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ON Φ -RECURRENT $N(k)$-CONTACT METRIC MANIFOLDS

Dedicated to PROFESSOR DAVID E. BLAIR

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Abstract. In this paper we prove that a ϕ-recurrent $N(k)$-contact metric manifold is an η-Einstein manifold with constant coefficients. Next, we prove that a 3-dimensional ϕ-recurrent $N(k)$-contact metric manifold is of constant curvature. The existence of a ϕ-recurrent $N(k)$-contact metric manifold is also proved.

1. Introduction

The notion of local symmetry of a Riemannian manifold has been weakend by many authors in several ways to a different extent. As a weaker version of local symmetry, T.Takahashi [1] introduced the notion of local ϕ-symmetry on a Sasakian manifold. Generalizing the notion of local ϕ-symmetry, one of the authors, De, [2] introduced the notion of ϕ-recurrent Sasakian manifold. In the context of contact geometry the notion of ϕ-symmetry is introduced and studied by Boeckx, Bueken and Vanhecke [3] with several examples.

In the present paper we study ϕ-recurrent $N(k)$-contact metric manifold which generalizes the result of De, Shaikh and Biswas [2]. The paper is organized as follows:
Section 2 contains necessary details about contact metric manifolds, some preliminaries and a brief account of (k, μ) manifolds and the basic results. In Section 3, it is proved that a ϕ-recurrent $N(k)$-contact metric manifold is a special type of η-Einstein manifold. Also it is shown that the characteristic vector field of the $N(k)$-contact metric manifold and the vector field associated to the 1-form of recurrence are co-directional. In Section 4, it is also proved that a 3-dimensional ϕ-recurrent $N(k)$-contact metric manifold is of constant curvature. The last section provides the existence of the ϕ-recurrent $N(k)$-contact metric manifold by an example which is neither symmetric nor locally ϕ-symmetric.

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2. **Contact Metric Manifolds**

A $(2n+1)$-dimensional manifold M^{2n+1} is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

(2.1)
$$(a)\quad \phi^2 = -I + \eta \otimes \xi, \quad (b) \quad \eta(\xi) = 1, \quad (c) \quad \phi\xi = 0, \quad (d) \quad \eta \circ \phi = 0.$$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M^{2n+1} \times \mathbf{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbf{R}$ and f is a smooth function on $M \times \mathbf{R}$. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (2.1) it can be easily seen that

$$(a) g(X, \phi Y) = -g(\phi X, Y), (b) g(X, \xi) = \eta(X), \tag{2.3}$$

for all vector fields X, Y. An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y), \tag{2.4}$$

for all vector fields X, Y. The 1-form η is then a contact form and ξ is its characterstic vector field. We define a $(1,1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the Lie-differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. We have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Also,

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.5}$$

holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in TM, \tag{2.6}$$

where ∇ is the Levi-Civita connection of the Riemannian metric g. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is a Killing vector is said to be a K-contact manifold. A Sasakian manifold is K-contact but not conversely. However a 3-dimensional K-contact manifold is Sasakian [4]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X,Y)\xi = 0$ ([5]). On the other hand, on a Sasakian manifold the following holds:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.7}$$

As a generalization of both $R(X,Y)\xi = 0$ and the Sasakian case; D. Blair, T. Koufogiorgos and B. J. Papantoniou [6] considered the (k,μ)-nullity condition on a contact metric manifold and gave several reasons for studying it. The (k,μ)-nullity distribution $N(k,\mu)$ ([6], [7]) of a contact metric manifold M is defined by

$$\begin{aligned} N(k,\mu) : p \longrightarrow N_p(k,\mu) \\ = \{W \in T_pM : R(X,Y)W = (kI + \mu h)(g(Y,W)X - g(X,W)Y)\}, \end{aligned}$$

for all $X, Y \in TM$, where $(k,\mu) \in \mathbf{R}^2$. A contact metric manifold M^{2n+1} with $\xi \in N(k,\mu)$ is called a (k,μ)-manifold. In particular on a (k,μ)-manifold, we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.8}$$

On a (k,μ)-manifold $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and μ is indeterminant) and if $k < 1$, the (k,μ)-nullity condition determines the curvature of M^{2n+1} completely [6]. Infact, for a (k,μ)-manifold, the condition of being a Sasakian manifold, a K-contact manifold, $k = 1$ and $h = 0$ are all equivalent.

