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ON PERIODIC AND MULTIPLE AUTOREGRESSIONS[1]

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A methodology is presented for analyzing periodic autoregressions which is also applicable when inferring the second order properties of periodically correlated processes. In addition, capitalizing on the connection between periodic and multiple autoregressions, a method is set forth for analyzing the latter, which overcomes the usual requirements of a large number of both parameters and computer storage locations. This is achieved by introducing an orthogonal parametrization for multiple autoregressions.

1. Introduction. Given a time series $\{Y(t); t = 0, \pm 1, \cdots\}$ whose second order moments exist, define its mean function, $m(t) = EY(t)$, and its covariance kernel, $R(s, t) = E\{Y(s) - m(s)\}\{Y(t) - m(t)\}$. A class of nonstationary processes which readily lends itself to analysis, and is of practical importance (see [9], [5], [16], [17]) is the class of periodically correlated processes (see [2]).

DEFINITION. The process $Y(\cdot)$ is said to be *periodically correlated* of period d, if for some positive integer d and for all integers s, t,

$$m(t) = m(t + d), \qquad R(s, t) = R(s + d, t + d).$$

Since we propose dealing with the second order properties of the process, without loss of generality take $m(t) = 0$.

Periodically correlated processes are not only of interest in their own right, but, because of their connection with multivariate covariance stationary time series, they also provide insight into and modeling facility for these series. This claim is based, in the main, upon the following construction: define the jth component of the d-dimensional vector $\mathbf{X}(t)$ by (see [2]),

$$X_j(t) = Y(j + d(t - 1)) \tag{1.1}$$

and the covariance kernel of $\mathbf{X}(\cdot)$ by

$$R_{jk}(s, t) = EX_j(s)X_k(t), \qquad 1 \leqslant j, k \leqslant d, \ s, t = 0, \pm 1, \cdots .$$

By noting that

$$R_{jk}(s, t) = R(j + ds, k + dt), \tag{1.2}$$

THEOREM. (*Gladyshev* [2]). *The process* $Y(\cdot)$ *is periodically correlated of period* d *if, and only if, the process* $\mathbf{X}(\cdot)$ *is covariance stationary.*

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This theorem shows that, although not covariance stationary, the process $Y(\cdot)$, nonetheless, does not deviate too much from stationarity; and, in fact, its second order properties may be deduced from those of $\mathbf{X}(\cdot)$.

We can further capitalize on this connection but in the other direction: given a covariance stationary process $\mathbf{X}(\cdot)$ define the associated periodically correlated process, $Y(\cdot)$, by (1.1). This is especially useful when $\mathbf{X}(\cdot)$ is a multiple autoregression. For then by investigating $Y(\cdot)$, we actually effect an easily calculated orthogonal decomposition of the process $\mathbf{X}(\cdot)$, thus achieving the usual gain in simplicity and power associated with orthogonalization. These points are amplified in the remainder of the paper.

2. Autoregressions. Given a sample $\mathbf{X}(1), \cdots, \mathbf{X}(T)$ from a d-dimensional zero mean covariance stationary time series with spectral density matrix $f(\omega)$, $-\pi \leqslant \omega \leqslant \pi$, the problem is to estimate $f(\cdot)$. Under some very mild restrictions on $f(\cdot)$, [7], it can be written as the spectral density of an infinite order autoregression. Parzen [13] thus proposes treating the sample as one from a pth order autoregression,

$$\mathbf{X}(t) + \Sigma_{j=1}^{p} A(j)\mathbf{X}(t-j) = \boldsymbol{\varepsilon}(t) \tag{2.1}$$

where $\mathrm{Cov}(\boldsymbol{\varepsilon}(t)) = \Sigma$, and by choosing p large enough, $f(\cdot)$ can be arbitrarily closely approximated. Parzen [15] gives a method for choosing p which accommodates both the numerical approximation, which argues for a large p, and the requisite statistical estimation, which argues for a small p. The autoregressive approximants have been used in practice with success (see [4], [14] and references therein, also in geophysics [17] where a closely analogous method, the method of maximal entropy, is used). For these reasons it seems important to study autoregressive processes.

