INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 9 NO. 2 PAGE 70–79 (2016)

On ϕ-Ricci Recurrent Almost Kenmotsu Manifolds with Nullity Distributions

U.C. De* and Krishanu Mandal

(Communicated by Cihan Özgür)

ABSTRACT

The purpose of this paper is to investigate ϕ-Ricci recurrent and ϕ-Ricci symmetric almost Kenmotsu manifolds with its characteristic vector field ξ belonging to some nullity distributions. Also we obtain several corollaries. Finally, we give an example of a 5-dimensional almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$.

Keywords: *Almost Kenmotsu manifold; ϕ-Ricci recurrence; ϕ-Ricci symmetry; generalized nullity distribution; nullity distribution.*

AMS Subject Classification (2010): *Primary: 53C25 ; Secondary: 53D15.*

1. Introduction

The notion of k-nullity distribution ($k \in \mathbb{R}$) was introduced by Gray [12] and Tanno [16] in the study of Riemannian manifolds (M, g), which is defined for any $p \in M$ and $k \in \mathbb{R}$ as follows:

$$N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}, \tag{1.1}$$

for any $X, Y \in T_pM$, where T_pM denotes the tangent vector space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type $(1,3)$.

Recently, Blair, Koufogiorgos and Papantoniou [3] introduced the (k,μ)-nullity distribution which is a generalized notion of the k-nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ and defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu) = \{Z \in T_pM^{2n+1} : R(X,Y)Z &= k[g(Y,Z)X - g(X,Z)Y] \\ &+\mu[g(Y,Z)hX - g(X,Z)hY]\}, \end{aligned} \tag{1.2}$$

where $h = \frac{1}{2}\pounds_\xi \phi$ and $\pounds$ denotes the Lie differentiation.

Next, Dileo and Pastore [10] introduced another generalized notion of the k-nullity distribution which is named the $(k,\mu)'$-nullity distribution on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ and is defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu)' = \{Z \in T_pM^{2n+1} : R(X,Y)Z &= k[g(Y,Z)X - g(X,Z)Y] \\ &+\mu[g(Y,Z)h'X - g(X,Z)h'Y]\}, \end{aligned} \tag{1.3}$$

where $h' = h \circ \phi$.

In [9], Dileo and Pastore studied locally symmetric almost Kenmotsu manifolds. We refer the reader to ([9],[10],[11]) for more related results on $(k,\mu)'$-nullity distribution and (k,μ)-nullity distribution on almost Kenmotsu manifolds. In recent papers ([17],[18],[19],[20],[21]) Wang and Liu study almost Kenmotsu manifolds with nullity distributions. In [19], Wang and Liu studied ϕ-recurrent almost Kenmotsu manifolds with the characteristic vector field ξ belonging to some nullity distributions.

On the other hand, Kenmotsu [13] introduced a special class of almost contact metric manifolds named Kenmotsu manifolds nowadays. The notion of locally ϕ-symmetry was introduced by Takahashi [15] in the study of Sasakian manifolds as a weaker version of local symmetric of such manifolds. De et al [6] introduced

Received : *15-06-2016,* ***Accepted :*** *21-09-2016*
*** Corresponding author***

a generalized version of local ϕ-symmetry, called ϕ-recurrence on Sasakian manifolds. In [5], De and Sarkar studied ϕ-Ricci symmetric Sasakian manifolds. For more related results we refer the reader to De [4] and De et al ([7],[8]). Motivated by the above studies in this paper we investigate ϕ-Ricci recurrent and ϕ-Ricci symmetric almost Kenmotsu manifolds.
The paper is organized as follows:
In section 2, we give some basic knowledge on almost Kenmotsu manifolds. A few well-known results on almost Kenmotsu manifolds with ξ belonging to some nullity distributions are provided in section 3. In the next section, we study ϕ-Ricci recurrent almost Kenmotsu manifolds with ξ belonging to some nullity distributions. Also we obtain several corollaries. In the final section, we give an example of a 5-dimensional almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$-nullity distribution and $h' \neq 0$.

2. Almost Kenmotsu manifolds

Let M be a $(2n+1)$-dimensional differentiable manifold admits a (ϕ, ξ, η)-structure or an almost contact structure, where ϕ is an $(1,1)$ tensor field, ξ a characteristic vector field and η an 1-form such that ([1, 2])

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{2.1}$$

where I denote the identity endomorphism. It is customary to include also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (2.1).