In a (k,μ)-manifold the following relations hold ([6], [8]):

$$h^2 = (k-1)\phi^2, \quad k \leq 1, \tag{2.9}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.10}$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.11}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.12}$$

$$\begin{aligned} S(X,Y) =& [2(n-1) - n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \\ &+ [2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \quad n \geq 1, \end{aligned} \tag{2.13}$$

$$r = 2n(2n - 2 + k - n\mu), \tag{2.14}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX,Y), \tag{2.15}$$

where S is the Ricci tensor of type $(0,2)$, Q is the Ricci-operator, that is, $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold. From (2.5), it follows that

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y). \tag{2.16}$$

Also in a (k,μ)-manifold

$$\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + \mu[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)] \tag{2.17}$$

holds.

The k-nullity distribution $N(k)$ of a Riemannian manifold M [9] is defined by

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = g(Y,Z)X - g(X,Z)Y\},$$

k being a constant. If the characterstic vector field $\xi \in N(k)$, then we call a contact metric manifold an $N(k)$-contact metric manifold [10]. If $k = 1$, then $N(k)$-contact metric manifold is Sasakian and if $k = 0$, then $N(k)$-contact metric manifold is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$. If $k < 1$, the scalar curvature is $r = 2n(2n-2+k)$. If $\mu = 0$, then a (k,μ)-contact metric manifold reduces to a $N(k)$-contact metric manifold.

In [11], $N(k)$-contact metric manifold were studied in some detail. For more details we reffer to [12] [13].

In $N(k)$-contact metric manifold the following relations hold:

$$h^2 = (k-1)\phi^2, \quad k \leq 1, \tag{2.18}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.19}$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X], \tag{2.20}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.21}$$

$$S(X,Y) = 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) \tag{2.22}$$

$$+ [2(1-n) + 2nk]\eta(X)\eta(Y), \quad n \geq 1, \tag{2.23}$$

$$r = 2n(2n-2+k), \tag{2.24}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 4(n-1)g(hX,Y), \tag{2.25}$$

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y), \tag{2.26}$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{2.27}$$

$$\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{2.28}$$

3. ϕ-recurrent $N(k)$-contact metric manifolds

Definition 1. ([1]) A Sasakian manifold is said to be locally ϕ-symmetric if the relation

$$\phi^2((\nabla_W R)(X,Y)Z)=0$$

holds for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2. ([2]) A $N(k)$-contact metric manifold is said to be ϕ-recurrent if and only if there exists a non-zero 1-form A such that

$$\phi^2((\nabla_W R)(X,Y)Z)=A(W)R(X,Y)Z, \tag{3.1}$$

for all vector fields X, Y, Z, W. Here X, Y, Z, W are arbitary vector fields which are not necessarily orthogonal to ξ.

If the 1-form A vanishes identically, then the manifold is said to be a locally ϕ-symmetric manifold.

Definition 3. ([6]) A contact manifold is said to be η-Einstein if the Ricci tensor S of type $(0,2)$ satisfies the condition

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y), \tag{3.2}$$

where a and b are smooth funtions on M^{2n+1}.

Now we prove the main theorem of the paper.