The periodically correlated analogue of an autoregression is given by (see [5]),

DEFINITION. A process $Y(\cdot)$ is said to be a *periodic autoregression* of period d and order $(p_1, \cdots, p_d)$ if for all integers t,

$$Y(t) + \Sigma_{j=1}^{p_t} \alpha_t(j) Y(t-j) = \varepsilon(t) \tag{2.2}$$

where the $\varepsilon(\cdot)$ are uncorrelated with mean zero and $E\varepsilon^2(t) = \sigma_t^2$, $p_t = p_{t+d}$, $\sigma_t^2 = \sigma_{t+d}^2$, and $\alpha_t(j) = \alpha_{t+d}(j)$, $j = 1, \cdots, p_t$.

THEOREM 1. *If* $\mathbf{X}(\cdot)$ *and* $Y(\cdot)$ *are associated by* (1.1), *then* $\mathbf{X}(\cdot)$ *is an autoregression of order p with positive definite* Σ *if, and only if,* $Y(\cdot)$ *is a periodic autoregression of period d and order* $(p_1, \cdots, p_d)$ *with positive* $\sigma_1^2, \cdots, \sigma_d^2$, *and,* $p = \max_j[(p_j - j)/d] + 1$, *where, for integral j,* $[x] = j$ *for* $j \leqslant x < j+1$.

PROOF. If $Y(\cdot)$ is a periodic autoregression of order $(p_1, \cdots, p_d)$ then it may be written as (see Theorem 3)

$$L\mathbf{X}(t) + \Sigma_{j=1}^{p} \mathbf{A}'(j) X(t-j) = \boldsymbol{\varepsilon}'(t) \tag{2.3}$$

where L is a unit (ones on the diagonal) lower triangular matrix, p satisfies the condition of the theorem, and the $\varepsilon'(\cdot)$ are uncorrelated with a diagonal covariance matrix, $E\boldsymbol{\varepsilon}'(t)\boldsymbol{\varepsilon}'^T(t) = D = \text{diag}(\sigma_1^2, \cdots, \sigma_d^2)$. Therefore,

$$\mathbf{X}(t) + \Sigma_{j=1}^{p} A(j)\mathbf{X}(t-j) = \boldsymbol{\varepsilon}(t)$$

where

$$A(j) = L^{-1}A'(j) \qquad j = 1, \cdots, p$$
$$\boldsymbol{\varepsilon}(t) = L^{-1}\boldsymbol{\varepsilon}'(t) \qquad t = 0, \pm 1, \cdots,$$

and the $\boldsymbol{\varepsilon}(\cdot)$ are thus uncorrelated and have covariance matrix $\Sigma = L^{-1}DL^{-T}$, which is positive definite since D is.

Conversely, suppose

$$\mathbf{X}(t) + \Sigma_{j=1}^{p} A(j)\mathbf{X}(t-j) = \boldsymbol{\varepsilon}(t)$$

where the $\boldsymbol{\varepsilon}(\cdot)$ are uncorrelated with positive definite covariance matrix Σ. Define the unique modified Cholesky decomposition of Σ by $\Sigma = LDL^T$, where L is unit lower triangular and D is positive definite diagonal, $D = \text{diag}(\sigma_1^2, \cdots, \sigma_d^2)$. Then L^{-1} is unit lower triangular and $\mathbf{X}(\cdot)$ satisfies

$$L^{-1}\mathbf{X}(t) + \Sigma_{j=1}^{p} A'(j)\mathbf{X}(t-j) = \boldsymbol{\varepsilon}'(t)$$

where $A'(j) = L^{-1}A(j), j = 1, \ldots,$ and the $\boldsymbol{\varepsilon}'(t) = L^{-1}\boldsymbol{\varepsilon}(t)$, and are thus uncorrelated with diagonal positive definite covariance matrix, proving the theorem. □

Note that without too much difficulty we could accommodate a process for which Σ is singular.