If a manifold M with a (ϕ, ξ, η)-structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X and Y of T_pM^{2n+1}, then M is said to have an almost contact metric structure (ϕ, ξ, η, g). The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X, Y of T_pM^{2n+1}. An almost Kenmotsu manifold is defined as an almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ [1]. A normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X, Y. It is well known [13] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f = ce^t$ for some positive constant c. Let $\mathcal{D}$ be the distribution orthogonal to ξ and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold $\mathcal{D}$ is an integrable distribution as η is closed. Let the two tensor fields $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(\cdot, \xi)\xi$ on an almost Kenmotsu manifold M^{2n+1}. The tensor fields l and h are symmetric and satisfy the following relations [14]

$$h\xi = 0, \ l\xi = 0, \ tr(h) = 0, \ tr(h\phi) = 0, \ h\phi + \phi h = 0, \tag{2.2}$$

$$\nabla_X \xi = -\phi^2 X - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{2.3}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.4}$$

$$R(X, Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{2.5}$$

for any vector fields X, Y. On the other hand, according to Takahashi [15] and De and Sarkar [5] we have the following definitions.

Definition 2.1. An almost Kenmotsu manifold is said to be ϕ-symmetric, if it satisfies

$$\phi^2((\nabla_W R)(X, Y)Z) = 0,$$

for any vector fields $W, X, Y, Z \in T_pM$. In addition, if the vector fields W, X, Y, Z are orthogonal to ξ, then the manifold is called locally ϕ-symmetric manifold.

Definition 2.2. An almost Kenmotsu manifold is said to be ϕ-Ricci recurrent if it satisfies

$$\phi^2((\nabla_W Q)Y) = A(W)QY, \tag{2.6}$$

for any vector fields $W, Y \in T_pM$, where A is the 1-form on M^{2n+1} and Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$. In addition, if the vector fields W, Y are orthogonal to ξ, then the manifold is called locally ϕ-Ricci recurrent manifold.

Definition 2.3. An almost Kenmotsu manifold is said to be ϕ-Ricci symmetric if it satisfies

$$\phi^2((\nabla_W Q)Y) = 0, \tag{2.7}$$

for any vector fields $W, Y \in T_pM$, where Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$. In addition, if the vector fields W, Y are orthogonal to ξ, then the manifold is called locally ϕ-Ricci symmetric manifold.

3. Properties of the nullity conditions

In this section we provide some related results on almost Kenmotsu manifolds with ξ belonging to some nullity distributions. The $(1, 1)$-type symmetric tensor field $h' = h \circ \phi$ is anticommuting with ϕ and $h'\xi = 0$. Also it is clear that

$$h = 0 \Leftrightarrow h' = 0, \ \ h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{3.1}$$

Let $X \in \mathcal{D}$ be the eigen vector of h' corresponding to the eigen value λ. It follows from (3.1) that $\lambda^2 = -(k+1)$, a constant. Therefore $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces associated with h' corresponding to the non-zero eigen value λ and $-\lambda$ respectively. We have following lemmas.

Lemma 3.1. *(Proposition 3.1 and Proposition 5.1 of [14]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold satisfying either the generalized (k, μ)-nullity condition or the generalized $(k, \mu)'$-nullity condition (the term generalized means k, μ both are smooth functions), with $h \neq 0$. Then, one has*

$$h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2), \tag{3.2}$$

$$S(X, \xi) = 2nk\eta(X), \tag{3.3}$$

for any vector field X on M^{2n+1}. Furthermore, in the case of generalized (k, μ)-nullity condition, one has

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX] \tag{3.4}$$

and in the case of generalized $(k, \mu)'$-nullity condition, one has

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(h'X, Y)\xi - \eta(Y)h'X], \tag{3.5}$$

for any $X, Y \in T_pM$. In addition if $n > 1$ then one has

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X) - (\mu+2)\eta(X)h'Y, \tag{3.6}$$

for any $X, Y \in T_pM$.

Lemma 3.2. *(Proposition 4.1 and Proposition 4.3 of [10]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and Spec $(h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigen value and $\lambda = \sqrt{-k-1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvature are given as following:*

(a) $K(X, \xi) = k - 2\lambda$ *if* $X \in [\lambda]'$ *and*
$K(X, \xi) = k + 2\lambda$ *if* $X \in [-\lambda]'$,
(b) $K(X, Y) = k - 2\lambda$ *if* $X, Y \in [\lambda]'$;
$K(X, Y) = k + 2\lambda$ *if* $X, Y \in [-\lambda]'$ *and*
$K(X, Y) = -(k+2)$ *if* $X \in [\lambda]', Y \in [-\lambda]'$,
(c) M^{2n+1} *has constant negative scalar curvature* $r = 2n(k - 2n)$.

