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ON PERIODIC SOLUTIONS OF THE NONLINEAR SUSPENSION BRIDGE EQUATION

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Introduction. In this paper we investigate nonlinear oscillations in the nonlinear suspension bridge equation, in $(-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}$, of the type

$$\begin{aligned} &-K_1 u_{xxtt} + u_{tt} + K_2 u_{xxxx} + K_3 u^+ = 1 + k\cos x + \epsilon h(x,t), \\ &u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0. \end{aligned} \tag{0.1}$$

The first term in (0.1), due to L. Rayleigh, represents the effect of rotary inertia, as can be traced from the derivation. In many applications, its effect is small.

McKenna and Walter [6] studied nonlinear oscillations in a nonlinear suspension bridge equation without the first term in (0.1):

$$\begin{aligned} &u_{tt} + u_{xxxx} + bu^+ = 1 + \epsilon h(x,t) \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R} \\ &u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0. \end{aligned} \tag{0.2}$$

This equation represented a bending beam supported by cables under a constant load $w = 1$. The constant b represented the restoring force if the cables were stretched. The nonlinearity u^+ models the fact that cables resist expansion but do not resist compression. They proved a counterintuitive result: if the cables were weak; that is, b is small, then there was only a unique solution. However, if b was large (that is, the cables were strengthened), then large scale oscillatory periodic solutions existed.

In this paper we improve this result in two ways. First, we generalize the beam equation to include the effect of rotary inertia. Second, we allow b to vary with x, as indeed it must be suspension bridges.

In Sections 1 and 2 we shall deal with the nonlinear bridge equation with constant coefficients, in $(-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}$

$$\begin{aligned} &-\tfrac{1}{4} u_{xxtt} + u_{tt} + u_{xxxx} + bu^+ = 1 + k\cos x + \epsilon h(x,t) \\ &u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \end{aligned} \tag{0.3}$$

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where $4 < b < 19$. The effect of the first term in (0.1) is small. So we took the small coefficient $-\frac{1}{4}$ in (0.3). We shall assume that h in (0.3) is even in x and t and periodic with period π, and we shall look for π-perodic solutions of (0.3).

In Section 3, we shall prove the existence of a positive solution of the nonlinear equation with a variable coefficient, in $(-\frac{\pi}{2}, \frac{\pi}{2})$

$$y^{(4)} + b(x)y^+ = 1, \quad y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0.$$

In Section 4, we shall deal with the nonlinear suspension bridge equation with a variable coeffcient

$$u_{tt} + u_{xxxx} + b(x)u^+ = 1, \quad u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0. \tag{0.4}$$

1. A priori bound. Let L be the differential operator

$$Lu = -\tfrac{1}{4}u_{xxtt} + u_{tt} + u_{xxxx}.$$

The eigenvalue problem for $u(x,t)$

$$\begin{aligned} &Lu = \lambda u \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\ &u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \\ &u(x,t) = u(-x,t) = u(x,-t) = u(x, t+\pi) \end{aligned} \tag{1.1}$$

has infinitely many eigenvalues λ_{mn} and corresponding eigenfunctions ϕ_{mn} $(m, n \geq 0)$ given by

$$\begin{aligned} \lambda_{mn} &= (2n+1)^4 - m^2((2n+1)^2 + 4), \\ \phi_{mn} &= \cos 2mt \cos(2n+1)x, \quad (m, n = 0, 1, 2, \cdots). \end{aligned}$$

We remark that all eigenvalues in the interval $(-36, 29)$ are given by

$$\lambda_{20} = -19 < \lambda_{10} = -4 < \lambda_{00} = 1.$$

The normalized eigenfunctions are denoted by

$$\theta mn = \frac{\phi_{mn}}{\|\phi_{mn}\|},$$

where $\|\phi_{mn}\| = \frac{\pi}{2}$ for $(m > 0)$, $\|\phi_{0n}\| = \frac{\pi}{\sqrt{2}}$. Let $\mathbb{Q}$ be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\mathbb{H}$ be the Hilbert space defined by

$$\mathbb{H} = \{u \in L_2(\mathbb{Q}) : u \text{ is even in } x \text{ and } t\}.$$

Then the set of $\{\theta_{mn}\}$ is an orthonormal base in $\mathbb{H}$.

We consider weak solutions of problems of the type

$$\begin{aligned} &Lu = f(u, x, t) \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\ &u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \end{aligned} \tag{1.2}$$

where u is even and π-periodic in t and even in x.

A weak solution of (1.2), which is also called a solution in $\mathbb{H}$, is of the form

$$u = \sum c_{mn}\theta_{mn} \quad \text{with} \quad Lu \in \mathbb{H};$$

i.e., $\sum \lambda_{mn}^2 c_{mn}^2$ is finite. Our function will be such that $u \in \mathbb{H}$ implies $f(u, x, t) \in \mathbb{H}$. The following symmetry theorem was proved in [3].

