ON PERTURBING LYAPUNOV FUNCTIONS

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1. Introduction: It is known [2,3] that in proving uniform boundedness of a differential system by means of Lyapunov functions, it is sufficient to impose conditions in the complement of a compact set in R^n, whereas, in the case of equiboundedness, the proofs demand that the assumptions hold everywhere in R^n.

We wish to present, in this paper, a new idea which permits us to discuss nonuniform properties of solutions of differential equations under weaker assumptions. Our results will show that the equiboundedness can be proved without assuming conditions everywhere in R^n (as in the case of uniform boundedness), provided we appropriately perturb the Lyapunov functions. Our results also imply that in those situations when the Lyapunov function found does not satisfy all the desired conditions, it is fruitful to perturb that Lyapunov function rather than discard it. We also discuss the corresponding situation relative to equistability.

We feel that the idea of perturbing Lyapunov functions introduced in this paper is a useful and important tool in the study of nonuniform properties of solutions as well as the preservation of those properties under constantly acting perturbations and therefore deserves further investigation.

2. Equiboundedness. We consider the differential system

$$x' = f(t,x), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C[R^+ \times R^n, R^n]$. Here R^+ denotes the nonnegative real line, R^n the euclidean space and $C[R^+ \times R^n, R^n]$ the class of continuous functions from $R^+ \times R^n$ to R^n. For any set $E \subset R^n$, we denote by $\overline{E}$, E^c and ∂E, the closure, the complement and the boundary of E respectively. For any $\rho > 0$, let $S(\rho) = [x \in R^n : ||x|| < \rho]$, $||\cdot||$ being any convenient norm in R^n. For the stability and boundedness definitions, see [2].

Theorem 1. Assume that

(i) $E \subset R^n$ is compact, $V_1 \in C[R^+ \times \overline{E^c}, R^+]$, $V_1(t,x)$ is locally Lipschitzian in x, bounded for $(t,x) \in R^+ \times \partial E$, and

$$D^+V_1(t,x) \equiv \limsup_{h \to 0^+} \frac{1}{h} \left[V_1(t+h, x+hf(t,x)) - V_1(t,x)\right]$$

$$\leq g_1(t, V_1(t,x)), \quad (t,x) \in R^+ \times \overline{E^c}, \tag{2.2}$$

where $g_1 \in C[R^+ \times R^+, R]$;

(ii) $V_2 \in C[R^+ \times S^c(\rho), R^+]$, $V_2(t,x)$ is locally Lipschitzian in x,

$$b(||x||) \leq V_2(t,x) \leq a(||x||), \quad (t,x) \in R^+ \times S^c(\rho), \tag{2.3}$$

where $a, b \in C[[\rho,\infty), R^+]$ such that $b(u) \to \infty$ as $u \to \infty$ (ρ may be sufficiently large) and for $(t,x) \in R^+ \times S^c(\rho)$,

$$D^+V_1(t,x) + D^+V_2(t,x) \le g_2(t,V_1(t,x) + V_2(t,x)) \tag{2.4}$$

where $g_2 \in C[R^+ \times R^+, R]$;

(iii) the scalar differential equations

$$u' = g_1(t,u), \quad u(t_0) = u_0 \ge 0, \tag{2.5}$$

and

$$v' = g_2(t,v), \quad v(t_0) = v_0 \ge 0 \tag{2.6}$$

are equibounded and uniformly bounded respectively. Then, the system (2.1) is equibounded.

Proof. Since E is compact, there exists a ρ (may be sufficiently large) such that $S(\rho) \supset S(E,\rho_0)$ for some $\rho_0 > 0$. Here $S(E,\rho_0) = [x \in R^n : d(x,E) < \rho_0]$, where $d(x,E) = \inf_{y \in E} ||x-y||$. Let $t_0 \in R^+$ and $\alpha \ge \rho$ be given. Let $\alpha_1 = \alpha_1(t_0,\alpha) = \max(\alpha_0,\alpha^*)$ where $\alpha_0 = \max[V_1(t_0,x_0) : x_0 \in \overline{S(\alpha) \cap E^c}]$ and $\alpha^* \ge V_1(t,x)$ for $(t,x) \in R^+ \times \partial E$. Since the equation (2.5) is equibounded, given $\alpha_1 > 0$ and $t_0 \in R^+$, there exists a $\beta_0 = \beta_0(t_0,\alpha_1)$ such that

