Gamma(x, y) > 0$ for all $x \ne y$. Furthermore, Γ is continuous and $X_{t_n} \sim Q$ and $\tilde{X}_{t_n} \sim \tilde{Q}$ for all $n \ge 1$. We thus infer from [6], Lemma 11, that $Q = \tilde{Q}$.

Step 3. We next prove point (ii). Consider two solutions $(X_t)_{t\ge 0}$ and $(\tilde{X}_t)_{t\ge 0}$ to (3), issued from x and $\tilde{x}$. Using our assumptions and (10)–(11), we get

$$\int_0^\infty \rho\big(|X_t - \tilde{X}_t|\big)\,\mathrm{d}t < \infty.$$

Hence, see e.g. [6], Lemma 10, there is a deterministic sequence $(t_n)_{n\ge 1}$ increasing to infinity such that $\rho(|X_{t_n} - \tilde{X}_{t_n}|)$ goes to 0 in probability. Since ρ is strictly increasing and vanishes only at 0, we deduce that $|X_{t_n} - \tilde{X}_{t_n}|$ goes to 0 in probability. We thus infer from (12), choosing e.g. $\beta = (\alpha - 1)/2$, that

$$\mathbb{E}\Big[\sup_{[t_n,\infty)} |X_t - \tilde{X}_t|^\beta \big| \mathcal{F}_{t_n}\Big] \le C|X_{t_n} - \tilde{X}_{t_n}|^\beta.$$

We used that conditionally on $\mathcal{F}_{t_n}$, $(X_{t_n+t})_{t\ge 0}$ and $(\tilde{X}_{t_n+t})_{t\ge 0}$ solve (3). We easily deduce that $\sup_{[t_n,\infty)} |X_t - \tilde{X}_t|$ tends to 0 in probability. Since finally $s \mapsto \sup_{[s,\infty)} |X_t - \tilde{X}_t|$ is non-increasing, it a.s. admits a limit as $s \to \infty$ and this limit can only be 0. □

5. The case with finite variation

We now study the case where $\alpha \in (0, 1)$. Here again, we first prove that we can apply Itô's formula with the function $|x|^\beta$.

Lemma 13. *Let $\alpha \in (0, 1)$, $0 \le a_- \le a_+$ and $\beta \in (0, \alpha)$. Assume that σ, b have at most linear growth and that for some constant $\kappa_0 \ge 0$, for some $\beta \in (0, 1]$,*

- *for all $x, y \in \mathbb{R}$, $\operatorname{sign}(x - y)(b(x) - b(y)) \le \kappa_0|x - y|$,*
- *σ is Hölder-continuous with index θ for some $\theta \in [\alpha - \beta, \alpha]$.*

Consider two solutions $(Y_t)_{t\ge 0}$ and $(\tilde{Y}_t)_{t\ge 0}$ to (4) started at x and $\tilde{x}$, driven by the same Poisson measure M. Put $\Delta_t = Y_t - \tilde{Y}_t$. Then for all $t \ge 0$, recall (5),

$$\begin{aligned}\mathbb{E}\big[|\Delta_t|^\beta\big] \le |x-\tilde{x}|^\beta + \beta\kappa_0 \int_0^t \mathbb{E}\big[|\Delta_s|^\beta\big]\,\mathrm{d}s \\ + \int_0^t \mathbb{E}\big[\big(\gamma(Y_s)-\gamma(\tilde{Y}_s)\big)_+ |\Delta_s|^{\beta-\alpha}\big[\mathbf{1}_{\{\Delta_s>0\}} I^{\alpha,\beta}_{a_+,a_-} + \mathbf{1}_{\{\Delta_s<0\}} I^{\alpha,\beta}_{a_-,a_+}\big]\big]\,\mathrm{d}s \\ + \int_0^t \mathbb{E}\big[\big(\gamma(\tilde{Y}_s)-\gamma(Y_s)\big)_+ |\Delta_s|^{\beta-\alpha}\big[\mathbf{1}_{\{\Delta_s>0\}} I^{\alpha,\beta}_{a_-,a_+} + \mathbf{1}_{\{\Delta_s<0\}} I^{\alpha,\beta}_{a_+,a_-}\big]\big]\,\mathrm{d}s\end{aligned}$$

with an equality and $\kappa_0 = 0$ if b is constant.

Proof. We define, for $y, \tilde{y} \in \mathbb{R}$ and $u \in \mathbb{R}_*$,

$$\Gamma(y,\tilde{y},u) = \mathbf{1}_{\{0<u<\gamma(y)\}} - \mathbf{1}_{\{\gamma(y)<u<0\}} - \mathbf{1}_{\{0<u<\gamma(\tilde{y})\}} + \mathbf{1}_{\{\gamma(\tilde{y})<u<0\}}.$$

Let also $\phi_\eta(x) = (\eta^2+x^2)^{\beta/2}$. Applying the Itô formula for jump processes, see e.g. [9], Theorem 4.57, p. 56, we get

$$\begin{aligned}\phi_\eta(\Delta_t) = \phi_\eta(x-\tilde{x}) + \int_0^t \phi'_\eta(\Delta_s)\big(b(Y_s)-b(\tilde{Y}_s)\big)\,\mathrm{d}s \\ + \int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}\big[\phi_\eta\big(\Delta_{s-}+z\Gamma(Y_{s-},\tilde{Y}_{s-},u)\big)-\phi_\eta(\Delta_{s-})\big]M(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u).\end{aligned}$$