Theorem 1.1. *Let $H = L_2(\Omega)$. Assume that $L : D(L) \subset H \to H$ is a linear, selfadjoint operator which possesses two closed invariant subspaces H_1 and $H_2 = H_1^{\perp}$. Let σ denote the spectrum of L and σ_i the spectrum of $L|_{H_i}$ ($i = 1, 2$; $\sigma = \sigma_1 \bigcup \sigma_2$). Let $\frac{\partial f}{\partial u}(u, x) = f_u$ be piecewise smooth and assume that $f_u \in [a, b]$ for $u \in \mathbb{R}$ and $x \in \Omega$. If $[a, b] \bigcap \sigma_2 = \phi$ and if the Nemytzki operator $u \mapsto Fu = f(u(x), x)$ maps H_1 into itself, then every solution of*

$$Lu = f(u, x) \text{ in } \mathbb{H}$$

is in H_1.

Lemma 1.1. *For $-1 < b < 19$ the problem*

$$Lu + bu^+ = 0 \text{ in } \mathbb{H} \tag{1.3}$$

has only the trivial solution $u = 0$.

Proof: The space $H_1 = \operatorname{span}\{\cos x \cos 2mt : m \geq 0\}$ is invariant under L and the map $u \mapsto bu^+$. The spectrum σ_1 of $L|_{H_1}$ in $(-19, 1)$ is only $\lambda_{10} = -4$. The spectrum σ_2 of L restricted to $H_2 = H_1^{\perp}$ does not intersect the interval $(-19, 1)$. From Theorem 1.1 we see that any solution of (1.3) belongs to H_1; i.e., it is of the form $y(t)\cos x$, where y satisfies

$$\tfrac{5}{4}y'' + by^+ + y = 0.$$

Any nontrivial periodic solution of this equation is periodic with period

$$\frac{\pi}{\sqrt{\frac{4}{5}(b+1)}} + \frac{\pi}{\sqrt{\frac{4}{5}}} > \pi,$$

which shows that there is no nontrivial solution of (1.3). ∎

We establish an *a priori* bound for solutions of (0.3), namely,

$$Lu + bu^+ = 1 + k\cos x + \epsilon h, \quad (k \geq 0) \text{ in } \mathbb{H}. \tag{1.4}$$

Lemma 1.2. *Let $k \geq 0$ be fixed. Let $h \in \mathbb{H}$ with $\|h\| = 1$ and $\alpha > 0$ be given. Then there exists $R_0 > 0$ (depending only on h and α) such that for all b with $-1 + \alpha \leq b \leq 19 - \alpha$ and all $\epsilon \in [-1, 1]$ the solutions of (1.4) satisfy $\|u\| < R_0$.*

Proof: We shall apply Lemma 1.1. Assume Lemma 1.2 does not hold. Then there is a sequence (b_n, ϵ_n, u_n) with $b_n \in [\alpha - 1, 19 - \alpha]$, $|\epsilon_n| \leq 1$, $\|u_n\| \to \infty$ such that

$$u_n = L^{-1}(1 + k\cos x + \epsilon_n h - b_n u_n^+).$$

Put $w_n = \frac{u_n}{\|u_n\|}$. Then

$$w_n = L^{-1}\left(\frac{1}{\|u_n\|} + \frac{k}{\|u_n\|}\cos x + \frac{\epsilon_n}{\|u_n\|}h - b_n w_n^+\right).$$

The operator L^{-1} is compact. Therefore we may assume that $w_n \to w_0$ and $b_n \to b_0 \in (-1, 19)$. Since $\|w_n\| = 1$ for all n, $\|w_0\| = 1$ and w_0 satisfies

$$w_0 = L^{-1}(-bw_0^+) \text{ or } Lw_0 + bw_0^+ = 0 \text{ in } \mathbb{H}.$$

This contradicts Lemma 1.1 and proves Lemma 1.2.

2. Existence of solutions of a nonlinear suspension bridge equation with a constant coefficient.

Our main result in this section is the following:

Theorem 2.1. *Let $h \in \mathbb{H}$ with $\|h\| = 1$ and $4 < b < 19$. Then there is $\epsilon_0 > 0$ such that if $|\epsilon| < \epsilon_0$ equation (1.4) has at least two solutions.*

In other words, equation (0.3) has at least two π-periodic solutions. The proof of Theorem 2.1 requires several lemmas. First we discuss the Leray-Schauder degree d_{LS}.

Lemma 2.1. *Under the assumptions and with the notation of Lemma 1.2,*

$$d_{LS}(u - L^{-1}(1 + k\cos x - bu^+ + \epsilon h), B_R, 0) = 1$$

for all $R \geq R_0$.

Proof: Let $b = 0$. Then we have

$$d_{LS}(u - L^{-1}(1 + k\cos x + \epsilon h), B_R, 0) = 1,$$

since the map is simply a translation of the identity and since $\|L^{-1}(1 + k\cos x + \epsilon h)\| < R_0$ by Lemma 1.2.

In case $b \neq 0$ $(-1 < b < 19)$, the result follows in the usual way by invariance under homotopy, since all solutions are in the open ball B_{R_0}. ∎

The following lemma was proved by McKenna and Walter [5].