$$u(t,t_0,u_0) < \beta_0, \quad t \ge t_0, \tag{2.7}$$

provided $u_0 < \alpha_1$, where $u(t, t_0,u_0)$ is any solution of (2.5) Also, uniform boundedness of the equation (2.6) yields that

$$v(t,t_0,v_0) < \beta_1(\alpha_2), \quad t \ge t_0, \tag{2.8}$$

provided $v_0 < \alpha_2$, where $v(t,t_0,v_0)$ is any solution of (2.6). We set $u_0 = v_1(t_0,x_0)$ and $\alpha_2 = a(\alpha) + \beta_0$. As $b(u) \to \infty$ with $u \to \infty$, we can choose a $\beta = \beta(t_0,\alpha)$ such that

$$b(\beta) > \beta_1(\alpha_2). \qquad (2.9)$$

We now claim that $x_0 \in S(\alpha)$ implies that any solution $x(t,t_0,x_0)$ satisfies $x(t,t_0,x_0) \in S(\beta)$, for $t \geq t_0$. If this is not true, there exists a solution $x(t,t_0,x_0)$ of (2.1) with $x_0 \in S(\alpha)$ such that for some $t^* > t_0$, $||x(t^*,t_0,x_0)|| = \beta$. Since $S(E,\rho_0) \subset S(\alpha)$, there are two possibilities to consider:

(i) $\quad x(t,t_0,x_0) \in E^c$ for $t \in [t_0,t^*]$;

(ii) there exists a $\tilde{t} \geq t_0$ such that $x(\tilde{t}, t_0,x_0) \in \partial E$ and $x(t,t_0,x_0) \in \overline{E^c}$ for $t \in [\tilde{t},t^*]$.

In case (i) holds, we can find $t_1 > t_0$ such that

$$\left.\begin{aligned} & x(t_1,t_0,x_0) \in \partial S(\alpha), \\ & x(t^*,t_0,x_0) \in \partial S(\beta), \\ \text{and} \quad & x(t, t_0,x_0) \in S^c(\alpha), \quad t \in [t_1,t^*]. \end{aligned}\right\} \qquad (2.10)$$

Setting $m(t) = V_1(t, x(t,t_0,x_0)) + V_2(t,x(t,t_0,x_0))$ for $t \in [t_1,t^*]$, it is easy to obtain, from (2.4), using standard arguments [see 1,2], the differential inequality

$$D^+m(t) \leq g_2(t,m(t)), \quad t \in [t_1,t^*].$$

Consequently, the theory of differential inequalities [Th. 1.4.1,2] gives

$$m(t) \le r_2(t,t_1,m(t)), \quad t \in [t_1,t^*] ,$$

where $r_2(t,t_1,v_0)$ is the maximal solution of (2.6) such that $r_2(t_1,t_1,v_0) = v_0$.

Thus,

$$\begin{aligned} &V_1(t^*, x(t^*,t_0,x_0)) + V_2(t^*, x(t^*,t_0,x_0)) \\ &\quad \le r_2\Big[t^*,t_1,V_1(t_1,x(t_1,t_0,x_0)) + V_2(t_1,x(t_1,t_0,x_0))\Big] . \end{aligned} \tag{2.11}$$

Similarly, because of (2.2), we also have

$$V_1(t_1,x(t_1,t_0,x_0)) \le r_1(t_1,t_0,V_1(t_0,x_0)) \tag{2.12}$$

where $r_1(t,t_0,u_0)$ is the maximal solution of (2.5). In view of the fact that $u_0 = V_1(t_0,x_0) < \alpha_1$, (2.7) yields

$$r_1(t_1,t_0,V_1(t_0,x_0)) \le \beta_0 .$$

Furthermore, $V_2(t_1,x(t_1,t_0,x_0)) \le a(\alpha)$ because of (2.3) and (2.10). Consequently, we have

$$\begin{aligned} v_0 &= V_1(t_1,x(t_1,t_0,x_0)) + V_2(t_1,x(t_1,t_0,x_0)) \\ &< \beta_0 + a(\alpha) = \alpha_2 . \end{aligned} \tag{2.13}$$

Hence, the inequality (2.11) gives, because of the relations (2.3), (2.8), (2.9), (2.10), (2.13) and the fact that $V_1 \ge 0$,

$$b(\beta) \le \beta_1(\alpha_2) < b(\beta) \tag{2.14}$$

which is a contradiction.