DEFINITION. A process $Y(\cdot)$ is said to be a *covariance stationary periodic autoregression* if it is a periodic autoregression and, furthermore, the associated vector process $\mathbf{X}(\cdot)$, (1.1), is covariance stationary.

As usual (cf. [3], page 21, footnote) by a stationary multiple autoregression, we mean a process $\mathbf{X}(\cdot)$ which obeys (2.1) and admits to an expression, in the mean square sense, in terms of the "past" $\boldsymbol{\varepsilon}(\cdot)$, i.e.,

$$\mathbf{X}(t) = \boldsymbol{\varepsilon}(t) + \Sigma_{k=1}^{\infty} B(k)\boldsymbol{\varepsilon}(t-k).$$

For this to be true, the zeroes of the determinantal polynomial, $\det(I_d + \Sigma_{j=1}^{p} A(j)z^j)$ must lie outside the unit circle.

One of the values of an autoregressive process is, of course, the facility with which one may perform linear prediction. It can easily be shown, see [19], that the least squares predictor of $Y(t+h)$, for positive integer h, on the basis of $Y(t)$, $Y(t-1), \cdots$, is

$$Y(t+h|t) = -\Sigma_{j=1}^{h-1}\alpha_{t+h}(j)Y(t+h-j|t) - \Sigma_{j=h}^{p_{t+h}}\alpha_{t-h}(j)Y(t+h-j)$$

if $Y(\cdot)$ is a covariance stationary periodic autoregression. To obtain the prediction error, one must formally solve (see (3.8)),

$$Y(t+h) = -\Sigma_{j=1}^{p_{t+h}}\alpha_{t+h}(j)Y(t+h-j) + \varepsilon(t+h) = \Sigma_{k=0}^{\infty}\beta_{t+h}(k)\varepsilon(t+h-k)$$

with $\beta_{t+h}(0) = 1$, in which case

$$E\{Y(t+h|t) - Y(t+h)\}^2 = \Sigma_{k=0}^{h-1}\beta_{t+h}^2(k)\sigma_{t+h-k}^2.$$

A side benefit is that the above method yields a solution to the problem of predicting only a subset of $\mathbf{X}(t+1), \mathbf{X}(t+2), \cdots$, in terms of $\mathbf{X}(t), \mathbf{X}(t-1), \cdots$, for an autoregressive $\mathbf{X}(\cdot)$.

3. Parameter estimation. Given a sample $Y(1), \cdots, Y(T)$ from a zero mean Gaussian covariance stationary periodic autoregression of order $(p_1, \cdots, p_d)$ and period d, we wish to make an inference about the parameters $\alpha_k(j)$ and σ_j^2, $j = 1, \cdots, p_k$, $k = 1, \cdots, d$. We show that the results of Mann and Wald [6] extend to this case.

To simplify the notation we take $T = Nd$, where N is a natural number. Then the problem is equivalent to having a sample $\mathbf{X}(1), \cdots, \mathbf{X}(N)$ from a stationary autoregression, where $\mathbf{X}(\cdot)$ and $Y(\cdot)$ are related by (1.1). So, define

$$R_N(k, v) = N^{-1}\Sigma_{j=0}^{m} Y(k+dj)Y(v+dj) \tag{3.1}$$

where $m = [N - \max(k, v)/d]$, for $k = 1, \cdots, d$, $v = 0, 1, \cdots, T-k-1$.

LEMMA 1. *If $Y(\cdot)$ is a periodically correlated Gaussian process, then, with $R_N(k, v)$ defined by* (3.1), *as $T \to \infty$, the $R_N(k, v)$ converge almost surely and in mean square to $R(k, v)$, and*

$$N\,\mathrm{Cov}\{R_N(k_1, v_1), R_N(k_2, v_2)\} \to \Sigma_{u=-\infty}^{\infty}\{R(k_1, k_2+du)R(v_1, v_2+du) + R(k_1, v_2+du)R(v_1, k_2+du)\}. \tag{3.2}$$

Furthermore, $N^{\frac{1}{2}}(R_N(k, v) - R(k, v))$ are asymptotically Gaussian, with zero mean and the above covariance function, for $k = 1, \cdots, d$ and $v = 0, 1, \cdots, q$ for any fixed q.