First, since $|\phi'_\eta(x)| \le \beta|x|^{\beta-1}$ and since $\mathrm{sign}(\phi'_\eta(x)) = \mathrm{sign}(x)$, we deduce from our assumption on b that for any $\eta > 0$, any $y, \tilde{y}$,

$$\begin{aligned}\phi'_\eta(y-\tilde{y})\big(b(y)-b(\tilde{y})\big) &= \big|\phi'_\eta(y-\tilde{y})\big|\,\mathrm{sign}(y-\tilde{y})\big(b(y)-b(\tilde{y})\big) \\ &\le \big|\phi'_\eta(y-\tilde{y})\big|\kappa_0|y-\tilde{y}| \\ &\le \beta\kappa_0|y-\tilde{y}|^\beta.\end{aligned}$$

We deduce that

$$\begin{aligned}\phi_\eta(\Delta_t) \le \phi_\eta(x-\tilde{x}) + \beta\kappa_0\int_0^t |\Delta_s|^\beta\,\mathrm{d}s \\ + \int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}\big[\phi_\eta\big(\Delta_{s-}+z\Gamma(Y_{s-},\tilde{Y}_{s-},u)\big)-\phi_\eta(\Delta_{s-})\big]M(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u)\end{aligned} \tag{13}$$

with of course an equality and $\kappa_0 = 0$ if b is constant. Observe now that for any $y, \tilde{y} \in \mathbb{R}$, any $u \in \mathbb{R}_*$,

$$\Gamma(y,\tilde{y},u) = \mathbf{1}_{\{\gamma(\tilde{y})<u<\gamma(y)\}} - \mathbf{1}_{\{\gamma(y)<u<\gamma(\tilde{y})\}}.$$

Hence integrating in u and recalling Lemma 10,

$$\begin{aligned}&\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}\big|\phi_\eta\big(\Delta_{s-}+z\Gamma(Y_{s-},\tilde{Y}_{s-},u)\big)-\phi_\eta(\Delta_{s-})\big|\nu^\alpha_{a_-,a_+}(\mathrm{d}z)\,\mathrm{d}u \\ &\quad= \big(\gamma(Y_s)-\gamma(\tilde{Y}_s)\big)_+\int_{\mathbb{R}_*}\big|\phi_\eta(\Delta_s+z)-\phi_\eta(\Delta_s)\big|\nu^\alpha_{a_-,a_+}(\mathrm{d}z) \\ &\qquad+ \big(\gamma(\tilde{Y}_s)-\gamma(Y_s)\big)_+\int_{\mathbb{R}_*}\big|\phi_\eta(\Delta_s-z)-\phi_\eta(\Delta_s)\big|\nu^\alpha_{a_-,a_+}(\mathrm{d}z)\end{aligned}$$

$$= \big(\gamma(Y_s)-\gamma(\tilde Y_s)\big)_+ K^{\alpha,\beta,\eta}_{a_+,a_-}(\Delta_s,\delta_s) + \big(\gamma(\tilde Y_s)-\gamma(Y_s)\big)_+ K^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta_s,\delta_s)$$
$$\le C\big|\gamma(Y_s)-\gamma(\tilde Y_s)\big|\cdot|\Delta_s|^{\beta-\alpha}.$$

Since γ is Hölder-continuous with index θ, this is bounded by $C|\Delta_s|^{\theta-\alpha+\beta}$. Using Proposition 2(ii) and that $\theta-\alpha+\beta\in[0,\alpha)$, we can thus take expectations in (13):

$$\mathbb{E}\big[\phi_\eta(\Delta_t)\big] \le \phi_\eta\big(|x-\tilde x|\big) + \beta\kappa_0\int_0^t \mathbb{E}\big[|\Delta_s|^\beta\big]\,\mathrm{d}s + \int_0^t \mathbb{E}\big[B^\eta_s\big]\,\mathrm{d}s \tag{14}$$

(with an equality and $\kappa_0=0$ if b is constant), where

$$B^\eta_s = \int_{\mathbb{R}_*}\int_{\mathbb{R}_*}\big[\phi_\eta\big(\Delta_{s-}+z\Gamma(Y_{s-},\tilde Y_{s-},u)\big)-\phi_\eta(\Delta_{s-})\big]\,\mathrm{d}u\,\nu^\alpha_{a_-,a_+}(\mathrm{d}z).$$