Lemma 3.3. *(Lemma 3 of [19]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the generalized $(k, \mu)'$-nullity distribution and $h' \neq 0$. If $n > 1$, then the Ricci operator Q of M^{2n+1} is given by*

$$Q = -2nid + 2n(k+1)\eta \otimes \xi + [\mu - 2(n-1)]h'. \tag{3.7}$$

Moreover, if both k and μ are constant, then we have

$$Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh'. \tag{3.8}$$

In both cases, the scalar curvature of M^{2n+1} is $2n(k - 2n)$.

Lemma 3.4. *(Proposition 4.2 of [10]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $h' \neq 0$ and ξ belongs to the $(k, -2)'$-nullity distribution. Then for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor satisfies:*

$$\begin{aligned}
R(X_\lambda, Y_\lambda)Z_{-\lambda} &= 0, \\
R(X_{-\lambda}, Y_{-\lambda})Z_\lambda &= 0, \\
R(X_\lambda, Y_{-\lambda})Z_\lambda &= (k+2)g(X_\lambda, Z_\lambda)Y_{-\lambda}, \\
R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda, \\
R(X_\lambda, Y_\lambda)Z_\lambda &= (k-2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\
R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= (k+2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].
\end{aligned}$$

Lemma 3.5. *(Lemma 4.1 of [10]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to the $(k, -2)'$-nullity distribution. Then for any $X, Y \in T_pM$,*

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X). \tag{3.9}$$

Lemma 3.6. *(Theorem 4.1 of [10]) Let M be an almost Kenmotsu manifold of dimension $2n+1$. Suppose that the characteristic vector field ξ belonging to the (k, μ)-nullity distribution. Then $k = -1$, $h = 0$ and M is locally a warped product of an open interval and an almost Kähler manifold.*

4. ϕ-Ricci recurrent almost Kenmotsu manifolds

This section is devoted to study ϕ-Ricci recurrent almost Kenmotsu manifolds with some nullity distributions. At first we prove the following theorem.

Theorem 4.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)(n > 1)$ be a ϕ-Ricci recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the $(k, \mu)'$-nullity distribution and $h' \neq 0$. Then the manifold M^{2n+1} is ϕ-Ricci symmetric and hence locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof. We suppose that the manifold M^{2n+1} is a ϕ-Ricci recurrent almost Kenmotsu manifold. Taking the covariant differentiation along arbitrary vector field $Y \in T_pM$ of (3.8) we have

$$(\nabla_Y Q)X = 2n(k+1)[(\nabla_Y \eta)X\xi + \eta(X)\nabla_Y \xi] - 2n(\nabla_Y h')X, \tag{4.1}$$

for any vector fields $X, Y \in T_pM$. Using (2.3) we obtain from the above equation

$$\begin{aligned}
(\nabla_Y Q)X \;=\; & 2n(k+1)[g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + Y\eta(X) \\
& + g(X, h'Y)\xi + h'Y\eta(X)] - 2n(\nabla_Y h')X,
\end{aligned} \tag{4.2}$$

for any vector fields $X, Y \in T_pM$. Applying ϕ^2 on both sides of (4.2) we obtain

$$\begin{aligned}
\phi^2((\nabla_Y Q)X) \;=\; & 2n(k+1)[-\eta(X)Y + \eta(X)\eta(Y)\xi - \eta(X)h'Y] \\
& - 2n\phi^2((\nabla_Y h')X),
\end{aligned} \tag{4.3}$$

for any vector fields $X, Y \in T_pM$. Making use of (3.9) we get from (4.3)

$$\begin{aligned}
\phi^2((\nabla_Y Q)X) \;=\; & 2n(k+1)[-\eta(X)Y + \eta(X)\eta(Y)\xi - \eta(X)h'Y] \\
& + 2n\eta(X)[-h'Y + (k+1)(Y - \eta(Y)\xi)],
\end{aligned} \tag{4.4}$$

from which we have

$$\phi^2((\nabla_Y Q)X) = -2n(k+2)\eta(X)h'Y, \tag{4.5}$$

for any vector fields $X, Y \in T_pM$. By virtue of equations (4.5) and (2.6) we obtain

