ON PERIODIC MAPS AND THE EULER CHARACTERISTICS OF ASSOCIATED SPACES

BY
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1. **Introduction.** Let X be a locally compact finite-dimensional Hausdorff space, and let T be a periodic map of prime period p operating on X. Let L denote the fixed point set of T, and let Y denote the orbit decomposition space of X and T, which has as elements the sets $[T^i(x) \mid i=0, 1, \cdots, p-1]$ for $x \in X$. Let us consider first the case in which X is a finite complex, T is simplicial, and the natural decomposition map $f: X \rightarrow Y$ is simplicial and therefore a homeomorphism on each simplex of X. Let v be a simplex of Y. If $f^{-1}(v) \subset L$, then $f^{-1}(v)$ contains exactly one simplex. Otherwise $f^{-1}(v)$ contains exactly p simplexes. As a consequence $\chi(X)+(p-1)\chi(L)=p\chi(Y)$, where χ indicates the Euler characteristic. The similarity of this formula to a result of G. T. Whyburn [8; p. 202].(1) concerning interior maps on 2-manifolds will be noted.

The main purpose of this paper is to provide an analogue of this formula under more general circumstances. We use for X and T any pair satisfying the requirements of the first sentence, with the restriction that the Čech homology groups $H_n(X)$, with the integers mod p as coefficient group, are all finitely generated. We prove that the same formula then holds if we define $\chi(A)$ to be $\sum(-1)^i \dim H_i(A)$ whenever this is defined, where $\dim H_i(A)$ denotes the minimum number of generators of $H_i(A)$ (that is, its dimension as a vector space over the integers mod p).

We use P. A. Smith's theory of special homology groups [4; 5; 6] to obtain the formula. We base our usage of the special groups on two exact homomorphism sequences, which we obtain in §§2 and 3. The first of these sequences, sequence (A), is implicit in the work of Smith (cf. [4]). However, the second, sequence (B), appears to be new. It is basic for our purpose in that it relates the structure of the special groups H_n^δ and H_m^σ.

In §4, we prove the main theorem. We also verify that $\sum \dim H_i(L) \leqq \sum \dim H_i(X)$. This result is closely related to results of Smith [5; p. 170] and of Richardson and Smith [2; p. 619].

In §5, we point out some applications of our results. The main theorem has as a consequence, of course, a theorem concerning existence of fixed points. This theorem generalizes the fixed point theorem of Smith [3]. We also prove that if X has the homology groups of an n-sphere over the integers mod p, p an odd prime, and if G is an abelian transformation group of order

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p^a, then the set L of points fixed under each $T \in G$ has the homology groups of an r-sphere where $n-r$ is even([2]). This solves a problem pointed out by Smith, who proved a weaker theorem for $a=1$ and a form of this theorem for the general case [7].

It must be noted that, throughout the paper, if T is periodic of period p in a given discussion, then all homology groups in the same discussion will have the integers mod p as coefficient group.

2. **Special homology groups for simplicial maps.** We treat in this section the special homology groups for simplicial periodic maps. To provide a notational basis, we give first the definitions of the special groups. These are due to Smith [4; 5; 6] and Richardson and Smith [2].

We assume throughout this section a finite complex X and a simplicial periodic map T on X of prime period p. We denote by L the fixed point set of T. We shall assume that L is a subcomplex of X.

Denote by $C_n(X)$, $Z_n(X)$, $B_n(X)$, and $H_n(X)$ the group of n-chains, n-cycles, bounding n-cycles, and the n-homology group of X with coefficient group I_p, the group of integers mod p. Denote by ∂ the boundary operator. Define $\tilde{Z}_0(X)$ to be the set of all $x \in Z_0(X)$ with coefficient sum 0. Define $\tilde{H}_0(X) = \tilde{Z}_0(X)/B_0(X)$. We denote by $Z_n(X, L)$, $B_n(X, L)$, and $H_n(X, L)$ the group of n-cycles of X mod L, and so on.

Let $\delta(t)$ denote the polynomial $1-t$. Then $\delta(T)=1-T$ will denote a chain mapping of $C_n(X)$ into itself for each n. We abbreviate $\delta(T)$ by δ. Also, for each positive integer i, we have the chain mapping $\delta^i \equiv \delta^i(T)$. Since the coefficient group is I_p, we have $\sigma \equiv \delta^{p-1} = 1+T+\cdots+T^{p-1}$ and $\delta^p=0$ [4, p. 614]. We use ρ to designate any one of the chain mappings δ^i, $i=1, \cdots, p-1$. If $\rho=\delta^i$, then define $\bar{\rho}=\delta^{p-i}$. Hence $\rho\bar{\rho}=\bar{\rho}\rho=0$.