First, we obviously have $\lim_{\eta\to0}\mathbb{E}[\phi_\eta(\Delta_t)]=\mathbb{E}[|\Delta_t|^\beta]$ and $\lim_{\eta\to0}\phi_\eta(x-\tilde x)=|x-\tilde x|^\beta$. Next, integrating in u as previously and recalling Lemma 10, we obtain

$$B^\eta_s = \big(\gamma(Y_s)-\gamma(\tilde Y_s)\big)_+\int_{\mathbb{R}_*}\big[\phi_\eta(\Delta_s+z)-\phi_\eta(\Delta_s)\big]\nu^\alpha_{a_-,a_+}(\mathrm{d}z)$$
$$+ \big(\gamma(\tilde Y_s)-\gamma(Y_s)\big)_+\int_{\mathbb{R}_*}\big[\phi_\eta(\Delta_s-z)-\phi_\eta(\Delta_s)\big]\nu^\alpha_{a_-,a_+}(\mathrm{d}z)$$
$$= \big(\gamma(Y_s)-\gamma(\tilde Y_s)\big)_+ J^{\alpha,\beta,\eta}_{a_+,a_-}(\Delta_s) + \big(\gamma(\tilde Y_s)-\gamma(Y_s)\big)_+ J^{\alpha,\beta,\eta}_{a_-,a_+}(\Delta_s).$$

For the first integral, we have used the substitution $x=-z$, so that $\nu^\alpha_{a_-,a_+}(\mathrm{d}z)=\nu^\alpha_{a_+,a_-}(\mathrm{d}x)$. Using finally Lemma 10, that γ is Hölder continuous with index $\theta\in[\alpha-\beta,\alpha]$, Proposition 2(ii) (since $\theta+\beta-\alpha\in[0,\alpha)$) and Lebesgue's dominated convergence theorem, we deduce that

$$\lim_{\eta\to0}\int_0^t \mathbb{E}\big[B^\eta_s\big]\,\mathrm{d}s$$
$$= \int_0^t \mathbb{E}\big[\big(\gamma(Y_s)-\gamma(\tilde Y_s)\big)_+|\Delta_s|^{\beta-\alpha}\big[\mathbf{1}_{\{\Delta_s>0\}}I^{\alpha,\beta}_{a_+,a_-}+\mathbf{1}_{\{\Delta_s<0\}}I^{\alpha,\beta}_{a_-,a_+}\big]\big]\,\mathrm{d}s$$
$$+ \int_0^t \mathbb{E}\big[\big(\gamma(\tilde Y_s)-\gamma(Y_s)\big)_+|\Delta_s|^{\beta-\alpha}\big[\mathbf{1}_{\{\Delta_s>0\}}I^{\alpha,\beta}_{a_-,a_+}+\mathbf{1}_{\{\Delta_s<0\}}I^{\alpha,\beta}_{a_+,a_-}\big]\big]\,\mathrm{d}s$$

as desired. □

We can now give the

Proof of Theorem 6. We consider $\alpha\in(0,1)$, $a_-\le a_+$ and two solutions $(Y_t)_{t\ge0}$ and $(\tilde Y_t)_{t\ge0}$ to (4), issued from x and $\tilde x$. We also fix $\beta\in(0,\alpha)$. We put $\Delta_t=Y_t-\tilde Y_t$. Applying Lemma 13 and recalling that γ is Hölder continuous with index α, a rough upperbound using only that $|I^{\alpha,\beta}_{a_-,a_+}|+|I^{\alpha,\beta}_{a_+,a_-}|<\infty$ yields that $\mathbb{E}[|\Delta_t|^\beta]\le|x-\tilde x|^\beta+C\int_0^t\mathbb{E}[|\Delta_s|^\beta]\,\mathrm{d}s$ and we conclude with the Gronwall Lemma. □

We conclude this section with the

Proof of Theorem 8. Let us thus assume that $\alpha\in(1/2,1)$, that $a_-<a_+$ with $a_-/a_+<-\cos(\pi\alpha)$ and let us set $\beta=\beta(\alpha,a_-/a_+)\in(0,\alpha)$. Consider two solutions $(Y_t)_{t\ge0}$ and $(\tilde Y_t)_{t\ge0}$ to (4), issued from x and $\tilde x$ and put $\Delta_t=Y_t-\tilde Y_t$. Applying Lemma 13 (γ is Hölder-continuous with index $\alpha-\beta$ by assumption) and recalling that $I^{\alpha,\beta}_{a_-,a_+}=0$ due to Lemma 9, we get

$$\mathbb{E}\big[|\Delta_t|^\beta\big]\le|x-\tilde x|^\beta+\beta\kappa_0\int_0^t\mathbb{E}\big[|\Delta_s|^\beta\big]\,\mathrm{d}s+\int_0^t\mathbb{E}\big[B^{\eta,1}_s+B^{\eta,2}_s\big]\,\mathrm{d}s$$

(with an equality and $\kappa_0 = 0$ if b is constant), where

$$B_s^{\eta,1} = I_{a_+,a_-}^{\alpha,\beta}\big(\gamma(Y_s) - \gamma(\tilde{Y}_s)\big)_+ |\Delta_s|^{\beta-\alpha}\mathbf{1}_{\{\Delta_s>0\}},$$
$$B_s^{\eta,2} = I_{a_+,a_-}^{\alpha,\beta}\big(\gamma(\tilde{Y}_s) - \gamma(Y_s)\big)_+ |\Delta_s|^{\beta-\alpha}\mathbf{1}_{\{\Delta_s<0\}}.$$

Step 1. We now prove point (i). Our assumption on γ guarantees that if $\Delta_s > 0$, then $(\gamma(Y_s) - \gamma(\tilde{Y}_s))_+ \le \kappa_1 |\Delta_s|^\alpha$, whence $B_s^{\eta,1} \le C|\Delta_s|^\beta$. Similarly, $B_s^{\eta,2} \le C|\Delta_s|^\beta$. We thus find $\mathbb{E}[|\Delta_t|^\beta] \le |x - \tilde{x}|^\beta + C\int_0^t \mathbb{E}[|\Delta_s|^\beta]\,\mathrm{d}s$ and we conclude with the Gronwall Lemma.