Theorem 3.1. *A ϕ-recurrent $N(k)$-contact metric manifold is an η-Einstein manifold with constant coefficients.*

Proof. By virtue of (2.1)(a) and (3.1) we have

$$-(\nabla_W R)(X,Y)Z+\eta((\nabla_W R)(X,Y)Z)\xi=A(W)R(X,Y)Z, \tag{3.3}$$

from which it follows that

$$\begin{aligned} &-g((\nabla_W R)(X,Y)Z,U)+\eta((\nabla_W R)(X,Y)Z)\eta(U) \\ =\ &A(W)g(R(X,Y)Z,U). \end{aligned} \tag{3.4}$$

Let $\{e_i\}$, $i=1,2,3,......,2n+1$, be an orthonormal basis of the tangent space at any point of the manifold. Putting $X=U=\{e_i\}$ in (3.4) and taking summation over i, $1\le i\le 2n+1$, we get

$$-(\nabla_W S)(Y,Z)+\sum_{i=1}^{2n+1}\eta((\nabla_W R)(e_i,Y)Z)\eta(e_i)=A(W)S(Y,Z). \tag{3.5}$$

The second term of (3.5) by putting $Z=\xi$ takes the form $g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi)$, which is denoted by E. In this case E vanishes.

Namely we have

$$\begin{aligned} g((\nabla_W R)(e_i, Y)\xi, \xi) &= g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(\nabla_W e_i, Y)\xi, \xi) \\ &- g(R(e_i, \nabla_W Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi) \end{aligned}$$

at $p \in M$. Using (2.3)(b) and (2.27) we obtain

$$\begin{aligned} g(R(e_i, \nabla_W Y)\xi, \xi) &= g(k[\eta(\nabla_W Y)e_i - \eta(e_i)\nabla_W Y], \xi) \\ &= k[\eta(\nabla_W Y)\eta(e_i) - \eta(e_i)\eta(\nabla_W Y)] = 0. \end{aligned}$$

Thus we obtain

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi).$$

In virtue of $g(R(e_i, Y)\xi, \xi) = g(R(\xi, \xi)e_i, Y) = 0$, we have

$$g(\nabla_W R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla_W \xi) = 0, \quad \text{since } (\nabla_W g) = 0,$$

which implies

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, \nabla_W \xi) - g(R(e_i, Y)\nabla_W \xi, \xi) = 0.$$

Using (2.5) and applying skew-symmetry of R we get

$$\begin{aligned} &g((\nabla_W R)(e_i, Y)\xi, \xi) \\ &= g(R(e_i, Y)\xi, \phi W + \phi hW) + g(R(e_i, Y)(\phi W + \phi hW), \xi) \\ &= g(R(\phi W + \phi hW, \xi)Y, e_i) + g(R(\xi, \phi W + \phi hW)Y, e_i). \end{aligned}$$

Hence we obtain

$$\begin{aligned} E = \sum_{i=1}^{2n+1} \Big[&g(R(\phi W + \phi hW, \xi)Y, e_i)g(\xi, e_i) \\ &+ g(R(\xi, \phi W + \phi hW)Y, e_i)g(\xi, e_i) \Big] \\ =& g(R(\phi W + \phi hW, \xi)Y, \xi) + g(R(\xi, \phi W + \phi hW)Y, \xi) = 0. \end{aligned}$$

Replacing Z by ξ in (3.5) and using (2.21) we have

$$-(\nabla_W S)(Y, \xi) = 2nkA(W)\eta(Y). \tag{3.6}$$

Now we have

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi).$$

Using (2.21) and (2.5) in the above relation, it follows that

$$(\nabla_W S)(Y, \xi) = 2nk(\nabla_W \eta)(Y) + S(Y, \phi W + \phi hW). \tag{3.7}$$

In virtue of (3.7), (2.26) and (2.3)(a) we get

$$(\nabla_W S)(Y, \xi) = -2nkg(\phi W + \phi hW, Y) + S(Y, \phi W + \phi hW). \tag{3.8}$$

By (3.6) and (3.8) we have

$$2nkg(\phi W + \phi hW, Y) - S(Y, \phi W + \phi hW) = 2nkA(W)\eta(Y). \tag{3.9}$$