In case (ii) holds, we again arrive at the inequality (2.11), where $t_1 > \tilde{t}$ satisfies (2.10). We now have, in place of (2.12), the relation

$$V_1(t_1,x(t_1,t_0,x_0)) \le r_1\Big[t_1,\tilde{t},V_1(\tilde{t},x(\tilde{t},t_0,x_0))\Big] .$$

Since $x(\tilde{t},t_0,x_0) \in \partial E$ and $V_1(\tilde{t},x(\tilde{t},t_0,x_0)) \le \alpha^* \le \alpha_1$, arguing as before, we arrive at the contradiction (2.14). This proves that if $x_0 \in S(\alpha)$, $\alpha \ge \rho$, $x(t,t_0,x_0) \in S(\beta)$, for $t \ge t_0$. For $\alpha < \rho$, we set $\beta(t_0,\alpha) = \beta(t_0,\rho)$ and hence the proof is complete.

Remarks: Theorem 1 improves significantly the equiboundedness result in [Th. 3.13.1,2]. Consider the special case $g_1 = g_2 \equiv 0$ which improves a similar result in [3] and Corollary 3.13.1 in [2]. The hypothesis (ii) together with $D^+V_1 \le 0$ is not enough to apply Corollary 3.13.1 because we will not have $D^+V_2 \le 0$. Also, hypothesis (i) is not sufficient to imply the stated result. We may be tempted to conclude, at a first glance, that by setting $V = V_1 + V_2$, all the assumptions of Corollary 3.13.1 in [2] are satisfied. This is not true because the right estimate in (2.3), namely $V(t,x) \le a(||x||)$, does not hold. As a result, the proof of Corollary 3.13.1 in [2] breaks down. Thus, our results demonstrate the advantage of perturbing Lyapunov functions.

3. <u>Equistability</u>: For the purpose of this section, it is enough to suppose that $f \in C[R^+ \times S(\rho), R^n]$, for some $\rho > 0$.

Theorem 2. Assume that

(i) $V_1 \in C[R^+ \times S(\rho), R^+]$, $V_1(t,x)$ is locally Lipschitzian in x, $V_1(t,o) \equiv 0$ and

$$D^+V_1(t,x) \leq g_1(t,V_1(t,x)),\ (t,x) \in R^+ \times S(\rho),$$

where $g_1 \in C[R^+ \times R^+, R]$ and $g_1(t,o) \equiv 0$;

(ii) for every $\eta > 0$, there exists a $V_{2,\eta} \in C[R^+ \times S(\rho) \cap S^c(\eta), R^+]$, $V_{2,\eta}$ is locally Lipschitzian in x,

$$b(||x||) \leq V_{2,\eta}(t,x) \leq a(||x||),\ (t,x) \in R^+ \times S(\rho) \cap S^c(\eta),$$

where $a, b \in C[(o,\rho), R^+]$, $a(u)$, $b(u)$ increasing in u and $a(u) \to 0$ as $u \to 0$ and

$$D^+V_1(t,x) + D^+V_2(t,x) \leq g_2(t,V_1(t,x) + V_2(t,x))$$

for $(t,x) \in R^+ \times S(\rho) \cap S^c(\eta)$, where $g_2 \in C[R^+ \times R^+, R]$, $g_2(t,o) \equiv 0$;

(iii) the trivial solution is equistable with respect to the equation

$$u' = g_1(t,u),\ u(t_0) = u_0 \geq 0, \tag{3.1}$$

and uniformly stable with respect to the equation

$$v' = g_2(t,v),\ v(t_0) = v_0 \geq 0. \tag{3.2}$$

Then, the trivial solution of the system (2.1) is equistable.