Lemma 2.2. *For $-1 < b$ the boundary value problem*

$$y^{(4)} + by = 1 \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2}), \; y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0 \tag{2.1}$$

has a unique solution y, which is even in x and positive, and satisfies

$$y'(-\frac{\pi}{2}) > 0 \text{ and } y'(\frac{\pi}{2}) < 0.$$

We can obtain an easy consequence of Lemma 2.2.

Lemma 2.3. *Let $k \geq 0$ be fixed. For $-1 < b$ the boundary value problem*

$$y^{(4)} + by = 1 + k\cos x \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2}), \; y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0 \tag{2.2}$$

has a unique solution y, which is even and positive. Also the solution y satisfies

$$y'(-\frac{\pi}{2}) > 0 \text{ and } y'(\frac{\pi}{2}) < 0.$$

Proof: The function

$$y_1 = y(x) - \frac{k}{1+b}\cos x$$

satisfies

$$y_1^{(4)} + by_1 = 1 \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2}), \; y_1(\pm\frac{\pi}{2}) = y_1''(\pm\frac{\pi}{2}) = 0. \tag{2.3}$$

By Lemma 2.2, we see that the solution y_1 is unique, even in x, positive, and satisfies

$$y_1'(-\frac{\pi}{2}) > 0 \quad \text{and} \quad y_1' < 0.$$

So the solution y is unique, even in x, positive, and satisfies

$$y'(-\frac{\pi}{2}) > 0 \quad \text{and} \quad y'(\frac{\pi}{2}) < 0.$$

Lemma 2.4. *For $-1 < b$ the boundary value problem*

$$y^{(4)} + by^{+} = 1 + k\cos x \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2}), \; y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0 \qquad (2.4)$$

has a unique solution.

Proof: The solution y of (2.2) is positive, hence it is also a solution of (2.4). Uniqueness follows from the contraction principle in the following familliar way. The eigenvalues of $My = \lambda y$, where $M = D^4$, with the boundary conditions as given in (2.2), are all ≥ 1. Hence for any $c < 1$,

$$\|(M - c)^{-1}\| = \frac{1}{1-c}.$$

Any problem $My = f(y, x)$ with $c \leq f_y \leq 1-\epsilon$ has a unique solution, since solutions y are characterized by

$$y = (M - c)^{-1}[f(y, x) - cy],$$

where the right-hand side is Lipschitz continuous with a Lipschitz constant $\leq (1 - \epsilon - c)/(1 - c) < 1$.

The following lemma was first used in [4].

Lemma 2.5. *Let K be a compact set in $L_2 = L_2(\Omega)$, and let $\phi \in L_2$ be positive almost everywhere. Then there exists a modulus of continuity δ depending only on K and ϕ such that*

$$\|(\eta \mid \psi \mid -\phi)^{+}\| \leq \eta\delta(\eta) \text{ for } \eta > 0 \text{ and } \psi \in K.$$

The following lemma is the final step in the proof of Theorem 2.1.

Lemma 2.6. *Let $4 < b < 19$. Then there exist $\gamma > 0$, $\epsilon_0 > 0$ such that*

$$d_{LS}(u - L^{-1}(1 + k\cos x - bu^{+}\epsilon h), B_\gamma(y), 0) = -1$$

for $\mid \epsilon \mid < \epsilon_0$, where $k \geq 0$ and y is the unique solution of (2.4).

Proof: Let K be the closure of $L^{-1}(B)$, where B is the closed unit ball in $\mathbb{H}$. Clearly K is compact. Let $\delta(\eta)$ be the modulus of continuity corresponding to K and y as in Lemma 2.5. We note that $\|L^{-1}\| = 1$. Let u be a solution of (1.4). If $u = y + \phi$ and $\|\phi\| = \gamma$, we see that

$$L\phi = \epsilon h + by - b(y + \phi)^{+} = \epsilon h - b\phi - b(y + \phi)^{-}, \qquad (2.5)$$

since $Ly + by = 1 + k\cos x$. Here we used the identity $u = u^{+} - u^{-}$. It follows that

$$\phi \in (\epsilon_0 + 2b\gamma)K \text{ for } \mid \epsilon \mid < \epsilon_0. \qquad (2.6)$$

We assume that $\epsilon_0 \leq \gamma$. Then $\psi = \frac{\phi}{\gamma}$ has the properties $\|\psi\| = 1$ and $\psi \in (2b+1)K$. Since ψ is in a compact set and different from zero and since $-b$ is not an eigenvalue of L, we get

$$\inf_{\psi} \|\psi + L^{-1}b\psi\| = \alpha > 0.$$

Hence $\|\phi + L^{-1}b\phi\| \geq \alpha\gamma$. It follows from (2.5) that

$$\phi + L^{-1}b\phi = L^{-1}(\epsilon h - b(y+\phi)^-). \tag{2.7}$$

Since $w \in K$ satisfies $\|(\eta w - y)^+\| \leq \eta\delta(\eta)$, we get from (2.6)

$$\|(\phi + y)^-\| = \|(-\phi - y)^+\| \leq (\epsilon_0 + 2b\gamma)\delta(\epsilon_0 + 2b\gamma).$$

Denoting the two sides of equation (2.7) by LS and RS, and keeping in mind that $\|L^{-1}\| = 1$, we get, for $\epsilon_0 \leq \gamma \min(1, \frac{\alpha}{2})$,

$$\|LS\| \geq \alpha\gamma \text{ and } \|RS\| \leq \frac{1}{2}\alpha\gamma + (2b\gamma + \frac{1}{2}\alpha\gamma)\delta(2b\gamma + \frac{1}{2}\alpha\gamma).$$

Now we choose $\gamma > 0$ so small that the right-hand side is $< \alpha\gamma$. It follows that for this value of γ there is no solution of (1.4) of the form $u = y + \phi$ with $\|\phi\| = \gamma$.