Define $C_n^\rho(X) = [x \mid x \in C_n(X),\ \rho x = 0]$. Note that $C_n(L) \subset C_n^\delta(X)$; hence $C_n(L) \subset C_n^\rho(X)$ for all ρ.

The proof of the following basic property may be found in Richardson and Smith [2, p. 616].

(2.1) We have $x \in C_n^\rho(X)$ if and only if there exists $a \in C_n(X)$, $b \in C_n(L)$ with $x = \bar{\rho}a + b$.

Define $Z_n^\rho(X) = C_n^\rho(X) \cap Z_n(X)$, $B_n^\rho(X) = \partial(C_{n+1}^\rho(X))$, $H_n^\rho(X) = Z_n^\rho(X)/B_n^\rho(X)$. For $p=0$, we also define $\tilde{Z}_0^\rho(X) = \tilde{Z}_0(X) \cap C_0^\rho(X)$ and $\tilde{H}_n^\rho(X) = \tilde{Z}_0^\rho(X)/B_0^\rho(X)$.

We note explicitly another basic property, due to Smith [4, p. 357] and Richardson and Smith [2, p. 617]:

(2.2) *If* $\bar{\rho}a + b \in Z_n^\rho(X)$, *where* $a \in C_n(X)$, $b \in C_n(L)$, *then* $\bar{\rho}a$ *and* b *are*

([2]) After this paper was submitted, recent work of S. D. Liao, *A theorem on periodic maps of homology spheres*, Bull. Amer. Math. Soc. Abstract 57-5-420, to appear in Ann. of Math., came to the author's attention. Liao proves that if one adds the requirement that X be compact and have finitely generated integral cohomology groups and drops the restriction that p be odd, then $n-r$ is even or odd according as T is orientation preserving or orientation reversing.

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cycles. If $\bar{\rho}a+b\in B_n^\rho(X)$, *then* $\bar{\rho}a\in B_n^\rho(X)$ *and* $b\in B_n(L)$.

We use also the "relative" special groups. Define $C_n^\rho(X, L)=[x\,|\,x\in C_n(X), x=\bar{\rho}y$ for some $y]$, $Z_n^\rho(X, L)=C_n^\rho(X, L)\cap Z_n(X)$, $B_n^\rho(X, L)=\partial(C_{n+1}^\rho(X, L))$, and $H_n^\rho(X, L)=Z_n^\rho(X, L)/B_n^\rho(X, L)$. It is useful to note that $x\in Z_n^\rho(X, L)$ if and only if $x\in C_n^\rho(X)$, ∂x is contained in L, and x is 0 on each simplex of L [4, p. 356]. It should also be noted that our symbol $H_n^\rho(X, L)$ is equivalent to either the symbol $\mathfrak{H}_{\rho I}^n(X)$ or $\mathfrak{D}_\rho^n(X)$ of Smith [4, pp. 358 and 363].

Definition. Let $\bar{\rho}x\in Z_{n+1}^\rho(X, L)$. Then $\bar{\rho}(\partial x)=\partial\bar{\rho}x=0$. Hence $\partial x\in Z_n^{\bar{\rho}}(X)$. The function $\bar{\rho}x\rightarrow\partial x$ induces a homomorphism $\alpha: H_{n+1}^\rho(X, L)\rightarrow H_n^{\bar{\rho}}(X)$. It is clear that α maps $H_1^\rho(X, L)$ into $\tilde{H}_0^\rho(X)$.

The inclusion $x\rightarrow x$ of $C_n^{\bar{\rho}}(X)$ into $C_n(X)$ generates a homomorphism of $H_n^{\bar{\rho}}(X)$ into $H_n(X)$, which we denote by $\beta: H_n^{\bar{\rho}}(X)\rightarrow H_n(X)$. We note that β maps $\tilde{H}_0^{\bar{\rho}}(X)$ into $\tilde{H}_0(X)$.

Moreover, the chain mapping $y\rightarrow\bar{\rho}y$ of $C_n(X)$ into $C_n^\rho(X, L)$ induces a homomorphism of $H_n(X)$ into $H_n^\rho(X, L)$ which we indicate by $\gamma: H_n(X) \rightarrow H_n^\rho(X, L)$.

The proof that each of these is a well-defined homomorphism is straightforward. The homomorphisms α and β are due to Smith [4, pp. 358–359].