$$-2n(k+2)\eta(X)h'Y = A(Y)QX, \tag{4.6}$$

for any vector fields $X, Y \in T_pM$. Substituting $X = \xi$ in (4.6) yields

$$-(k+2)h'Y = kA(Y)\xi, \tag{4.7}$$

for any vector field $Y \in T_pM$. Taking inner product of (4.7) with ξ we have

$$kA(Y) = 0, \tag{4.8}$$

for any vector field $Y \in T_pM$. Dileo and Pastore [10] proved that in an almost Kenmotsu manifold with the characteristic vector field ξ belonging to the $(k,\mu)'$-nullity distribution with $h' \neq 0$ then $k < -1$. Hence (4.8) implies $A = 0$, that is, the manifold M^{2n+1} is ϕ-Ricci symmetric. Using the relation $A = 0$, it follows from (4.7) that $k = -2$. Noticing the fact $\lambda^2 = -(k+1)$ and $k = -2$ we have $\lambda = \pm 1$. Without losing generality we may choose $\lambda = 1$. Then we can write from Lemma 3.4

$$\begin{aligned} R(X_\lambda, Y_\lambda)Z_\lambda &= -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\ R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= 0, \end{aligned}$$

for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also it follows from Lemma 3.2 that $K(X,\xi) = -4$ for any $X \in [\lambda]'$ and $K(X,\xi) = 0$ for any $X \in [-\lambda]'$. Again from Lemma 3.2 we see that $K(X,Y) = -4$ for any $X, Y \in [\lambda]'$; $K(X,Y) = 0$ for any $X, Y \in [-\lambda]'$ and $K(X,Y) = 0$ for any $X \in [\lambda]', Y \in [-\lambda]'$. As is shown in [10] that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. This completes the proof. □

Since ϕ-recurrent implies ϕ-Ricci recurrent, we have the following:

Corollary 4.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)(n > 1)$ be a ϕ-recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. Then the manifold is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

The above corollary have been proved by Wang and Liu [19].

Theorem 4.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)(n > 1)$ be a ϕ-Ricci recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the generalized $(k,\mu)'$-nullity distribution and $h' \neq 0$. Then the following statements are equivalent:*

(i) M^{2n+1} is ϕ-Ricci symmetric;

(ii) k is a constant;

(iii) ξ belongs to the $(k,\mu)'$-nullity distribution.

Proof. Let us suppose M^{2n+1} be a ϕ-Ricci recurrent almost Kenmotsu manifold with ξ belonging to the generalized $(k,\mu)'$-nullity distribution and $h' \neq 0$. Taking the covariant differentiation along arbitrary vector field $W \in T_pM$ of (3.7) we get

$$\begin{aligned} (\nabla_W Q)Y &= 2nW(k)\eta(Y)\xi + 2n(k+1)[(\nabla_W \eta)Y\xi + \eta(Y)\nabla_W \xi] \\ &\quad + W(\mu)h'Y + [\mu - 2(n-1)](\nabla_W h')Y, \end{aligned} \tag{4.9}$$

for any vector fields $W, Y \in T_pM$. Applying ϕ^2 on both sides of (4.9), we have

$$\begin{aligned} \phi^2((\nabla_W Q)Y) &= -W(\mu)h'Y + [\mu - 2(n-1)]\phi^2((\nabla_W h')Y) \\ &\quad + 2n(k+1)\eta(Y)\{-W + \eta(W)\xi - h'W\}, \end{aligned} \tag{4.10}$$

for any vector fields $W, Y \in T_pM$. In view of (4.10) and (2.6) we have

$$\begin{aligned} &-W(\mu)h'Y + [\mu - 2(n-1)]\phi^2((\nabla_W h')Y) \\ &+ 2n(k+1)\eta(Y)\{-W + \eta(W)\xi - h'W\} = A(W)QY, \end{aligned} \tag{4.11}$$

for any vector fields $W, Y \in T_pM$. Making use of (3.6) and (4.11) we obtain

$$\begin{aligned} &-W(\mu)h'Y + [\mu - 2(n-1)][-\eta(Y)\{-h'W + (k+1)(W - \eta(W)\xi)\} \\ &+ (\mu+2)\eta(W)h'Y] - 2n(k+1)[\eta(Y)W - \eta(Y)\eta(W)\xi + \eta(Y)h'W] \\ &= A(W)QY, \end{aligned} \tag{4.12}$$

for any vector fields $W, Y \in T_pM$. Taking inner product with ξ of (4.12) yields

$$kA(W) = 0, \tag{4.13}$$

for any vector field $W \in T_pM$. From (3.2) we see that the smooth function k satisfies $k < -1$. Hence it follows from (4.13) that $A = 0$. Thus M^{2n+1} is ϕ-Ricci symmetric. Therefore $(i) \Rightarrow (ii)$ is proved.
Taking into account $A = 0$ we have from (4.12)