The same conclusion holds for solutions $u = y + \phi$ of the equation

$$Lu + bu = 1 + k\cos x + \lambda(\epsilon h - bu^-),$$

where $0 \leq \lambda \leq 1$. Here $\lambda = 1$ gives the equation (1.4), while for arbitrary λ the function $\phi = u - y$ satisfies (2.7) with a factor λ on the right-hand side. Hence we have the same conclusion: There is no solution $u = y + \phi$ with $\|\phi\| = \gamma$. Since the degree is invariant under homotopy, we get

$$d_{LS}(u - L^{-1}(1 + k\cos x - bu^+ + \epsilon h), B_\gamma(0), 0) = d_{LS}(u - L^{-1}(1 + k\cos x - bu), B_\gamma(y), 0).$$

The equation $u - L^{-1}(1 + k\cos x - bu) = 0$ has the unique solution $u = y$ in $B_\gamma(y)$, and hence the degree on the right-hand side is equal to

$$d_{LS}(u + L^{-1}bu, B_\gamma(0), 0).$$

The eigenvalues ρ of the operator $u + L^{-1}bu$ are related to the eigenvalues λ of L, namely,

$$u + L^{-1}bu = \rho u \Leftrightarrow Lu = \frac{b}{\rho - 1}u$$

or $\rho = 1 + (b/\rho)$. It follows from (1.2) that there is just one negative eigenvalue ρ which corresponds to $\lambda_{10} = -4$. Thus the usual method of approximating on finite-dimensional subspaces spanned by eigenvectors with dimension going to infinity (see [7]) shows that the desired degree is -1.

Proof of Theorem 2.1: Equation (1.4) can be written in the form

$$Su := u - L^{-1}(1 + k\cos x - bu^+ + \epsilon h) = 0.$$

The degree of Su on a large ball of radius $R > R_0$ is $+1$ by Lemma 2.1. We know from Lemma 2.6 that the degree on the ball $B_\gamma(y)$ is -1. Choosing $R > R_0$ so large that $B_R \supset B_\gamma(y)$, we can conclude that

$$d_{LS}(Su, B_R - B_\gamma(y), 0) = 2.$$

Therefore, equation (1.4) has at least two solutions, one in $B_\gamma(y)$ and the other one in $B_R - B_\gamma(y)$. This concludes the proof of Theorem 2.1. ∎

In the rest of this section, we generalize Theorem 2.1, replacing $1 + k\cos x (k \geq 0)$ by $k_1 + k_2\cos x$ ($k_1 > 0$, $k_2 \geq 0$) in the right-hand side of (1.4); namely,

$$Lu + bu^+ = k_1 + k_2\cos x + \epsilon h \text{ in } \mathbb{H}, \tag{2.8}$$

where $k_1 > 0$, $k_2 \geq 0$. This equation is more nonsteady state than equation (1.4). To do that we need several lemmas. First we make a generalization of Lemma 2.2 with replacing 1 by k_1 ($k_1 > 0$) in the right-hand side of (2.1).

Lemma 2.7. *Let $k_1 > 0$ and $b > -1$. Then the boundary value problem*

$$y^{(4)} + by = k_1 \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2}), \quad y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0 \tag{2.9}$$

has a unique solution y, which is even in x, positive, and satisfies

$$y'(-\frac{\pi}{2}) > 0 \text{ and } y'(\frac{\pi}{2}) < 0.$$

Proof: We consider only for $b \geq 4$ (cf. Theorem B of [6] or [8; p. 100]). It is convenient to write $b = 4\beta^4$, $a = \frac{\pi}{2}\beta$ and to introduce a function

$$z(x) = \frac{k}{4\beta^4} - y(\frac{x}{\beta}).$$

The solution y is a solution of (2.9) if and only if z satisfies

$$z^{(4)} + 4z = 0 \text{ in } \mid x \mid \leq a, \quad z(\pm a) = \frac{k}{4\beta^4}, \; z''(\pm a) = 0, \tag{2.10}$$

and $y > 0$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ if and only if $z(x) < z(\pm a)$ in $\mid x \mid < a$. Also

$$y'(-\frac{\pi}{2}) > 0 \text{ and } y'(\frac{\pi}{2}) < 0 \text{ iff } z'(-a) < 0 \text{ and } z'(a) > 0.$$

On the other hand, the solution z of (2.10) is explicitly determinded (up to a positive constant) by

$$z(x) = C_1 \sin x \sinh x + C_2 \cos x \cosh x,$$

where $C_1 = \sin a \sinh a$, $C_2 = \cos a \cosh a$. This z satisfies the above conditions. For the detailed proof of this lemma we refer to [6; p. 172].