Theorem 2.3. *The sequence*

$$\text{(A)}\qquad \begin{aligned} \cdots \rightarrow H_{n+1}^\rho(X, L) \xrightarrow{\alpha} H_n^{\bar{\rho}}(X) \xrightarrow{\beta} H_n(X) \xrightarrow{\gamma} H_n^\rho(X, L) \rightarrow \cdots \\ \rightarrow H_0(X, L) \rightarrow 0 \end{aligned}$$

is exact. If $L\neq 0$ *and if* X *is connected, then* $H_0^{\bar{\rho}}(X)$, $H_0(X)$ *may be replaced by* $\tilde{H}_0^{\bar{\rho}}(X)$, $\tilde{H}_0(X)$ *respectively and the sequence is still exact.*

Proof. The proofs that $\beta\alpha=0$, $\gamma\beta=0$, $\alpha\gamma=0$ are trivial and will be omitted. Let now $x\in$kernel β. If $\rho a+b$ represents x we then have $\rho a+b=\partial c$ for some chain c. Then $\partial(\bar{\rho}c)=\bar{\rho}(\partial c)=0$, so that $\bar{\rho}c\in Z_n^\rho(X, L)$. If $\bar{\rho}c$ represents $y \in H_{n+1}^\rho(X, L)$, then $\alpha(y)=x$. Hence image $\alpha=$kernel β.

Next suppose $x\in$kernel γ. Then if a represents x we have $\bar{\rho}a=\partial\bar{\rho}c$ for some chain c. Hence $\bar{\rho}(a-\partial c)=0$, so that $a-\partial c\in Z_n^{\bar{\rho}}(X)$. If $a-\partial c$ represents $y \in H_n^{\bar{\rho}}(X)$, then $\beta(y)=x$ and image $\beta=$kernel γ.

Now let $x\in$kernel α. If $\bar{\rho}a$ represents x, then $\partial a=\partial(\rho c+d)$ for some c in X and d in L. Then $a-\rho c-d\in Z_n(X)$. If $a-\rho c-d$ represents $y\in H_n(X)$, then $\gamma(y)=x$. Then image $\gamma=$kernel α and exactness is proven.

Suppose now that X is connected and that $L\neq 0$. Then $\tilde{H}_0(X)=0$. To prove the latter part of the theorem, we must prove $H_0^\rho(X, L)=0$ and that $\alpha: H_1^\rho(X, L)\rightarrow\tilde{H}_0^{\bar{\rho}}(X)$ is onto. To prove α onto, let $x\in\tilde{H}_0^{\bar{\rho}}(X)$. Then $\beta(x) \in\tilde{H}_0(X)$, so $\beta(x)=0$. Then there exists $y\in H_1^\rho(X, L)$ with $\alpha(y)=x$. To prove $H_0^\rho(X, L)=0$, it is sufficient to prove that if v is a vertex of $X-L$, then $\bar{\rho}v=\partial(\bar{\rho}c)$ for some chain c. Let a denote a vertex of L. Let $c\in C_1(X)$ be such that $\partial c=v-a$. Then $\partial\bar{\rho}c=\bar{\rho}v-\bar{\rho}a=\bar{\rho}v$, so that $H_0^\rho(X, L)=0$.

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DEFINITIONS. Let m be one of the numbers 2, 3, $\cdots$, $p-1$. We note that $C_n^\delta(X, L) \subset C_n^{\delta^m}(X, L)$. For if $x=\delta^{p-1}c \in C_n^\delta(X, L)$, then $x=\delta^{p-m}(\delta^{m-1}c) \in C_n^{\delta^m}(X, L)$. The inclusion of $C_n^\delta(X, L)$ into $C_n^{\delta^m}(X, L)$ generates a homomorphism $\xi: H_n^\delta(X, L) \to H_n^{\delta^m}(X, L)$.

The transformation $x \to \delta x$ of $C_n^{\delta^m}(X, L)$ into $C_n^{\delta^{m-1}}(X, L)$ generates a homomorphism of $H_n^{\delta^m}(X, L)$ into $H_n^{\delta^{m-1}}(X, L)$ which we denote by η.

The transformation $\delta^{p-m+1}c \to \partial\delta^{p-m}c$ of $Z_n^{\delta^{m-1}}(X, L)$ into $Z_{n-1}^\delta(X, L)$ generates a homomorphism of $H_n^{\delta^{m-1}}(X, L)$ into $H_{n+1}^\delta(X, L)$ which we indicate by τ. For if $\delta^{p-m+1}c \in Z_n^{\delta^{m-1}}(X, L)$, then

$$\partial\delta^{p-m+1}c = \delta(\partial\delta^{p-m}c) = 0, \quad \text{so} \quad \partial\delta^{p-m}c \in Z_n^\delta(X).$$

But $\partial\delta^{p-m}c=0$ on L, so $\partial\delta^{p-m}c \in Z_n^\delta(X, L)$.