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in R^+$ be given. Since the trivial solution is uniformly stable relative to the equation (3.2), given $b(\varepsilon) > 0$ and $t_0 \in R^+$, there exists a $\delta_0 = \delta_0(\varepsilon) > 0$ such that

$$v(t,t_0,v_0) < b(\varepsilon), \quad t \geq t_0, \tag{3.3}$$

provided $v_0 < \delta_0$, where $v(t,t_0,v_0)$ is any solution of (3.2). In view of the hypothesis on $a(u)$, there is a $\delta_2 = \delta_2(\varepsilon) > 0$ such that

$$a(\delta_2) < \frac{\delta_0}{2} \quad . \tag{3.4}$$

Also, since the trivial solution of equation (3.1) is equistable, given $\frac{\delta_0}{2} > 0$ and $t_0 \in R^+$, there exists a $\delta^* = \delta^*(t_0,\varepsilon)$ such that

$$u(t,t_0,u_0) < \frac{\delta_0}{2}, \quad t \geq t_0, \tag{3.5}$$

whenever $u_0 < \delta^*$, $u(t,t_0,u_0)$ being any solution of (3.1).

Choose $u_0 = V_1(t_0, x_0)$. Since $V_1(t,x)$ is continuous and $V_1(t,o) \equiv 0$, there exists a $\delta_1 > 0$ such that

$$||x_0|| < \delta_1 \quad \text{and} \quad V_1(t_0,x_0) < \delta^* \tag{3.6}$$

hold simultaneously. We set $\delta = \min(\delta_1,\delta_2)$. Then, we claim that $||x_0|| < \delta$ implies $||x(t,t_0,x_0)|| < \varepsilon$ for $t \geq t_0$. If this were false, there would exist a solution $x(t,t_0,x_0)$ of (2.1) with $||x_0|| < \delta$ and $t_1,t_2 > t_0$ such that

$$\left.\begin{aligned} &x(t_1,t_0,x_0) \in \partial S(\delta_2), \\ &x(t_2,t_0,x_0) \in \partial S(\varepsilon) \\ \text{and} \qquad &x(t, t_0,x_0) \in \overline{S(\varepsilon) \cap S(\delta_2)}, \quad t \in [t_1,t_2]. \end{aligned}\right\} \tag{3.7}$$

We let $\delta_2 = \eta$ so that the existence of a $V_{2,\eta}$ satisfying hypothesis (ii) is assured. Hence, setting $m(t) = V_1(t,x(t,t_0,x_0)) + V_{2,\eta}(t,x(t,t_0,x_0))$, $t \in [t_1,t_2]$, we obtain the differential inequality

$$D^+m(t) \le g_2(t,m(t)), \quad t \in [t_1,t_2],$$

which yields

$$V_1(t_2,x(t_2,t_0,x_0)) + V_{2,\eta}(t_2,x(t_2,t_0,x_0))$$

$$\le r_2\Big[t_2,t_1,V_1(t_1,x(t_1,t_0,x_0)) + V_{2,\eta}(t_1,x(t_1,t_0,x_0))\Big],$$

$r_2(t,t_1,v_0)$ being the maximal solution of (3.2) such that $r_2(t_1,t_1,v_0) = v_0$. We also have

$$V_1(t_1,x(t_1,t_0,x_0)) \le r_1(t_1,t_0,V_1(t_0,x_0)),$$

where $r_1(t, t_0,u_0)$ is the maximal solution of (3.1). By (3.5) and (3.6), we get

$$V_1(t_1,x(t_1,t_0,x_0)) < \frac{\delta_0}{2} \; . \tag{3.8}$$

Also, by (3.4), (3.7) and the assumptions on $V_{2,\eta}$, we have

$$V_{2,\eta}(t_1,x(t_1,t_0,x_0)) \le a(\delta_2) < \frac{\delta_0}{2} \, . \tag{3.9}$$

The inequalities (3.8), (3.9) together with (3.3), (3.7), $V_1 \geq 0$ and $V_{2,\eta} \geq b(||x||)$ lead to the contradiction $b(\varepsilon) < b(\varepsilon)$. Hence, the proof of the theorem is complete.

Remarks: Known results [1,2] on equistability require that the assumptions hold everywhere in $S(\rho)$. Here, in Theorem 2, we have relaxed this requirement considerably. One can make comments similar to those in section 2 for this situation also. We shall not repeat them.

References

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