With Lemma 2.7 and several other lemmas we can obtain the generalization of Theorem 2.1.

Theorem 2.2. *Let $h \in \mathbb{H}$ with $\|h\| = 1$ and $4 < b < 19$. Then there is $\epsilon_0 > 0$ such that if $\mid \epsilon \mid < \epsilon_0$ equation (2.8) has at least two solutions.*

Proof: We can prove Lemmas 3, 5, 6, and 8 with replacing $1 + k\cos x$ $(k \geq 0)$ by $k_1 + k_2 \cos x$ $(k_1 > 0,\ k_2 \geq 0)$, respectively. Finally the proof of Theorem 2.2 is similar to that of Theorem 2.1.

3. Nonlinear equations with a variable coefficient. In this section we investigate the solutions of the equation

$$y^{(4)} + b(x)y^+ = 1, \quad y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0. \tag{3.1}$$

Uniqueness results for a class of equations similar to this have also been obtained by Aftabizadeh [1] and Yang [9]. To deal with the equation (3.1) we need the following powerful result of Schröder [8; p. 100].

Theorem 3.1. *For any positive right-hand side f the Green's fuction for the boundary value problem*

$$y^{(4)} + by = f(x), \ \ y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0, \tag{3.2}$$

in $(-\frac{\pi}{2}, \frac{\pi}{2})$, is nonnegative if and only if $-1 < b < c_0 = 4\kappa^4/\pi^4$, where κ is the smallest positive zero of the function $\tan x - \tanh x$. We have $\kappa = 3.9266$ and $c_0 = 9.762$.

Lemma 3.1. *Let $b(x)$ be even and $0 < b(x) < 9.7$ for all x in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Then the boundary value problem*

$$y^{(4)} + b(x)y = 1, \ \ y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0 \tag{3.3}$$

has a positive solution y, which is even.

Proof: Let P be the differential operator

$$Py = y^{(4)} + 9.7y.$$

Let $y^0 \equiv 0$ and y^{n+1} $(n = 0, 1, 2, \cdots)$ be the solution of the boundary value problem

$$Py^{n+1} = 1 + (9.7 - b(x))y^n, \ \ y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0.$$

It follows from Theorem 3.1 that each y^n $(n > 0)$ is even and positive. Now y^n $(n = 0, 1, 2, \cdots)$ is an increasing sequence of functions, since this sequence satisfies the following equations with the boundary conditions

$$P(y^{n+2} - y^{n+1}) = (9.7 - b(x))(y^{n+1} - y^n), \ \ n \geq 0.$$

On the other hand, the sequence $\{y^n(x)\}$ is bounded. In fact, let $\tilde{y}$ solve $y_{xxxx} = 1$ with the boundary condition. Then $y_0 \leq \tilde{y}$. Since

$$P(\tilde{y} - y^{n+1}) = 9.7(\tilde{y} - y^n) + b(x)y^n,$$

$y^n \leq \tilde{y}$ implies $y^{n+1} \leq \tilde{y}$ for all $n = 0, 1, 2, \cdots$. Hence y^n converges to y, which is a solution of (3.3). ∎

The eigenvalue problem for $y(x)$, in $(-\frac{\pi}{2}, \frac{\pi}{2})$

$$y^{(4)} = \lambda y, \ \ y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0, \ \ y(x) = y(-x)$$

has infinitely many eigenvalues λ_j and corresponding eigenfunctions ϕ_j $(j = 0, 1, 2, \cdots)$ given by

$$\lambda_j = (2j+1)^4, \phi_j(x) = \cos(2j+1)x.$$

The normalized eigenfunctions are denoted by

$$\theta_j = \frac{\phi_j}{\|\phi_j\|},$$

where $\|\phi_j\| = \sqrt{\pi/2}$ for $j = 0, 1, \cdots$. Let Q_1 be the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and H the Hilbert space defined by

$$H = \{y \in L_2(Q_1) : y \text{ is even}\}.$$

Then the set of functions $\{\theta_j\}$ is an orthonormal base in $\mathbb{H}$.

Lemma 3.2. *Let $b > -1$. If $f(x)$ is even and belongs to $L_2(Q_1)$, then the Green's function for equation (3.2) can be exactly determined as follows*

$$y = \sum_{j=0}^{\infty} c_j \theta_j, \quad c_j = \frac{1}{(2j+1)^4 + b} \int f \theta_j. \tag{3.4}$$

Proof: Let $y = \sum_{j=0}^{\infty} c_j \theta_j$ and define the inner product in H

$$\langle u, v \rangle = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} uv.$$

Then we can obtain from (3.2) that

$$\left\langle y^{(4)}, \theta_j \right\rangle + b \langle y, \theta_j \rangle = \langle f, \theta_j \rangle .$$

Hence we have, for all $j = 0, 1, 2, \cdots$,

$$c_j = \frac{1}{(2j+1)^4 + b} \langle f, \theta_j \rangle .$$

Lemma 3.3. *Let $f(x)$ be even and $0 \le f(x) \le 1$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. If y is a solution of the following equation*

$$y^{(4)} + 9.7y = f(x), \quad y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0, \tag{3.5}$$

then y is even and $0 \le y(x) < \frac{1}{7}$.