Let us now prove the existence of τ. Suppose $\delta^{p-m+1}x$ and $\delta^{p-m+1}y$ represent the same element of $H_n^{\delta^{m-1}}(X, L)$. Then for some c

$$\delta^{p-m+1}x - \delta^{p-m+1}y = \partial(\delta^{p-m+1}c),$$

so that $\delta(\delta^{p-m}x - \delta^{p-m}y - \partial\delta^{p-m}c) = 0$. Then

$$\delta^{p-m}x - \delta^{p-m}y - \partial\delta^{p-1}c = \delta^{p-1}a + b \qquad \text{by (2.1).}$$

Operating with ∂, $\partial\delta^{p-m}x - \partial\delta^{p-m}y = \partial\delta^{p-1}a$ since the left-hand side is contained in $X-L$. Therefore $\partial\delta^{p-m}x$ and $\partial\delta^{p-m}y$ represent the same element of $H_n^\delta(X, L)$. So τ is well-defined.

THEOREM 2.4. *The sequence*

$$\text{(B)} \qquad \begin{aligned} \cdots \to H_n^\delta(X, L) \xrightarrow{\xi} H_n^{\delta^m}(X, L) \xrightarrow{\eta} H_n^{\delta^{m-1}}(X, L) \xrightarrow{\tau} H_{n-1}^\delta(X, L) \\ \to \cdots \textit{ is exact.} \end{aligned}$$

Proof. To prove that $\eta\xi=0$, let $\delta^{p-1}c$ represent an element x of $H_n^\delta(X, L)$. Then $\eta\xi(x)$ is represented by $\delta(\delta^{p-1}c) = \delta^p c = 0$. Hence $\eta\xi=0$. To prove $\tau\eta=0$, let $\delta^{p-m}c$ represent $x \in H_n^{\delta^m}(X, L)$. Then $\delta^{p-m+1}c$ represents $\eta(x)$ and $\partial\delta^{p-m}c$ represents $\tau\eta(x)$. But $\delta^{p-m}c$ is a cycle, so $\tau\eta(x)=0$. To prove $\xi\tau=0$, let $\delta^{p-m+1}c$ represent an element $x \in H_n^{\delta^{m-1}}(X, L)$. Then $\partial\delta^{p-m}c$ represents $\xi\tau(x)$ in $H_n^{\delta^m}(X, L)$. But then $\xi\tau(x)=0$.

Next, let $x \in$ kernel η. If x is represented by $\delta^{p-m}c$, then $\delta^{p-m+1}c = \partial\delta^{p-m+1}b$ for some chain b. Then $\delta(\delta^{p-m}c - \partial\delta^{p-m}b)=0$ and $d=\delta^{p-m}c - \partial\delta^{p-m}b \in Z_n^\delta(X, L)$. For $d \in Z_n^\delta(X)$ and d is 0 on L. If $y \in H_n^\delta(X, L)$ is represented by d, then it follows that $\xi(y)=x$, and hence image ξ = kernel η.

Let $x \in$ kernel τ. If x is represented by $\delta^{p-m+1}c$, then $\partial\delta^{p-m}c = \partial(\delta^{p-1}b)$ for some chain b. Then $\partial(\delta^{p-m}c - \delta^{p-1}b)=0$, so that $d=\delta^{p-m}c - \delta^{p-1}b \in H_n^{\delta^m}(X, L)$. But $\delta d = \delta^{p-m+1}c - \delta^p b = \delta^{p-m+1}c$. Hence if $y \in H_n^{\delta^m}(X, L)$ is represented by d, then $\eta(y)=x$. Hence kernel τ = image η.

Finally, let $x \in$ kernel ξ. Then if $\delta^{p-1}c$ represents x, we have $\delta^{p-1}c = \partial\delta^{p-m}b$

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for some chain b. Operating on both sides with δ, we have $\partial\delta^{p-m+1}b=\delta^p c=0$. Hence $\delta^{p-m+1}b\in Z_n^{\delta^{m-1}}(X, L)$. Suppose $\delta^{p-m+1}b$ represents $y\in H_n^{\delta^{m-1}}(X, L)$. Then $\partial\delta^{p-m}b=\delta^{p-1}c$ represents $\tau(y)$ so that $\tau(y)=x$. Hence exactness is proven.