Replacing Y by ϕY in (3.9) and using (2.1)(d), (2.2), (2.25) we get

$$2nkg(\phi W + \phi hW, \phi Y) - S(\phi Y, \phi W + \phi hW) = 0$$

or,

$$\begin{aligned} &2nk[g(W + hW, Y) - \eta(W + hW)\eta(Y)] - S(Y, W + hW) \\ &+2nk\eta(W + hW)\eta(Y) + 4(n-1)g(hY, W + hW) = 0 \end{aligned}$$

or,

$$\begin{aligned} &2nkg(Y, W) + 2nkg(Y, hW) - S(Y, W) - S(Y, hW) \\ &+4(n-1)g(Y, hW) + 4(n-1)g(Y, h^2W) = 0 \end{aligned}$$

since, $g(X, hY) = g(hX, Y)$. Now by (2.23), (2.18) and (2.1)(a) this implies

$$\begin{aligned} S(Y, W) + S(Y, hW) &= 2nkg(Y, W) + [2nk + 4(n-1)]g(Y, hW) \\ &\quad + 4(n-1)(k-1)g(Y, -W + \eta(W)\xi) \end{aligned}$$

or,

$$\begin{aligned} &S(Y, W) + 2(n-1)g(Y, hW) - 2(n-1)(k-1)g(Y, W) \\ &+2(n-1)(k-1)\eta(Y)\eta(W) = [2nk - 4(n-1)(k-1)]g(Y, W) \\ &+[2nk + 4(n-1)]g(Y, hW) + 4(n-1)(k-1)\eta(Y)\eta(W), \end{aligned}$$

which implies,

$$\begin{aligned} &S(Y, W) = 2(n + k - 1)g(Y, W) \\ &+2(nk + n - 1)g(Y, hW) + 2(n-1)(k-1)\eta(Y)\eta(W). \end{aligned} \tag{3.10}$$

Replacing W by hW and using (2.23), (2.18) and (2.1)(a) we get from (3.10)

$$-2kg(Y, hW) = -2nk(k-1)g(Y, W) + 2nk(k-1)\eta(Y)\eta(W).$$

Since we may assume that $k \neq 0$, this implies

$$g(Y, hW) = n(k-1)g(Y, W) - n(k-1)\eta(Y)\eta(W). \tag{3.11}$$

From (3.10) and (3.11) we get

$$\begin{aligned} S(Y, W) &= 2[(n + k - 1) + n(k-1)(nk + n - 1)]g(Y, W) \\ &+ 2[(n-1)(k-1) - n(k-1)(nk + n - 1)]\eta(Y)\eta(W) \end{aligned}$$

or,

$$S(Y, W) = ag(Y, W) + b\eta(Y)\eta(W), \tag{3.12}$$

where $a = 2[(n+k-1) + n(k-1)(nk+n-1)]$, $b = 2[(n-1)(k-1) - n(k-1)(nk+n-1)]$ are constant. So, the manifold is an η-Einstein manifold with constant coefficients. Hence the theorem is proved. □

Now, from (3.3) we have

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z. \tag{3.13}$$

From (3.13) and the second Bianchi identity we get

$$A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) = 0. \tag{3.14}$$

Using (2.28), we get from (3.14)

$$\begin{aligned} &k[A(W)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) + A(X)(g(W,Z)\eta(Y) \\ &-g(Y,Z)\eta(W)) + A(Y)(g(X,Z)\eta(W) - g(W,Z)\eta(X))] = 0. \end{aligned} \tag{3.15}$$

Putting $Y = Z = \{e_i\}$ in (3.15) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$k(2n-1)[A(W)\eta(X) - A(X)\eta(W)] = 0,$$

which implies that

$$A(W)\eta(X) = A(X)\eta(W). \tag{3.16}$$

Replacing X by ξ in (3.16), it follows that

$$A(W) = \eta(\rho)\eta(W), \tag{3.17}$$

for any vector field W, where $A(\xi) = g(\xi,\rho) = \eta(\rho)$, ρ being the vector field associated to the 1-form A, that is, $g(X,\rho) = A(X)$. Hence we can state the following theorem:

Theorem 3.2. *In a ϕ-recurrent $N(k)$-contact metric manifold (M^{2n+1}, g), $n > 1$, the charaterstic vector field ξ and the vector field ρ associated to the 1-form A are co-directional and the 1-form A is given by (3.17).*

4. 3-dimensional ϕ-recurrent $N(k)$-contact metric manifolds

In a 3-dimensional Riemannian manifold we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &\quad -S(X,Z)Y + \frac{r}{2}[g(X,Z)Y - g(Y,Z)X], \end{aligned} \tag{4.1}$$

where Q is the Ricci-operator, that is, $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold. Now putting $Z = \xi$ in (4.1) and using (2.3)(b) and (2.21), we get

$$\begin{aligned} R(X,Y)\xi &= \eta(Y)QX - \eta(X)QY \\ &\quad +2k[\eta(Y)X - \eta(X)Y] + \frac{r}{2}[\eta(X)Y - \eta(Y)X]. \end{aligned} \tag{4.2}$$

Using (2.27) in (4.2), we have

$$(k - \frac{r}{2})[\eta(Y)X - \eta(X)Y] = \eta(X)QY - \eta(Y)QX. \tag{4.3}$$

Puting $Y = \xi$ in (4.3) and using (2.21), we get

$$QX = (\frac{r}{2} - k)X + (3k - \frac{r}{2})\eta(X)\xi. \tag{4.4}$$

Therefore, it follows from (4.4) that

$$S(X,Y) = (\frac{r}{2} - k)g(X,Y) + (3k - \frac{r}{2})\eta(X)\eta(Y). \tag{4.5}$$

Thus from (4.1), (4.4) and (4.5), we get

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} - 2k)[g(Y,Z)X - g(X,Z)Y] \\ &\quad +(3k - \frac{r}{2})[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{4.6}$$

Taking the covariant differentiation to the both sides of the equation (4.6), we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z = \frac{dr(W)}{2}&[g(Y,Z)X - g(X,Z)Y - g(Y,Z)\eta(X)\xi \\ &+ g(X,Z)\eta(Y)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y] \\ &+ (3k - \frac{r}{2})[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\nabla_W\xi \\ &+ (3k - \frac{r}{2})[\eta(Y)X - \eta(X)Y](\nabla_W\eta)(Z) \\ &+ (3k - \frac{r}{2})[g(Y,Z)\xi - \eta(Z)Y](\nabla_W\eta)(X) \\ &- (3k - \frac{r}{2})[g(X,Z)\xi - \eta(Z)X](\nabla_W\eta)(Y). \end{aligned} \tag{4.7}$$

Noting that we may assume that all vector fields X, Y, Z, W are orthogonal to ξ and using (2.1)(b), we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z = \tfrac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ +(3k - \tfrac{r}{2})[g(Y,Z)(\nabla_W\eta)(X) - g(X,Z)(\nabla_W\eta)(Y)]\xi. \end{aligned} \tag{4.8}$$

Applying ϕ^2 to the both sides of (4.8) and using (2.1)(a) and (2.1)(c), we get

$$\phi^2(\nabla_W R)(X,Y)Z = \frac{dr(W)}{2}[g(X,Z)Y - g(Y,Z)X]. \tag{4.9}$$

By (3.1) the equation (4.9) reduces to

$$A(W)R(X,Y)Z = \frac{dr(W)}{2}[g(X,Z)Y - g(Y,Z)X]. \tag{4.10}$$

Putting $W = \{e_i\}$, where $\{e_i\}$, $i = 1, 2, 3$, is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \leq i \leq 3$, we obtain

$$R(X,Y)Z = \lambda[g(X,Z)Y - g(Y,Z)X], \tag{4.11}$$

where $\lambda = \frac{dr(e_i)}{2A(e_i)}$ is a scalar, since A is a non-zero 1-form. Then by Schur's theorem λ will be a constant on the manifold. Therefore, M^3 is of constant curvature λ. Thus we get the following theorem:

Theorem 4.1. *A 3-dimensional ϕ-recurrent $N(k)$-contact metric manifold is of constant curvature.*

5. Existence of ϕ-recurrent $N(k)$-contact metric manifolds

In this section we give an example of ϕ-recurrent $N(k)$-contact metric manifold which is neither symmetric nor locally ϕ-symmetric. We take the 3-dimensional manifold $M = \{(x,y,z) \in \mathbf{R}^3 : x \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbf{R}^3$.Let $\{E_1, E_2, E_3\}$ be linearly independent global frame on M given by

$$E_1 = \frac{2}{x}\frac{\partial}{\partial y}, \quad E_2 = 2\frac{\partial}{\partial x} - \frac{4z}{x}\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(E_1,E_3) &= g(E_2,E_3) = g(E_1,E_2) = 0, \\ g(E_1,E_1) &= g(E_2,E_2) = g(E_3,E_3) = 1. \end{aligned}$$

Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$.Let ϕ be the $(1,1)$ tensor field defined by $\phi E_1 = E_2$, $\phi E_2 = -E_1$, $\phi E_3 = 0$. Then using the linearity of ϕ and g we have $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U,W) - \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Moreover $hE_1 = -E_1$, $hE_2 = E_2$ and $hE_3 = 0$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines a contact metric structure on M. Hence we have $[E_1, E_2] = 2E_3 + \frac{2}{x}E_1$, $[E_1, E_3] = 0$, $[E_2, E_3] = 2E_1$.

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

Taking $E_3 = \xi$ and using the above formula for Riemannian metric g, it can be easily calculated that

$$\nabla_{E_1}E_3 = 0, \quad \nabla_{E_2}E_3 = 2E_1, \quad \nabla_{E_3}E_3 = 0, \quad \nabla_{E_3}E_1 = 0, \quad \nabla_{E_1}E_2 = \frac{2}{x}E_1,$$

$$\nabla_{E_2}E_1 = -2E_3, \quad \nabla_{E_2}E_2 = 0, \quad \nabla_{E_3}E_2 = 0, \quad \nabla_{E_1}E_1 = -\frac{2}{x}E_2.$$

From the above it can be easily seen that (ϕ, ξ, η, g) is a $N(k)$-contact metric manifold with $k = -\frac{4}{x} \neq 0$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(E_2, E_3)E_2 = -\frac{4}{x}E_1, \quad R(E_2, E_3)E_1 = \frac{4}{x}E_2,$$

and the components which can be obtained from these by symmetry property. We shall now show that in such a $N(k)$-contact metric manifold the curvature tensor R is ϕ-recurrent. Since $\{E_1, E_2, E_3\}$ form a basis of M^3, any vector field $X \in \chi(M)$ can be taken as

$$X = a_1E_1 + a_2E_2 + a_3E_3$$

where $a_i \in \mathbf{R}^+$ (= the set of all positive real numbers),$i = 1, 2, 3$. Thus the covariant derivatives of the curvature tensor are given by

$$(\nabla_X R)(E_2, E_3)E_1 = -\frac{8a_2}{x^2}E_2,$$

$$(\nabla_X R)(E_2, E_3)E_2 = \frac{8a_2}{x^2}E_1.$$

Let us now consider the non-vanishing 1-form $A(X) = \frac{2a_2}{x}$, at any point $p \in M$. In our M^3, (2.1) reduces with the 1-form to the following equations:

$$\phi^2((\nabla_X R)(E_2, E_3)E_1) = A(X)R(E_2, E_3)E_1, \tag{5.1}$$

$$\phi^2((\nabla_X R)(E_2, E_3)E_2) = A(X)R(E_2, E_3)E_2. \tag{5.2}$$

This implies that the manifold under consideration is a ϕ-recurrent $N(k)$-contact metric manifold, which is neither symmetric nor locally ϕ-symmetric. So, we can state the following:

Theorem 5.1. *There exists a ϕ-recurrent $N(k)$-contact metric manifold, which is neither symmetric nor locally ϕ-symmetric.*

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