PROOF. By using (1.1) and (1.2) and Gladyshev's theorem, we see that the consistency and asymptotic covariance are merely an extension of Slutsky's result; see [3], pages 209, 210. The asymptotic normality follows from [3], page 228. □

In a periodic autoregression, the α and σ are related to the covariance kernel R in a modified Yule-Walker form:

THEOREM 2. *If $Y(\cdot)$ is a covariance stationary periodic autoregression of order $(p_1, \cdots, p_d)$ with covariance kernel $R(\cdot, \cdot)$, then for $k = 1, \cdots, d$, and using the Kronecker delta,*

$$R(k, k-v) + \Sigma_{j=1}^{p_k}\alpha_k(j)R(k-j, k-v) = \delta_{v0}\sigma_k^2, \qquad v \geqslant 0. \tag{3.3}$$

PROOF. We must first show that

$$Y(t) = -\Sigma_{j=1}^{p_t}\alpha_t(j)Y(t-j) + \varepsilon(t) = \varepsilon(t) + \Sigma_{k=1}^{\infty}\beta_t(k)\varepsilon(t-k).$$

This result is immediately available from Theorem 1. Therefore, for positive v,

$Y(t - v)$ is uncorrelated with $\varepsilon(t)$. Multiplying both sides of (2.2) by $Y(t - v)$ and taking expected values yields the theorem. □

We are thus immediately led to Fisher-consistent estimators of the α and σ^2; in (3.3) replace the R by the appropriate R_N and solve the resulting linear equations. We proceed to find the properties of these estimators.

If $Y(\cdot)$ is a periodically correlated process, denote by $\mathcal{I}$ the Fisher information matrix, and by $\mathcal{I}(\alpha, \beta)$ the element in this matrix corresponding to the parameters α and β.

THEOREM 3. *If $Y(\cdot)$ is a Gaussian covariance stationary periodic autoregression of order $(p_1, \cdots, p_d)$, then the information matrix is block diagonal,*

$$\mathcal{I}(\alpha_k(j), \alpha_m(l)) = \delta_{km} R(k - j, k - l)/\sigma_k^2,$$
$$\mathcal{I}(\alpha_k(j), \sigma_m^2) = 0$$
$$\mathcal{I}(\sigma_k^2, \sigma_m^2) = \delta_{km}/2\sigma_k^4,$$

for $j = 1, \cdots, p_k$, $l = 1, \cdots, p_m$, $k, m = 1, \cdots, d$.

PROOF. Using (1.1) then,

$$L\mathbf{X}(t) + \Sigma_{j=1}^{p} A(j)\mathbf{X}(t - j) = \boldsymbol{\varepsilon}(t)$$

where

$$L_{kj} = \alpha_k(k - j), \qquad j < k$$
$$A_{kj}(v) = \alpha_k(dv + k - j), \qquad v = 1, 2, \cdots, p$$

and p as in Theorem 1. Then, defining $D = \operatorname{diag}(\sigma_1^2, \cdots, \sigma_d^2)$, and,

$$G(z) = L + \Sigma_{j=1}^{p} A(j) z^j, \qquad z = e^{i\omega},$$

we have that the spectral density matrix of the $\mathbf{X}(\cdot)$ process is

$$f(\omega) = G^{-1}(z) D G^{-*}(z)/2\pi,$$

where (minus) the asterisk denotes the (inverse of the) complex conjugate transpose. We have, see [18], that

$$\mathcal{I}(\alpha, \beta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \operatorname{tr}\left(\frac{\partial f^{-1}}{\partial \alpha} f \frac{\partial f^{-1}}{\partial \beta} f \right)(\omega)\, d\omega. \tag{3.4}$$