3. **The special groups for locally compact spaces.** Let X be a compact Hausdorff space and let T be a periodic map on X of prime period p. By a covering of X will be meant a finite open covering. We denote by $[U_\mu]$ the collection of primitive special coverings of X [4, pp. 350–353]. Denote by X_μ the nerve of U_μ. Then there is generated a simplicial periodic map T_μ on X_μ defined by $T_\mu(u)=T(u)$ for each vertex u of X_μ. If U_μ refines U_λ, then there exists a projection $\pi_{\mu\lambda}: X_\mu\to X_\lambda$ such that $\pi_{\mu\lambda}T_\mu=T_\lambda\pi_{\mu\lambda}$ [4, p. 351] (such a projection is called a T-projection). We call L_μ the fixed point set of T_μ.

We have for each U_μ the exact sequence

$$(\mathrm{A}_\mu)\qquad \cdots\to H^{\rho}_{n+1}(X_\mu, L_\mu)\xrightarrow{\alpha_\mu} H^{\bar\rho}_n(X_\mu)\xrightarrow{\beta_\mu} H_n(X_\mu)\xrightarrow{\gamma_\mu} H^{\rho}_n(X_\mu, L_\mu)\to\cdots .$$

If U_μ refines U_λ then a T-projection $\pi_{\mu\lambda}$ defines a set of homomorphisms of the exact sequence (A_μ) into the exact sequence (A_λ) [4, p. 361]. That these homomorphisms are independent of the particular T-projection $\pi_{\mu\lambda}$ has been proved by Smith [4, p. 360]. That the generated homomorphisms commute with α, β, γ follows from the relations $\pi_{\mu\lambda}\partial=\partial\pi_{\mu\lambda}$ and $\pi_{\mu\lambda}\rho=\rho\pi_{\mu\lambda}$. We note that each (A_μ) is made up of compact (actually finite) groups. So we may consider the inverse limit of the exact sequences (A_μ) [1, pp. 694–695], connected by the homomorphisms generated by the T-projections $\pi_{\mu\lambda}$. We denote the exact sequence of inverse limits by

$$(\mathrm{A})\qquad \cdots\to H^{\rho}_{n+1}(X, L)\xrightarrow{\alpha} H^{\bar\rho}_n(X)\xrightarrow{\beta} H_n(X)\xrightarrow{\gamma} H^{\rho}_n(X, L)\to\cdots ,$$

which serves to define α, β, γ as well as the ρ-homology groups. The definitions for the groups and for α, β have been given by Smith [4].

THEOREM 3.1. *If X is a compact Hausdorff space, and if T is a periodic map on X of prime period p, then* (A) *is exact. Moreover, if X is connected and $L\neq 0$, then* (A) *remains exact when $H^{\bar\rho}_0(X)$, $H_0(X)$ are replaced by $\tilde H^{\bar\rho}_0(X)$, $\tilde H_0(X)$.*

Proof. The exactness follows from a general theorem on inverse limits of exact homomorphism sequences of compact groups [1, p. 695].

Moreover, for each U_μ we have the exact sequence

$$(\mathrm{B}_\mu)\ \cdots\to H^{\delta}_n(X_\mu, L_\mu)\xrightarrow{\xi_\mu} H^{\delta^m}_n(X_\mu, L_\mu)\xrightarrow{\eta_\mu} H^{\delta^{m-1}}_n(X_\mu, L_\mu)\xrightarrow{\tau_\mu} H^{\delta}_{n-1}(X_\mu, L_\mu)\to\cdots .$$

If U_μ refines U_λ, then since $\pi_{\mu\lambda}\partial=\partial\pi_{\mu\lambda}$ and $\pi_{\mu\lambda}\rho=\rho\pi_{\mu\lambda}$, the map $\pi_{\mu\lambda}$ generates a set of homomorphisms of the exact sequence (B_μ) of compact groups into the exact sequence (B_λ). The homomorphism is independent of the particular

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T-projection $\pi_{\mu\lambda}$ [4, p. 360]. Let

$$\text{(B)} \qquad \cdots \to H^{\delta}_n(X, L) \xrightarrow{\xi} H^{\delta^m}_n(X, L) \xrightarrow{\eta} H^{\delta^{m-1}}_n(X, L) \xrightarrow{\tau} H^{\delta}_{n-1}(X, L) \to \cdots$$

denote the exact sequence of inverse limits of the exact sequence (B_μ), which serves to define ξ, η, τ. Then, as in (3.1),

THEOREM 3.2. *If X is a compact Hausdorff space, and if T is periodic of prime period p, then the sequence* (B) *is exact.*