Step 2. We now check point (ii), assuming that b is constant and that γ is non-increasing. Then $\Delta_s > 0$ implies $\gamma(Y_s) - \gamma(\tilde{Y}_s) \le 0$, whence $B_s^{\eta,1} = 0$. Similarly, $B_s^{\eta,2} = 0$ and we obtain $\mathbb{E}[|\Delta_t|^\beta] = |x - \tilde{x}|^\beta$ as desired. □

6. Weak existence and equivalence of the two equations

We start this section with the equivalence in law between (3) and (4).

Proof of Lemma 1. We fix $\alpha \in (0,1)$, $a_-, a_+ \in [0,\infty)$ and we start with point (i). We thus consider a solution $(Y_t)_{t\ge0}$ to (4) driven by a Poisson measure M with intensity measure $\mathrm{d}s\,\nu_{a_-,a_+}^\alpha(\mathrm{d}z)\,\mathrm{d}u$. Recall that $\gamma(x) = \mathrm{sign}(\sigma(x))\cdot|\sigma(x)|^\alpha$. Set

$$Z_t = \int_0^t \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \bigg\{\mathbf{1}_{\{\sigma(Y_{s-})\neq0\}}\frac{z}{\sigma(Y_{s-})}[\mathbf{1}_{\{0<u<\gamma(Y_{s-})\}} - \mathbf{1}_{\{\gamma(Y_{s-})<u<0\}}] + \mathbf{1}_{\{\sigma(Y_{s-})=0\}} z\mathbf{1}_{\{0<u<1\}}\bigg\} M(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u).$$

Then we obviously have

$$\int_0^t \sigma(Y_{s-})\,\mathrm{d}Z_s = \int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} z[\mathbf{1}_{\{0<u<\gamma(Y_{s-})\}} - \mathbf{1}_{\{\gamma(Y_{s-})<u<0\}}]M(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u).$$

It only remains to prove that $(Z_t)_{t\ge0}$ is a (α, a_-, a_+)-stable process. But $(Z_t)_{t\ge0}$ is a pure jump process without drift, so that we only need to check that, for $J = \{s \in [0,\infty), \Delta Z_s \neq 0\}$, $\sum_{s>0}\mathbf{1}_{\{s\in J\}}\delta_{(s,\Delta Z_s)}$ is a Poisson measure on $[0,\infty)\times\mathbb{R}_*$ with intensity measure $\mathrm{d}s\,\nu_{a_-,a_+}^\alpha(\mathrm{d}z)$. Denote by $q(\mathrm{d}s\,\mathrm{d}z)$ its compensator. It is enough (see Jacod and Shiryaev [9], Theorem 4.8, p. 104) to show that $q(\mathrm{d}s\,\mathrm{d}z) = \nu_{a_-,a_+}^\alpha(\mathrm{d}z)\,\mathrm{d}s$. By Definition of $(Z_t)_{t\ge0}$, we clearly have (recall that $\mathrm{sign}(\sigma(x)) = \mathrm{sign}(\gamma(x))$), for all measurable $\phi : [0,\infty)\times\mathbb{R}_* \mapsto \mathbb{R}$ sufficiently integrable (e.g. ϕ compactly supported in $[0,\infty)\times\mathbb{R}_*$),

$$\begin{aligned}\int_0^t\int_{\mathbb{R}_*}\phi(s,z)q(\mathrm{d}s\,\mathrm{d}z) &= \int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}\mathbf{1}_{\{\sigma(Y_s)>0\}}\phi\big(s, z/\sigma(Y_s)\big)\mathbf{1}_{\{0<u<\gamma(Y_s)\}}\,\mathrm{d}u\,\nu_{a_-,a_+}^\alpha(\mathrm{d}z)\,\mathrm{d}s\\ &\quad+\int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}\mathbf{1}_{\{\sigma(Y_s)<0\}}\phi\big(s, -z/\sigma(Y_s)\big)\mathbf{1}_{\{\gamma(Y_s)<u<0\}}\,\mathrm{d}u\,\nu_{a_-,a_+}^\alpha(\mathrm{d}z)\,\mathrm{d}s\\ &\quad+\int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}\mathbf{1}_{\{\sigma(Y_s)=0\}}\phi(s,z)\mathbf{1}_{\{0<u<1\}}\,\mathrm{d}u\,\nu_{a_-,a_+}^\alpha(\mathrm{d}z)\,\mathrm{d}s.\end{aligned}$$