Proof: In the formula (3.4) the solution of (3.5) is given by

$$y = \sum_{j=0}^{\infty} c_j \phi_j, \quad c_j = \frac{2}{((2j+1)^4 + 9.7)\pi} \int f \phi_j \, dx.$$

Let us estimate c_j $(j \ge 0)$:

$$|c_0| = \frac{2}{10.7\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x) \cos x \, dx < \frac{1}{8},$$

$$|c_1| = \Big|\frac{2}{90.7\pi} \int_{-\frac{\pi}{2}}^{\pi 2} f(x) \cos 3x \, dx\Big| < \frac{1}{100},$$

$$|c_2| = \Big|\frac{2}{(5^4 + 9.7)\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x) \cos 5x \, dx\Big| < \frac{1}{100},$$

$$\sum_{j=3}^{\infty} |c_j| \le \sum_{j=3}^{\infty} \frac{2}{((2j+1)^4 + 9.7)\pi} \frac{2j+2}{2j+1} < \frac{1}{750} .$$

Hence we have

$$\sum_{j=0}^{\infty} |c_j| < \frac{1}{7} .$$

On the other hand y is nonnegative by Theorem 3.1. Therefore $0 \le y(x) < \frac{1}{7}$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Corollary. *If y is a solution of the boundary equation, in $(-\frac{\pi}{2}, \frac{\pi}{2})$,*

$$y^4 + 9.7y = 1, \quad y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0,$$

then $y(x)$ is even and $0 < y(x) < \frac{1}{7}$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Lemma 3.4. *Let $b(x)$ be even and $9.7 < b(x) < 16.7$ for all x in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Then there is a solution y of the boundary value equation, in $(-\frac{\pi}{2}, \frac{\pi}{2})$,*

$$y^{(4)} + b(x)y = 1, \quad y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0, \tag{3.6}$$

which is even and nonnegative.

Proof: Let P be the differential operator for $y(x)$

$$Py = y^{(4)} + 9.7y.$$

Put $y_0 = 0$ and let y_{j+1} $(j = 0, 1, 2, \cdots)$ be the solution of the boundary value problem

$$Py = 1 + (9.7 - b(x))y_j(x), \quad y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0. \tag{3.7}$$

Then $0 \le y_j(x) < \frac{1}{7}$ for all $j = 0, 1, 2, \cdots$ and all x in $(-\frac{\pi}{2}, \frac{\pi}{2})$. On the other hand

$$P(y_{j+1} - y_j) = (9.7 - b(x))(y_j - y_{j-1}).$$

Since $y_1 - y_0 > 0$, $y_2 - y_1 \le 0$ by Theorem 3.1. By induction, we have

$$y_{2j+1} - y_{2j} \ge 0 \ (j = 0, 1, \cdots), \quad y_{2j} - y_{2j-1} \le 0 \ (j = 1, 2, \cdots).$$

We claim that, in $(-\frac{\pi}{2}, \frac{\pi}{2})$,

$$y_{2j+2} - y_{2j} \ge 0, \quad j = 0, 1, 2, \cdots, \tag{3.8}$$

$$y_{2j+1} - y_{2j-1} \le 0, \quad j = 1, 2, \cdots. \tag{3.9}$$

In fact, $y_2 - y_0 \ge 0$. From

$$P(y_3 - y_1) = (9.7 - b(x))(y_2 - y_0)$$

we have $y_3 - y_1 \le 0$. From

$$P(y_4 - y_2) = (9.7 - b(x))(y_3 - y_1)$$

we have $y_4 - y_2 \ge 0$. Continuing this method, we can obtain (3.8) and (3.9). The sequence $\{y_{2j}\}$ is increasing and $0 \le y_{2j} < \frac{1}{7}$. Also $\{y_{2j-1}\}$ is decreasing and $0 \le y_{2j-1} < \frac{1}{7}$. Hence $\{y_{2j}\}$ and $\{y_{2j-1}\}$ converge. Let $y_{2j} \to Y_0$ and $y_{2j-1} \to Y_1$. From (3.7),

$$y_{2j} = P^{-1}(1 + (9.7 - b(x))y_{2j-1}), \quad y_{2j+1} = P^{-1}(1 + (9.7 - b(x))y_{2j}).$$

Letting $j \to \infty$, we obtain

$$Y_0 = P^{-1}(1 + (9.7 - b(x))Y_1), \quad Y_1 = P^{-1}(1 + (9.7 - b(x))Y_0).$$

Hence

$$P(Y_0 + Y_1) = 2 + (9.7 - b(x))(Y_0 + Y_1).$$

Therefore $\frac{1}{2}(Y_0 + Y_1)$ is a solution of (3.6) and nonnegative. ∎

With the method similar to the proof of Lemma 3.4, we can also prove the following lemma.