Let E_{jk} denote the d-dimensional square matrix with a one in the (j, k) component and zeroes elsewhere. Then

$$\frac{\partial f^{-1}}{\partial A_{kj}(v)} f = E_{jk} G^{-*} z^{-v} + G^* D E_{kj} f z^{v} \tag{3.5}$$

$$\frac{\partial f^{-1}}{\partial L_{kj}} f = E_{jk} G^{-*} + G^* D E_{kj} f, \qquad j < k \tag{3.6}$$

$$\frac{\partial f^{-1}}{\partial \sigma_k^2} f = -G^* E_{kk} G^{-*}/\sigma_k^2. \tag{3.7}$$

Using the formula,

$$G^{-1}(z) = L^{-1} + \Sigma_{k=1}^{\infty} B(k) z^k, \tag{3.8}$$

where the $B(\cdot)$ decay exponentially to zero because of the assumption on the zeroes of $\det(G(z))$, we see that the terms of the form

$$\operatorname{tr} \int_{-\pi}^{\pi} E_{kj} G^{-1} E_{mn} G^{-1} e^{\pm i(v+u)\omega}\, d\omega = 0$$

for all k, j, m, n if $v + u \neq 0$, and for $k > j$ or $m > n$ if $v + u = 0$ (see [10]). Upon substitution of the appropriate (3.5), (3.6) and/or (3.7) into (3.4), and some straightforward algebra, the proof follows. □

If, with the R_N defined in (3.1), we define the $\hat{\alpha}$ and $\hat{\sigma}$ as solutions to the normal equations,

$$R_N(k, k - v) + \Sigma_{j=1}^{p_k} \hat{\alpha}_k(j) R_N(k - j, k - v) = \delta_{v0} \hat{\sigma}_k^2 \tag{3.9}$$

for $v = 0, \cdots, p_k$ and $k = 1, \cdots, d$, and the vectors $\hat{\boldsymbol{\alpha}}_k = (\hat{\alpha}_k(1), \cdots, \hat{\alpha}_k(p_k))^T$, then,

THEOREM 4. *If $Y(1), \cdots, Y(T)$ is a sample from a covariance stationary Gaussian periodic autoregression of order $(p_1, \cdots, p_d)$, then the $\hat{\alpha}$ and $\hat{\sigma}$ defined by* (3.9) *are almost surely consistent estimators ($T \to \infty$). And $N^{\frac{1}{2}}(\hat{\boldsymbol{\alpha}}_k - \boldsymbol{\alpha}_k)$ $k = 1, \cdots, d$ have an asymptotic distribution which is Gaussian with mean zero and covariance matrix $\mathcal{I}^{-1}$, where $\mathcal{I}$ is the appropriate block of the information matrix given in Theorem* 3. *Thus the estimators are, in this sense, asymptotically efficient.*

PROOF. The consistency follows from Lemma 1 and Theorem 2. To show the asymptotic distribution, as in [12], we have from (3.3) and (3.9) for $k = 1, \cdots, d$ and $v = 1, \cdots, p_k$,

(3.10)

$$\Sigma_{j=0}^{p_k} R_N(k - j, k - v)\{\hat{\alpha}_k(j) - \alpha_k(j)\} = \Sigma_{j=0}^{p_k} \alpha_k(j)\{R_N(k - j, k - v) - R(k - j, k - v)\}$$

defining $\alpha_k(0) = \hat{\alpha}_k(0) = 1$. From Lemma 1 the random variables on the right of (3.10) are asymptotically Gaussian with mean zero. To find their asymptotic covariance matrix, consider for v_1 and v_2 positive,

$$N \Sigma_{j_1=0}^{p_{k_1}} \Sigma_{j_2=0}^{p_{k_2}} \alpha_{k_1}(j_1) \alpha_{k_2}(j_2) \operatorname{Cov}\{R_N(k_1 - j_1, k_1 - v_1), R_N(k_2 - j_2, k_2 - v_2)\} = \sigma_{k_1}^2 \delta_{k_1, k_2} R(k_1 - v_1, k_1 - v_2),$$