For the remainder of the section, we shall assume that X is a locally compact Hausdorff space, and that T is a periodic map of prime period p on X. Let $[A_\mu]$ denote the collection of compact subsets of X with $T(A_\mu)=A_\mu$. Note that if A is any compact subset of X, then $A\cup T(A)\cup \cdots \cup T^{p-1}(A)$ is a compact invariant subset. That is, if $[A]$ denotes the collection of compact subsets of X partially ordered by inclusion, then $[A_\mu]$ is cofinal in $[A]$. Let now A_μ contain A_λ. Denote by T_μ the periodic map $T|A_\mu$. Let $f_{\lambda\mu}$ denote the inclusion map of A_λ into A_μ. Then obviously $T_\mu f_{\lambda\mu}=f_{\lambda\mu}T_\lambda$. Denote the fixed point set of T_μ by L_μ; that is, $L_\mu=A_\mu\cap L$. According to a construction of Smith [4, p. 370], $f_{\lambda\mu}$ generates homomorphisms of the special homology groups of A_λ, T_λ into those for A_μ, T_μ. We have the exact sequences

$$(\mathrm{A}_\mu) \qquad \cdots \to H^{\rho}_{n+1}(A_\mu, L_\mu) \xrightarrow{\alpha_\mu} H^{\bar\rho}_n(A_\mu) \xrightarrow{\beta_\mu} H_n(A_\mu) \xrightarrow{\gamma_\mu} H^{\rho}_n(A_\mu, L_\mu) \to \cdots,$$

$$(\mathrm{B}_\mu) \qquad \cdots \to H^{\delta}_n(A_\mu, L_\mu) \xrightarrow{\xi_\mu} H^{\delta^m}_n(A_\mu, L_\mu) \xrightarrow{\eta_\mu} H^{\delta^{m-1}}_n(A_\mu, L_\mu) \xrightarrow{\tau_\mu} H^{\delta}_{n-1}(A_\mu, L_\mu) \to \cdots$$

of (3.1) and (3.2).

For $A_\lambda \subset A_\mu$, $f_{\lambda\mu}$ generates a set of homomorphisms of (A_λ) into (A_μ), and of (B_λ) into (B_μ). To note commutativity, see the remarks by Smith [6, p. 370]. We define the direct limit of the exact homomorphism sequences (A_μ), (B_μ) by

$$\text{(A)} \qquad \cdots \to H^{\rho}_{n+1}(X, L) \xrightarrow{\alpha} H_n(X) \xrightarrow{\beta} H_n(X) \xrightarrow{\gamma} H^{\rho}_n(X, L) \to \cdots,$$

$$\text{(B)} \qquad \cdots \to H^{\delta}_n(X, L) \xrightarrow{\xi} H^{\delta^m}_n(X, L) \xrightarrow{\eta} H^{\delta^{m-1}}_n(X, L) \xrightarrow{\tau} H^{\delta}_{n+1}(X, L) \to \cdots.$$

Let us clarify the last statement. Define $H^{\rho}_n(X, L)$ to be the direct limit of the system $[H^{\rho}_n(A_\mu, L_\mu); f^1_{\lambda\mu}]$ where $f_{\lambda\mu}$ denotes the inclusion of A_λ into A_μ and where $f^1_{\lambda\mu}$ denotes the generated homomorphism of $H^{\rho}_n(A_\lambda, L_\lambda)$ into $H^{\rho}_n(A_\mu, L_\mu)$. Similarly for the other special groups and for $H_n(X)$. Then define α, β, and so on as the homomorphisms induced on these as in [1, p. 689], where α is generated by α_μ, β by β_μ, and so on. It should be noted that $H_n(X)$ is then essentially the group of Čech cycles on compact subsets of X modulo the Čech cycles which bound on compact subsets of X.

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Theorem 3.3. *If X is a locally compact Hausdorff space and if T is a periodic map on X of prime period p, then the sequences* (A) *and* (B) *are exact. Moreover,* (A) *is exact if we replace $H_0^{\bar{\rho}}(X)$, $H_0(X)$ by $\tilde{H}_0^{\bar{\rho}}(X)$, $\tilde{H}_0(X)$ respectively if $\tilde{H}_0(X)=0$ and $L\neq 0$.*

The proof follows from [1, p. 689].

We say that a locally compact Hausdorff space X is *finite-dimensional* if and only if there exists an integer n such that if A is any compact subset of X, then the dimension of A (in the covering sense) is less than n.

We note here three important results of P. A. Smith that we use. We let X be a finite-dimensional locally compact Hausdorff space, and let T be a periodic map of prime period p. The results have been proved by Smith for the compact case; they may then be extended in a straightforward manner to the locally compact case.