Lemma 3.5. *Let $0 < b_0 \leq 9.7$. Suppose that for any even function $f(x)$ with $0 \leq f(x) \leq 1$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$, the boundary value problem*

$$y^{(4)} + b_0 y = f, \quad y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0$$

has a solution y, which is even and satisfies $0 \leq y(x) \leq \frac{1}{c_0}$ for some c_0. If $b_0 < b(x) \leq b_0 + c_0$, then the boundary value problem

$$y^{(4)} + b(x)y = 1, \quad y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0$$

has a solution y, which is even and positive in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

The proof of the following uniqueness theorem is similar to that of Lemma 2.4.

Theorem 3.2. *Assume that b_0 and c_0 are the same as in Lemma 3.5. Let $b_0 < b(x) < b_0 + c_0$. Then the nonlinear equation*

$$y^{(4)} + b(x)y^+ = 1, \quad y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0$$

has a unique solution y, which is even and positive in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

4. Nonlinear suspension bridge equation with a variable coefficient. In this section we investigate nonlinear oscillations in a differential equation with a variable coefficient

$$u_{tt} + u_{xxxx} + b(x)u^+ = 1 \quad \text{in } \mathbb{H}, \tag{4.1}$$

where the Hilbert space $\mathbb{H}$ was defined in Section 1. We note that the set $\{\theta_{mn} : m, n = 0, 1, \cdots\}$ is an orthonormal base in $\mathbb{H}$.

In earlier sections and in [6], the fact that b is constant is crucial to the proof of an *a priori* bound. Here we show that if b depends on x, we can still get an *a priori* bound, although with more restrictions on $b(x)$.

From now on, let L be the differential operator

$$Lu(t, x) = u_{tt} + u_{xxxx}.$$

Then the eigenvalue problem for $u(t, x)$

$$Lu = \lambda u \quad \text{in } \mathbb{H}$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n + 1)^4 - 4m^2 \quad (m, n = 0, 1, 2, \cdots)$$

and corresponding eigenfunctions θ_{mn}, which were defined in Section 1. We note that all eigenvalues in the interval $(-19, 45)$ are given by

$$\lambda_{20} = -15 < \lambda_{10} = -3 < \lambda_{00} = 0 < \lambda_{41} = 17.$$

We consider the problem

$$Lu + b(x)u^+ = 0 \quad \text{in } \mathbb{H}. \tag{4.2}$$

Let $H_1 = \text{span}\,\{\theta_{10}\}$ and $H_2 = H_1^{\perp}$. Let P denote the orthogonal projection on H_1.

Theorem 4.1. *Let $b(x)$ be even and $1.65 \le b(x) \le 3.35$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Then equation (4.2) has only the trivial solution.*

Proof: Let $c = 1.65$. Then

$$\|(L+c)^{-1}\|_{H_1} = \frac{1}{1.35}, \quad \|(L+c)^{-1}\|_{H_2} = \frac{1}{2.65}.$$

Let us rewrite (4.2) as follows

$$(L+c)u + (b(x)-c)u^+ + cu^- = 0 \quad \text{in } \mathbb{H}.$$

Let $g(x) = b(x)u^+ + cu^-$ for all $u \in \mathbb{H}$. Then we have

$$0 \le (b(x)-c)|u| \le g(x) \le 1.7|u|$$

and $\|g\| \le 1.7\|u\|$. Decompose g as $g = v + w$, $v = Pg$, $w = (I-P)g$. Then there exists δ $(0 \le \delta \le 1)$ such that

$$\|v\|^2 = \delta^2(1.7)^2\|u\|^2, \quad \|w\|^2 \le (1-\delta^2)(1.7)^2\|u\|^2. \tag{4.3}$$

We note that $v = \alpha \cos 2t \cos x$ for some $\alpha \in \mathbb{R}$ and hence $\|v^+\|^2 = \frac{1}{2}\|v\|^2$. Thus we get

$$\|v\|^2 = \int gv = \int gv^+ - \int gv^- \le 1.7 \int |u|v^+ \le \frac{1.7}{\sqrt{2}}\|u\|\,\|v\|,$$

from which

$$\|u\| \le \frac{(1.7)^2}{2}\|u\|^2. \tag{4.4}$$

From (4.3) and (4.4) we obtain $0 \le \delta^2 \le \frac{1}{2}$. Summing up, we get

$$\begin{aligned}\|(L+c)^{-1}g(x)\|^2 &\le \|u\|^2(1.7)^2\left(\frac{\delta^2}{(1.35)^2} + \frac{1-\delta^2}{(2.65)^2}\right)\\ &\le \|u\|(1.7)^2\frac{1}{2}\left(\frac{1}{(1.35)^2} + \frac{1}{(2.65)^2}\right) < \|u\|^2,\end{aligned}$$

since $0 \le \delta^2 \le \frac{1}{2}$. Thus the equation

$$u + (L+c)^{-1}g(x) = 0 \quad \text{in } \mathbb{H}$$

has a unique solution, which is the trivial solution. That is, (4.2) has only the trivial solution.