Integrating in u, we deduce that

$$\begin{aligned}\int_0^t\int_{\mathbb{R}_*}\phi(s,z)q(\mathrm{d}s\,\mathrm{d}z) &= \int_0^t\int_{\mathbb{R}_*}\mathbf{1}_{\{\sigma(Y_s)>0\}}\phi\big(s, z/\sigma(Y_s)\big)\gamma(Y_s)\nu_{a_-,a_+}^\alpha(\mathrm{d}z)\,\mathrm{d}s\\ &\quad+\int_0^t\int_{\mathbb{R}_*}\mathbf{1}_{\{\sigma(Y_s)<0\}}\phi\big(s, -z/\sigma(Y_s)\big)\big|\gamma(Y_s)\big|\nu_{a_-,a_+}^\alpha(\mathrm{d}z)\,\mathrm{d}s\\ &\quad+\int_0^t\int_{\mathbb{R}_*}\mathbf{1}_{\{\sigma(Y_s)=0\}}\phi(s,z)\nu_{a_-,a_+}^\alpha(\mathrm{d}z)\,\mathrm{d}s.\end{aligned}$$

We perform the substitution $x = z/|\sigma(Y_s)|$, for which $\nu^\alpha_{a_-,a_+}(\mathrm{d}z) = |\sigma(Y_s)|^{-\alpha}\nu^\alpha_{a_-,a_+}(\mathrm{d}x)$, in the two first integrals. Recalling that $|\sigma(Y_s)|^{-\alpha}|\gamma(Y_s)| = 1$, we conclude that $\int_0^t\int_{\mathbb{R}_*}\phi(s,z)q(\mathrm{d}s\,\mathrm{d}z) = \int_0^t\int_{\mathbb{R}_*}\phi(s,z)\nu^\alpha_{a_-,a_+}(\mathrm{d}z)\,\mathrm{d}s$ as desired.

We now check point (ii). Let thus $(X_t)_{t\geq 0}$ solve (3) with some (α, a_-, a_+)-stable process $(Z_t)_{t\geq 0}$. Put $N = \sum_{s>0}\mathbf{1}_{\{s\in J\}}\delta_{(s,\Delta Z_s)}$, which is a Poisson measure on $[0,\infty)\times\mathbb{R}_*$ with intensity measure $\mathrm{d}s\,\nu^\alpha_{a_-,a_+}(\mathrm{d}z)$. On an enlarged probability space, we consider a Poisson measure $O(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u)$ on $[0,\infty)\times\mathbb{R}_*\times\mathbb{R}_*$ with intensity measure $\mathrm{d}s\,\nu^\alpha_{a_-,a_+}(\mathrm{d}z)\,\mathrm{d}u$ such that $N(\mathrm{d}s\,\mathrm{d}z) = O(\mathrm{d}s\,\mathrm{d}z\times[0,1])$. We finally introduce the random point measure $M(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u)$ on $[0,\infty)\times\mathbb{R}_*\times\mathbb{R}_*$ defined by

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}\varphi(s,z,u)M(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u)\\
&\quad=\int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}\mathbf{1}_{\{\sigma(X_{s-})\neq 0\}}\varphi\bigl(s, z\bigl|\sigma(X_{s-})\bigr|, u\gamma(X_{s-})\bigr)O(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u)\\
&\qquad+\int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}\mathbf{1}_{\{\sigma(X_{s-})=0\}}\varphi(s,z,u)O(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u)
\end{aligned}$$

for all measurable φ: $[0,\infty)\times\mathbb{R}_*\times\mathbb{R}_*$ sufficiently integrable. Then we have

$$\begin{aligned}
\int_0^t\sigma(X_{s-})\,\mathrm{d}Z_s &= \int_0^t\int_{\mathbb{R}_*} z\sigma(X_{s-})N(\mathrm{d}s\,\mathrm{d}z)\\
&= \int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} z\sigma(X_{s-})\mathbf{1}_{\{u\in[0,1]\}}O(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u)\\
&= \int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} z\cdot\operatorname{sign}\bigl(\sigma(X_{s-})\bigr)\mathbf{1}_{\{\sigma(X_{s-})\neq 0\}}\mathbf{1}_{\{u/\gamma(X_{s-})\in[0,1]\}}M(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u)\\
&= \int_0^t\int_{\mathbb{R}_*} z[\mathbf{1}_{\{0<u<\gamma(X_{s-})\}} - \mathbf{1}_{\{\gamma(X_{s-})<u<0\}}]M(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u).
\end{aligned}$$

We finally used that $\operatorname{sign}(\sigma(x)) = \operatorname{sign}(\gamma(x))$ and that $\sigma(x) = 0$ implies $\gamma(x) = 0$. It thus only remains to check that M is a Poisson measure with intensity measure $\mathrm{d}s\,\nu^\alpha_{a_-,a_+}(\mathrm{d}z)\,\mathrm{d}u$. Let us call p the compensator of M and observe that