$$\begin{aligned} &-W(\mu)h'Y + [\mu - 2(n-1)][-\eta(Y)\{-h'W + (k+1)(W - \eta(W)\xi)\} \\ &+(\mu+2)\eta(W)h'Y] - 2n(k+1)[\eta(Y)W - \eta(Y)\eta(W)\xi + \eta(Y)h'W] \\ &= 0, \end{aligned} \tag{4.14}$$

for any vector fields $W, Y \in T_pM$. Substituting $Y = \xi$ in (4.14) we obtain

$$[\mu - 2(n-1)][h'W - (k+1)(W - \eta(W)\xi)] - 2n(k+1)[W - \eta(W)\xi + h'W] = 0, \tag{4.15}$$

for any vector field $W \in T_pM$. Now taking inner product of (4.15) with the vector field X we have

$$\begin{aligned} &[\mu - 2(n-1)][g(h'W, X) - (k+1)(g(W, X) - \eta(W)\eta(X))] \\ &-2n(k+1)[g(W, X) - \eta(W)\eta(X) + g(h'W, X)] = 0, \end{aligned} \tag{4.16}$$

for any vector fields $W, X \in T_pM$. Consider a local orthonormal basis $\{e_i : i = 1, 2, ..., 2n+1\}$ of tangent space at each point of the manifold M^{2n+1}. Setting $X = W = e_i$ in (4.16) and taking summation over $i : 1 \leq i \leq 2n+1$, we get

$$(k+1)(\mu+2) = 0. \tag{4.17}$$

Again from (3.2) we see that the smooth function k satisfies $k < -1$. Therefore it follows from (4.17) that $\mu = -2$. Pastore and Saltarelli [14] proved that in an almost Kenmotsu manifold with generalized $(k, \mu)'$-nullity distribution and $h' \neq 0$, then

$$\xi(k) = -2(k+1)(\mu+2) \tag{4.18}$$

holds. In view of $\mu = -2$ we have from (4.18), k=constant. Thus $(ii) \Rightarrow (iii)$ is proved. Conversely, $(iii) \Rightarrow (i)$ is proved in Theorem 4.1. This completes the proof. □

Since ϕ-recurrent implies ϕ-Ricci recurrent, we have the following:

Corollary 4.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)(n > 1)$ be a ϕ-recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the generalized $(k, \mu)'$-nullity distribution and $h' \neq 0$. Then the following statements are equivalent:*

(i) M^{2n+1} is ϕ-symmetric;

(ii) k is a constant;

(iii) ξ belongs to the $(k, \mu)'$-nullity distribution.

The above corollary have been proved by Wang and Liu [19].
Now we prove some special theorems on almost Kenmotsu manifolds with ξ belonging to the (k, μ)-nullity and generalized (k, μ)-nullity distributions.

Theorem 4.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a ϕ-Ricci recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the (k, μ)-nullity distribution. Then the manifold M^{2n+1} is an Einstein one.*

Proof. Let us suppose that the manifold M^{2n+1} is an ϕ-Ricci recurrent almost Kenmotsu manifold with ξ belonging to the (k, μ)-nullity distribution. In view of Lemma 3.6 and the equation (2.5) we get

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{4.19}$$

for any $X, Y \in T_pM$. Contracting Y in (4.19) we have

$$S(X, \xi) = -2n\eta(X), \tag{4.20}$$

for any $X, Y \in T_pM$. Thus, we have

$$Q\xi = -2n\xi. \tag{4.21}$$

Taking the covariant differentiation along arbitrary vector field $Z \in T_pM$ of (4.21) we get

$$(\nabla_Z Q)\xi = -2n(\nabla_Z \xi). \tag{4.22}$$

Using Lemma 3.6 the Equation (2.3) becomes

$$\nabla_Z \xi = Z - \eta(Z)\xi, \tag{4.23}$$

for any $Z \in T_pM$. From (4.22) and (4.23) we have

$$(\nabla_Z Q)\xi = -2n[Z - \eta(Z)\xi], \tag{4.24}$$

where Z be any vector field on M. Applying ϕ^2 on both sides of (4.24) we see that