from Lemma 1 and a repeated application of (3.3). Now, from Lemma 1 and Cramér's theorem, ([1] page 254) the random variables on the left of (3.10) have the same joint asymptotic distribution as

$$\Sigma_{j=0}^{p_k} R(k - j, k - v)\{\hat{\alpha}_k(j) - \alpha_k(j)\}, \qquad v = 1, \cdots, p_k, \quad \text{and} \quad k = 1, \cdots, d$$

and the theorem is proved. □

A noteworthy result contained in the last two theorems is that, asymptotically, the $\hat{\boldsymbol{\alpha}}_k$, $k = 1, \cdots, d$, are independent. Thus, when analyzing a multivariate autoregression, the parametrization in terms of the $\boldsymbol{\alpha}_k$, as opposed to the $A(j)$ in (2.1), allows us to analyze each channel separately.

A further advantage to dealing with the $\boldsymbol{\alpha}_k$, as opposed to the $A(j)$, is a practical one. Since the different orders of the channels are *not* necessarily such that $dp = p_k - k + 1$, $k = 1, \cdots, d$, we can model a multivariate autoregression of order p (with $A(p) \neq 0$) with fewer than $d^2p + d(d+1)/2$ parameters, and we thus have a general methodology for systematically reducing the number of parameters required. The number of parameters in terms of the $A(j)$ can be sizable; for example, if $d = 12$ (monthly data), to introduce some correlation structure into the data at the lowest level, a zero order autoregression, would require 78 parameters; if a zero order is judged inadequate (if the spectrum should not be flat) then a first order would require 222 parameters. This is the difficulty expressed by Whittle [18], "... of analyzing a d-tuple series [which] may be said to increase roughly as d^2 (the number of auto- and cross-correlograms which must be calculated, and the order of the number of parameters to be estimated), while the number of observations increases only as d." In terms of the $\boldsymbol{\alpha}_k$ we overcome this difficulty; for example, we can obtain a nonflat spectrum with only one of the $\boldsymbol{\alpha}_k$ nonzero, $\alpha_1(1)$. It could be said that we have defined multiple autoregressions of noninteger orders.

4. Discussion. Viewing a multiple autoregression as a periodic autoregression clearly displays the effects of prewhitening each channel before doing a joint analysis. If the prewhitening is done with an autoregressive filter on each channel, then this can be viewed as, first, fitting a periodic autoregression with $\alpha_k(d), \alpha_k(2d), \cdots$, $k = 1, \cdots, d$, as the only nonzero coefficients, and, second, performing a periodic autoregression on the residuals. An alternative approach would be to include a subset regression option when fitting the periodic autoregression (such as in [8], for example) and this would obviate the need for prewhitening.

In analyzing atmospheric data, Jones and Brelsford [5] achieved a reduction in the number of parameters by expanding the α in Fourier series,

$$\alpha_k(j) = \Sigma_{n=0}^{m}\{c_{jn}\cos(2\pi nk/d) + s_{jn}\sin(2\pi nk/d)\}, \tag{4.1}$$

taking m small (relative to $d/2$), arguing that the α were slowly varying (with respect to k) and periodic of period d. Using the asymptotic distribution in Theorem 4, one can obtain efficient estimators of the c's and s's by performing a weighted regression of the $\hat{\alpha}$, in (3.9), on the c's and s's (as in [11]). Indeed, these estimators are not too different from those in [5] but this approach would provide a method for systematically testing the hypothesis exhibited in (4.1).

The Gaussian assumptions made in Section 3 can clearly be relaxed. Another generalization can be achieved by considering periodically correlated q-dimensional vector processes $\mathbf{Y}(\cdot)$, in which case $\mathbf{X}(\cdot)$ would be a dq-dimensional covariance stationary time series [2].

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