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}\phi(s,z,u)p(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u)\\
&\quad=\int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}\mathbf{1}_{\{\sigma(X_{s-})\neq 0\}}\phi\bigl(s, z\bigl|\sigma(X_{s-})\bigr|, u\gamma(X_{s-})\bigr)\,\mathrm{d}u\,\nu^\alpha_{a_-,a_+}(\mathrm{d}z)\,\mathrm{d}s\\
&\qquad+\int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}\mathbf{1}_{\{\sigma(X_{s-})=0\}}\phi(s,z,u)\,\mathrm{d}u\,\nu^\alpha_{a_-,a_+}(\mathrm{d}z)\,\mathrm{d}s
\end{aligned}$$

for all measurable ϕ: $[0,\infty)\times\mathbb{R}_*\times\mathbb{R}_*$ sufficiently integrable. Performing the substitution $v = z|\sigma(X_{s-})|$, $w = u\gamma(X_{s-})$ and recalling that $|\sigma(X_{s-})|^{-\alpha}|\gamma(X_{s-})| = 1$, we easily conclude that $p(\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}u) = \mathrm{d}s\,\nu^\alpha_{a_-,a_+}(\mathrm{d}z)\,\mathrm{d}u$, which ends the proof. □

We finally end this paper with weak existence and moment estimates for (3).

Proof of Proposition 2. Let us for example treat the case where $\alpha\in(1,2)$. We divide the proof into several steps.

Step 1. Consider the equation

$$Y_t = x + \int_0^t\int_{|z|\leq 1}\sigma(Y_{s-})z\tilde{N}(\mathrm{d}s\,\mathrm{d}z) + \int_0^t c(Y_s)\,\mathrm{d}s, \tag{15}$$

where $c(x)=b(x)-\sigma(x)\int_{|z|\geq 1} z\nu^\alpha_{a_-,a_+}(\mathrm{d}z)$. If b and σ have at most linear growth, one immediately checks, using that $\int_{|z|\leq 1} z^2\nu^\alpha_{a_-,a_+}(\mathrm{d}z)<\infty$, that for any $T>0$, for some constant C_T not depending on x, any solution to (15) satisfies

$$\mathbb{E}\Big[\sup_{[0,T]} Y_t^2\Big]\leq C_T\big(1+x^2\big). \tag{16}$$

Finally observe that we can rewrite (3) as

$$X_t=x+\int_0^t\int_{|z|\leq 1}\sigma(X_{s-})z\tilde{N}(\mathrm{d}s\,\mathrm{d}z)+\int_0^t c(X_s)\,\mathrm{d}s+\int_0^t\int_{|z|>1}\sigma(X_{s-})zN(\mathrm{d}s\,\mathrm{d}z). \tag{17}$$

Step 2. We now prove the moment estimates. We have not found a direct proof relying on stochastic calculus. Fix $\beta\in(0,\alpha)$, $T>0$ and assume only that b,σ have at most linear growth. Consider a solution $(X_t)_{t\geq 0}$ to (3) and rewrite it as (17).

The last integral of (17) generates jumps at some discrete instants: write the restriction of N to $[0,\infty)\times\{|z|\geq 1\}$ as $\sum_{n\geq 1}\delta_{(T_n,Z_n)}$, where $0<T_1<T_2<\cdots$ are the jump instants of a Poisson process with parameter $\lambda=\int_{|z|\geq 1}\nu^\alpha_{a_-,a_+}(\mathrm{d}z)$ and where the random variables $(Z_n)_{n\geq 1}$ are i.i.d. with law $\lambda^{-1}\nu^\alpha_{a_-,a_+}(\mathrm{d}z)$. Hence (3) reduces to (15) on each time interval (T_n,T_{n+1}).

Denote by $\mathcal{G}=\sigma(T_1,T_2,\ldots)$. Then X_t solves (15) during $[0,T_1)$. Hence we have

$$\mathbb{E}\Big[\sup_{[0,T_1\wedge T)} X_t^2\big|\mathcal{G}\Big]\leq C_T\big(1+x^2\big)\quad\text{whence } \mathbb{E}\Big[\sup_{[0,T_1\wedge T)}|X_t|^\beta\big|\mathcal{G}\Big]\leq K_T\big(1+|x|^\beta\big).$$

Furthermore, $X_{T_1}=X_{T_1-}+\sigma(X_{T_1-})Z_1$, whence, since σ has at most linear growth, $|X_{T_1}|^\beta\leq L(1+|X_{T_1-}|^\beta)(1+|Z_1|^\beta)$. Consequently, we have

$$\mathbb{E}\Big[\sup_{[0,T_1\wedge T]}|X_t|^\beta\big|\mathcal{G}\Big]\leq M_T\big(1+|x|^\beta\big),$$

where $M_T=K_T+L(1+K_T)\mathbb{E}[1+|Z_1|^\beta]<\infty$ (here we need that $\beta<\alpha$ to have $\mathbb{E}[|Z_1|^\beta]<\infty$). Exactly in the same way, since $(X_t)_{t\geq 0}$ solves (15) during (T_k,T_{k+1}) for any $k\geq 1$, one can prove that

$$\mathbb{E}\Big[\sup_{[T_k\wedge T,T_{k+1}\wedge T]}|X_t|^\beta\big|\mathcal{G}\Big]\leq M_T\big(1+\mathbb{E}\big[|X_{T_k}|^\beta|\mathcal{G}\big]\big)$$