(3.4) There exists an integer k such that for $i>k$ all the groups $H_i^{\rho}(X, L)$ vanish [4, p. 362, Remark 9.5];

(3.5) $H_i^{\rho}(X)$ is isomorphic with the direct sum of $H_i^{\rho}(X, L)$ and $H_i(L)$ [4, p. 363];

(3.6) If Y denotes the orbit decomposition space and if L^* denotes the subset of Y generated by L, then $H_i(Y, L^*)\approx H_i^{\delta}(X, L)$ [6, p. 144, Theorems 3.19 and 3.20].

It should be noted that in [6], in which the proof of (3.6) is given for the compact case, there is a standing hypothesis that every open subset of X is an F_σ. It may be seen, however, that this property was not used in the proof of (3.6).

4. **The main theorem.** If C is a vector space over the field F, then we denote by dim C the dimension of the vector space over F. If $[C_i | i=0, \pm 1, \cdots]$ is a sequence of finite-dimensional vector spaces all but a finite number of which vanish, then we define $k(C)$ to be $\sum(-1)^i \dim C_i$.

The following lemma is essentially the same as a result of Kelley and Pitcher [1, p. 688].

Lemma 4.1. *Let*

$$\cdots \to K_{i+1} \to G_i \to H_i \to K_i \to \cdots$$

be an exact sequence of vector spaces and linear operators. If G_i and K_i are finite-dimensional, then so is H_i. If all the elements are finite-dimensional and all but a finite number of them vanish, then $k(H)=k(G)+k(K)$.

Proof. For purposes of proof, we consider an exact sequence

$$\cdots \to C_{i+1} \xrightarrow{f_{i+1}} C_i \xrightarrow{f_i} C_{i-1} \to \cdots,$$

where each C_i is a vector space over F, and f_i is linear. Then, $C_i/\text{kernel } f_i \approx \text{image } f_i$. By exactness, we have kernel $f_i=$ image f_{i+1}, and so dim C_i

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$=\dim f_{i+1}(C_{i+1})+\dim f_i(C_i)$. If each C_i is finite-dimensional and almost all vanish, it clearly follows that $\sum(-1)^i \dim C_i=0$. The lemma follows.

THEOREM 4.2. *Let X be a finite-dimensional locally compact Hausdorff space, and let T be a periodic map of prime period p on X. Suppose that the vector spaces $H_i(X)$ are of finite dimension for each i. Let L be the fixed point set of X, and let Y denote the orbit decomposition space of X, T. Then all the vector spaces $H_i(L)$ and $H_i(Y)$ are of finite dimension and*

$$\chi(X) + (p-1)\chi(L) = p\chi(Y),$$

where

$$\chi(A) = \sum (-1)^i \dim H_i(A).$$

Proof. We use first the exact sequence (A)

$$\cdots \rightarrow H^{\rho}_{i+1}(X, L) \rightarrow H^{\bar{\rho}}_i(X) \rightarrow H_i(X) \rightarrow H^{\rho}_i(X, L) \rightarrow \cdots .$$

Suppose it has been proved that the vector spaces $H^{\rho}_i(X, L)$ are finite-dimensional for all ρ (and hence all $\bar{\rho}$) and for all $i\geqq n+1$. It then follows from Lemma 4.1 that $H^{\bar{\rho}}_i(X)$ is finite-dimensional for all ρ and all $i\geqq n$ since both $H^{\rho}_{i+1}(X, L)$ and $H_i(X)$ are finite-dimensional. We would then have that $H^{\rho}_i(X)$ is finite-dimensional for all ρ and all $i\geqq n$. But for n sufficiently large, all the groups $H^{\rho}_i(X, L)=0$ for $i>n$ by (3.4). Hence by induction all the groups $H^{\rho}_i(X)$ are finite-dimensional. But then so are the vector spaces $H_i(L)$, they being isomorphic with subspaces of $H^{\rho}_i(X)$. Moreover, the groups $H_i(Y, L^*)$ are finite-dimensional, being isomorphic with $H^{\delta}_i(X, L)$ by (3.6). But then, using the homology sequence of the pair (Y, L^*),

$$\cdots \rightarrow H_i(L^*) \rightarrow H_i(Y) \rightarrow H_i(Y, L^*) \rightarrow \cdots ,$$

we see that all the entries except possibly the vector spaces $H_i(Y)$ are finite-dimensional. But then by Lemma 4.1 all the entries are finite-dimensional. Moreover

$$\begin{aligned} k(H(Y)) &= k(H(L^*)) + k(H(Y, L^*)) \\ &= k(H(L)) + k(H^{\delta}(X, L)) \end{aligned}$$

so that (i) $k(H^{\delta}(X, L))=k(H(Y))-k(H(L))$.