We establish *a priori* bounds for solutions of (4.1).

Lemma 4.1. *There exists $R_0 > 0$ such that for all $b(x)$ with $1.65 \le b(x) \le 3.35$ the solutions u of (4.1) satisfy $\|u\| < R_0$.*

Proof: If not, there exists a sequence $(b_n(x), u_n)$ with $1.65 \le b_n(x) \le 3.35$ such that $\|u_n\| \to \infty$ and

$$u_n = L^{-1}(1 - b_n(x)u_n^+).$$

The functions $w_n = u_n/\|u_n\|$ satisfy

$$w_n = L^{-1}\left(\frac{1}{\|u_n\|} - b_n(x)w_n^+\right).$$

Since the operator L^{-1} is compact, we may assume that $w_n \to w_0$ and $b_n(x) \to b_0(x)$ with $1.65 \le b_0(x) \le 3.35$. We note that $\|w_n\| = 1$ and hence $\|w_0\| = 1$. Therefore we have

$$w_0 = L^{-1}(-b_0(x)w_0^+) \quad \text{or} \quad Lw_0 + b_0(x)w_0^+ = 0 \quad \text{in } \mathbb{H}.$$

This contradicts Theorem 4.1 and hence proves the lemma. ∎

In [6] *a priori* bounds for solutions of

$$Lu + bu^+ = 1 \quad \text{in } \mathbb{H} \tag{4.5}$$

were established for a constant coefficient b. That is,

Lemma 4.2. *There exists $R_1 > 0$ such that for all b with $0 \le b \le 14$ the solutions u of (4.5) satisfy $\|u\| < R_1$.*

Lemma 4.3. *Let $R_2 = \max\{R_0, R_1\}$. Under the assumptions and with the notation of Lemmas* 4.1, 4.2,

$$d_{LS}(u - L^{-1}(1 - b(x)u^+), B_R, 0) = 1$$

for all $R \ge R_2$.

Proof: We consider the homotopy

$$u - L^{-1}(1 - (3(1-t) + tb(x))u^+) = 0 \quad \text{in } \mathbb{H}. \tag{4.6}$$

If $1.65 \le b(x) \le 3.35$, then $1.65 \le 3(1-t) + tb(x) \le 3.35$. From Lemma 4.1 we know that (4.6) has no solution on $\|u\| = R \ge R_2$. Hence for $R \ge R_2$

$$d_{LS}(u - L^{-1}(1 - b(x)u^+), B_R, 0) = d_{LS}(u - L^{-1}(1 - 3u^+), B_R, 0).$$

Applying Lemma 4.3 and the invariance under homotopy, we have

$$d_{LS}(u - L^{-1}(1 - 3u^+), B_R, 0) = d_{LS}(u - L^{-1}(1), B_R 0) = d_{LS}(u, B_R, 0) = 1.$$

We know from Lemma 2.6 that for $3 < b(x) \le 3.35$ equation (4.1) has a positive solution $y(x)$.

Lemma 4.4. *Let $b(x)$ be even and $3 < b(x) \le 3.35$. Then there exists $\gamma > 0$ such that*

$$d_{LS}(u - L^{-1}(1 - b(x)u^+), B_\gamma(y), 0) = -1,$$

where y is the positive solution of (4.1).

Proof: If we follow the method of the proof of Lemma 2.6, then we can obtain

$$d_{LS}(u - L^{-1}(1 - b(x)u^+), B_\gamma(y), 0) = d_{LS}(u + L^{-1}(b(x)u), B_\gamma(0), 0)$$

for some small $\gamma > 0$. We consider the homotopy, $0 \leq t \leq 1$,

$$u + L^{-1}(3.3(1-t)u + tb(x)u) = 0 \quad \text{in } \mathbb{H}. \tag{4.8}$$

Since $3 < 3.3(1-t) + tb(x) \leq 3.35$ for all $0 \leq t \leq 1$, (4.8) has only the trivial solution and no solution on $\|u\| = \gamma$. Therefore

$$d_{LS}(u + L^{-1}(b(x)u), B_\gamma(0), 0) = d_{LS}(u + L^{-1}(3.3u), B_\gamma(0), 0) = -1.$$

For the last equality in the above equation we can refer to the end of the proof of Lemma 2.6. ∎

When $3 < b(x) \leq 3.35$, in Lemma 4.3

$$d_{LS}(u - L^{-1}(1 - b(x)u^+), B_R(0), 0) = 1$$

and in Lemma 4.4

$$d_{LS}(u - L^{-1}(1 - b(x)u^+), B_\gamma(y), 0) = -1.$$

Thus we can see that if we take $R > 0$ such that $B_R \supset B_\gamma(y)$, then

$$d_{LS}(u - L^{-1}(1 - b(x)u^+), B_R \setminus B_\gamma(y), 0) = 2.$$

Therefore (4.1) has at least two solutions, one in $B_\gamma(y)$ and the other one in $B_R(0) \setminus B_\gamma(y)$. From this fact we obtain our main result in this section.

Theorem 4.2. *Let $3 < b(x) \leq 3.35$ be even. Then (4.1) has at least two solutions.*

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