with the same constant M_T. Put $u_k=\mathbb{E}[\sup_{[T_k\wedge T,T_{k+1}\wedge T]}|X_t|^\beta|\mathcal{G}]$ for $k\geq 0$ (set $T_0=0$). We have proved that $u_0\leq M_T(1+|x|^\beta)$ and that $u_{k+1}\leq M_T(1+u_k)$. We classically deduce that for some constant $A_T>1$, depending on x, $u_k\leq A_T^{k+1}$. Consequently, for any $k\geq 1$,

$$\mathbb{E}\Big[\sup_{[0,T_k\wedge T]}|X_t|^\beta\big|\mathcal{G}\Big]\leq u_0+\cdots+u_{k-1}\leq A_T+\cdots+A_T^k\leq\frac{A_T^{k+1}}{A_T-1}.$$

Finally, we find

$$\begin{aligned}\mathbb{E}\Big[\sup_{[0,T]}|X_t|^\beta\Big]&\leq\sum_{k\geq 0}\mathbb{E}\Big[\mathbf{1}_{\{T_k<T<T_{k+1}\}}\mathbb{E}\Big(\sup_{[0,T_{k+1}\wedge T]}|X_t|^\beta\big|\mathcal{G}\Big)\Big]\\&\leq\frac{1}{A_T-1}\sum_{k\geq 0}A_T^{k+2}\frac{(\lambda T)^k}{k!}\mathrm{e}^{-\lambda T}<\infty.\end{aligned}$$

Step 3. To prove weak existence for (3), we assume that b and σ are continuous with at most linear growth. We introduce the approximate equation, for $n \geq 1$,

$$X_t^n = x + \int_0^t \sigma\big(X_{s-}^n\big)\,\mathrm{d}Z_s^n + \int_0^t b\big(X_s^n\big)\,\mathrm{d}s, \tag{18}$$

where $(Z_t^n)_{t\geq 0}$ is a Lévy process with Lévy measure $\mathbf{1}_{\{|z|\leq n\}}\nu^\alpha_{a_-,a_+}(\mathrm{d}z)$ (compensated, without drift and without Brownian part). Since $\int_{|z|\leq n} z^2 \nu^\alpha_{a_-,a_+}(\mathrm{d}z) < \infty$, we can apply Theorem 175 of Situ [16] to deduce that for each $n \geq 1$, (18) has a weak solution $(X_t^n)_{t\geq 0}$. Exactly as in Step 2, for any $\beta \in (0,\alpha)$, any $T > 0$,

$$\sup_{n\geq 1} \mathbb{E}\Big[\sup_{[0,T]} \big|X_t^n\big|^\beta\Big] < \infty. \tag{19}$$

We now use Aldous' criterion, see Jacod and Shiryaev [9], Theorem 4.5, p. 356, to check that the sequence of processes $((X_t^n)_{t\geq 0})_{n\geq 1}$ is tight in $\mathbb{D}([0,\infty),\mathbb{R})$. For $T > 0$ and $\delta > 0$, we introduce the set $\mathcal{A}_T(\delta)$ of all pairs of stopping times (S,S') satisfying a.s. $0 \leq S \leq S' \leq S+\delta \leq T$. We have to check that for any $\eta > 0$,

$$\lim_{\delta\to 0} \sup_{n\geq 1} \sup_{(S,S')\in\mathcal{A}_T(\delta)} \Pr\big[\big|X_{S'}^n - X_S^n\big| > \eta\big] = 0. \tag{20}$$

To do so, we consider a Poisson measure $N_n(\mathrm{d}s\,\mathrm{d}z)$ on $[0,\infty)\times\mathbb{R}_*$ with intensity measure $\mathrm{d}t\,\mathbf{1}_{\{|z|\leq n\}}\nu^\alpha_{a_-,a_+}(\mathrm{d}z)$ such that $Z_t^n = \int_0^t \int_{\mathbb{R}_*} z\tilde{N}_n(\mathrm{d}s\,\mathrm{d}z)$. We also introduce, for $A > 0$, the stopping time $\tau_n^A = \inf\{t \geq 0\colon |X_t^n| \geq A\}$. An immediate computation shows that for any $A > 0$, any $n \geq 1$, any $(S,S') \in \mathcal{A}_T(\delta)$,

$$\begin{aligned}\big|X^n_{S'\wedge\tau_n^A} - X^n_{S\wedge\tau_n^A}\big| &\leq \left|\int_{S\wedge\tau_n^A}^{S'\wedge\tau_n^A}\int_{|z|\leq 1} \sigma\big(X_{s-}^n\big) z\tilde{N}_n(\mathrm{d}s\,\mathrm{d}z)\right| \\ &\quad + \int_{S\wedge\tau_n^A}^{S'\wedge\tau_n^A}\int_{|z|\geq 1} \big|\sigma\big(X_{s-}^n\big)\big||z| N_n(\mathrm{d}s\,\mathrm{d}z) + \int_{S\wedge\tau_n^A}^{S'\wedge\tau_n^A} \big|c_n\big(X_{s-}^n\big)\big|\mathrm{d}s \\ &=: I_n^{1,A} + I_n^{2,A} + I_n^{3,A},\end{aligned}$$