Using Lemma 4.1 on (A) for $\rho=\sigma$ we obtain

$$\text{(ii)} \qquad \begin{aligned} k(H(X)) &= k(H^{\delta}(X)) + k(H^{\sigma}(X, L)) \\ &= k(H(L)) + k(H^{\delta}(X, L)) + k(H^{\sigma}(X, L)). \end{aligned}$$

Using Lemma 4.1 on (B) we obtain

$$k(H^{\delta^m}(X, L)) = k(H^{\delta}(X, L)) + k(H^{\delta^{m-1}}(X, L)).$$

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Using this equation for $i=2, \cdots, p-1$, we get

$$k(H^{\sigma}(X, L)) = k(H^{\delta^{p-1}}(X, L)) = (p-1)k(H^{\delta}(X, L)).$$

Returning to (i) we then get

$$k(H(X)) = k(H(L)) + pk(H^{\delta}(X, L)).$$

Using (i), and changing notation, we get the desired theorem.

COROLLARY 4.3. *Using the notation of the above theorem, we have* $\chi(X) = \chi(L) \bmod p$.

THEOREM 4.4. *Let X be a locally compact finite-dimensional Hausdorff space. Let T denote a periodic map on X of prime period p, and denote by L the fixed point set of T. Then for each nonnegative integer n, we have* $\sum_n^{\infty} \dim H_i(L) \leqq \sum_n^{\infty} \dim H_i(X)$.

Proof. We prove that for each n and each ρ, $\dim H_n^{\rho}(X) + \sum_{n+1}^{\infty} \dim H_i(L) \leqq \sum_n^{\infty} \dim H_i(X)$. Suppose this has been proved for $n+1$. Then consider (A);

$$\cdots \rightarrow H_{n+1}^{\rho}(X, L) \xrightarrow{\alpha} H_n^{\bar{\rho}}(X) \xrightarrow{\beta} H_n(X) \rightarrow \cdots.$$

By exactness we have

$$\begin{aligned} \dim H_n^{\bar{\rho}}(X) &\leqq \dim H_{n+1}^{\rho}(X, L) + \dim H_n(X) \\ &\leqq \dim H_{n+1}^{\rho}(X) - \dim H_{n+1}(L) + \dim H_n(X); \end{aligned}$$

using the induction hypothesis, we obtain

$$\dim H_n(X) + \sum_{n+1}^{\infty} \dim H_i(L) \leqq \sum_{n}^{\infty} \dim H_i(X).$$

We note that this inequality reduces to $0 \leqq 0$ for n sufficiently large. The theorem then follows from the remark that $\dim H_n(L) \leqq \dim H_n^{\bar{\rho}}(X)$.

5. **Some applications.** The following is a generalization of a theorem of Smith [3].

THEOREM 5.1. *Let X be a locally compact finite-dimensional Hausdorff space. Let T be a periodic map on X of period* p^a, *where p is prime. If the groups* $H_n(X)$ *are finitely generated and if* $\sum(-1)^i \dim H_n(X) \neq 0 \bmod p$, *then T has at least one fixed point.*

Proof. Let us consider the case $a=1$. Then by (4.3) we have $\chi(X) = \chi(L) \bmod p$, and moreover the groups $H_n(L)$ are finitely generated. Then by an inductive device used often by Smith [3; 6], we have also in the general case $\chi(X) = \chi(L) \bmod p$, and the theorem follows.

It is convenient in the following theorem to agree, following Smith [4, p. 366], that the empty set is a (-1)-sphere.

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THEOREM 5.2. *Let X be a locally compact finite-dimensional Hausdorff space. Let G be an abelian transformation group on X, of finite order p^a where p is an odd prime. Suppose X has the homology groups of an n-sphere over I_p. Let L denote the set of all $x \in X$ with $T(x)=x$ for all $T \in G$. Then L has the homology groups of an r-sphere, $r \leqq n$, and $n-r$ is even.*

Proof. As Smith has pointed out [7, p. 358], it is sufficient to prove such a theorem for $a=1$, since the fixed point set L inherits all the properties postulated for X. That L has the homology groups of an r-sphere for some $r \leqq n$ has been proved by Smith [4, p. 366]. A proof could also be easily established using Theorems 4.2 and 4.4. Now by Corollary 4.3, we have $\chi(X) = \chi(L) \bmod p$. Since X and L are homological spheres and p is odd, then $\chi(X)=\chi(L)$. Hence $1+(-1)^n=1+(-1)^r$ and $n-r$ is even.

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