where $c_n(x) = b(x) - \sigma(x)\int_{1<|z|<n} \nu^\alpha_{a_-,a_+}(\mathrm{d}z)$. Since b and σ have at most linear growth, $\sup_{[0,\tau_n^A)}[|\sigma(X_s^n)| + |c_n(X_s^n)|] \leq C(1+A)$ a.s. for some constant C. Thus standard computations (recall that $0 \leq S \leq S' \leq S+\delta \leq T$ a.s.) show that $\mathbb{E}[(I_n^{1,A})^2] \leq C(1+A)^2\delta$, $\mathbb{E}[I_n^{2,A}] \leq C(1+A)\delta$ and $\mathbb{E}[I_n^{3,A}] \leq C(1+A)\delta$. Consequently, $\mathbb{E}[|X^n_{S'\wedge\tau_n^A} - X^n_{S\wedge\tau_n^A}|] \leq C(1+A)\sqrt{\delta}$ for all $\delta \in (0,1)$, the constant C depending only on $a_-, a_+, \alpha, b, \sigma$. Finally, using (19) with $\beta = 1$, we see that $\Pr[\tau_n^A \leq T] \leq \Pr[\sup_{[0,T]}|X_s^n| > A] \leq C_T/A$. Hence for any $\eta > 0$, any value of $A > 0$,

$$\begin{aligned}\Pr\big[\big|X_{S'}^n - X_S^n\big| > \eta\big] &\leq \Pr\big[\big|X^n_{S'\wedge\tau_n^A} - X^n_{S\wedge\tau_n^A}\big| > \eta\big] + \Pr\big[\tau_n^A \leq T\big] \\ &\leq \frac{C(1+A)\sqrt{\delta}}{\eta} + \frac{C_T}{A}.\end{aligned}$$

Choosing $A = \delta^{-1/4}$, we finally get, for all $\delta \in (0,1)$,

$$\Pr\big[\big|X_{S'}^n - X_S^n\big| > \eta\big] \leq \frac{C(1+\delta^{-1/4})\sqrt{\delta}}{\eta} + C_T\delta^{1/4} \leq C_T(1+1/\eta)\delta^{1/4},$$

where C_T depends only on $T, a_-, a_+, \alpha, b, \sigma$. This implies (20) and ends the step.

Step 4. We finally conclude. Consider, for each $n \geq 1$, a (weak) solution $(X_t^n)_{t\geq 0}$ to (18) driven by $(Z_t^n)_{t\geq 0}$. The sequence $((Z_t^n)_{t\geq 0})_{n\geq 1}$ is obviously tight, since it goes in law, in $\mathbb{D}([0,\infty),\mathbb{R})$, to the (α, a_-, a_+)-stable process

$(Z_t)_{t\geq 0}$. Using Step 3, we deduce that, up to extraction of a (not relabelled) subsequence, $(X_t^n, Z_t^n)_{t\geq 0}$ converges in law to some $(X_t, Z_t)_{t\geq 0}$ in $\mathbb{D}([0,\infty), \mathbb{R}^2)$. By continuity of b, σ, it follows that $(X_t^n, \sigma(X_t^n), b(X_t^n), Z_t^n)_{t\geq 0}$ converges in law $(X_t, \sigma(X_t), b(X_t), Z_t)_{t\geq 0}$ in $\mathbb{D}([0,\infty), \mathbb{R}^4)$.

Using [9], Corollary 6.30, p. 385, the sequence $(Z_t^n)_{t\geq 0}$ satisfies the P-UT property. Indeed, it suffices to check that $\sup_{n\geq 1} \mathbb{E}[\sup_{[0,t]} |\Delta Z_s^n|] < \infty$ for all $t > 0$. But with the notation of Step 3,

$$\mathbb{E}\Big[\sup_{[0,t]}\big|\Delta Z_s^n\big|\Big] \leq \mathbb{E}\bigg[1 + \int_0^t \int_{|z|\geq 1} |z| N_n(\mathrm{d}s\,\mathrm{d}z)\bigg]$$
$$\leq 1 + t\int_{1\leq |z|\leq n} |z|\nu^\alpha_{a_-,a_+}(\mathrm{d}z) \leq 1 + Ct$$

since $\alpha > 1$. We simply used that the supremum of the jumps is smaller than 1 plus the sum of all the jumps greater than 1.

Applying [9], Theorem 6.22, p. 383, the sequence $(X_t^n, \int_0^t \sigma(X_s^n)\,\mathrm{d}Z_s^n, \int_0^t b(X_s^n)\,\mathrm{d}s, Z_t^n)_{t\geq 0}$ thus converges in law to $(X_t, \int_0^t \sigma(X_{s-})\,\mathrm{d}Z_s, \int_0^t b(X_s)\,\mathrm{d}s, Z_t)_{t\geq 0}$ in $\mathbb{D}([0,\infty), \mathbb{R}^4)$. Passing to the limit in (18), we deduce that $X_t = x + \int_0^t \sigma(X_{s-})\,\mathrm{d}Z_s + \int_0^t b(X_s)\,\mathrm{d}s$. We have built a weak solution to (3). □

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