$$\phi^2((\nabla_Z Q)\xi) = -2n[-Z + \eta(Z)\xi], \tag{4.25}$$

for any $Z \in T_pM$. In view of (4.25) and (2.6) we obtain

$$-2n[-Z + \eta(Z)\xi] = A(Z)Q\xi, \tag{4.26}$$

where $Z \in T_pM$. Taking the inner product of (4.26) with ξ yields $A(Z) = 0$, for any $Z \in T_pM$, thus we have $A = 0$ and consequently M^{2n+1} is ϕ-Ricci symmetric. Taking into account $A = 0$, we can write from (4.25) and (4.26) that $\phi^2((\nabla_Z Q)\xi) = 0$, for any $Z \in T_pM$, from which it follows that

$$-(\nabla_Z Q)\xi + \eta((\nabla_Z Q)\xi)\xi = 0. \tag{4.27}$$

Taking the inner product of (4.27) with the vector field U and making use of $g((\nabla_Z Q)\xi, U) = (\nabla_Z S)(\xi, U)$ we have

$$(\nabla_Z S)(\xi, U) - \eta((\nabla_Z Q)\xi)\eta(U) = 0, \tag{4.28}$$

for any $U, Z \in T_pM$. Again using (4.24) we see from (4.28) that

$$(\nabla_Z S)(\xi, U) = 0, \tag{4.29}$$

which implies

$$\nabla_Z S(\xi, U) - S(\nabla_Z \xi, U) - S(\xi, \nabla_Z U) = 0, \tag{4.30}$$

for any $U, Z \in T_pM$. Making use of (4.20) and (4.23) into (4.30) we can write $S(U, Z) = -2ng(U, Z)$, for any $U, Z \in T_pM$. Therefore the manifold M^{2n+1} is an Einstein manifold. This completes the proof. ☐

In [19], Wang and Liu proved that a ϕ-recurrent almost Kenmotsu manifold M^{2n+1} with the characteristic vector field ξ belonging to the (k, μ)-nullity distribution is of constant sectional curvature -1, which implies that the manifold is an Einstein one of the form $S(U, Z) = -2ng(U, Z)$, for any $U, Z \in T_pM$. Also, ϕ-recurrent implies ϕ-Ricci recurrent. Thus, we have the following:

Corollary 4.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a ϕ-recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the (k, μ)-nullity distribution. Then the manifold M^{2n+1} is an Einstein one.*

Theorem 4.4. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a ϕ-Ricci recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the generalized (k, μ)-nullity distribution and $h \neq 0$. Then the manifold M^{2n+1} is an Einstein one.*

Proof. Suppose that an almost Kenmotsu manifold M^{2n+1} is ϕ-Ricci recurrent and ξ belongs to the generalized (k, μ)-nullity distribution with $h \neq 0$. Making use of (3.3) we get

$$Q\xi = 2nk\xi. \tag{4.31}$$

Taking the covariant differentiation along arbitrary vector field $Z \in T_pM$ of (4.31) we get

$$(\nabla_Z Q)\xi = 2nk(\nabla_Z \xi) + 2nZ(k)\xi. \tag{4.32}$$

Using (2.3) in (4.32) yields

$$(\nabla_Z Q)\xi = 2nk(Z - \eta(Z)\xi - \phi hZ) + 2nZ(k)\xi, \tag{4.33}$$

for any vector field $Z \in T_pM$. Applying ϕ^2 on both sides of (4.33) we have

$$\phi^2((\nabla_Z Q)\xi) = 2nk(-Z + \eta(Z)\xi + \phi hZ), \tag{4.34}$$

for any vector field $Z \in T_pM$. On the other hand, the assumption that M^{2n+1} is ϕ-Ricci recurrent implies $\phi^2((\nabla_Z Q)Y) = A(Z)QY$, for any $Y, Z \in T_pM$. Substituting $Y = \xi$ into this relation we obtain

$$\phi^2((\nabla_Z Q)\xi) = A(Z)Q\xi. \tag{4.35}$$

In view of (4.34) and (4.35) we get

$$2nk(-Z + \eta(Z)\xi + \phi hZ) = A(Z)Q\xi, \tag{4.36}$$

for any vector field $Z \in T_pM$. Taking inner product of (4.36) with ξ implies $kA(Z) = 0$ for any vector field $Z \in T_pM$. Also, from (3.2) we see that the smooth function k satisfies the condition $k < -1$. Therefore, $A = 0$ and consequently M^{2n+1} is ϕ-Ricci symmetric. Taking into account $A = 0$, we can write from (4.35) that $\phi^2((\nabla_Z Q)\xi) = 0$, for any $Z \in T_pM$, from which it follows that

$$-(\nabla_Z Q)\xi + \eta((\nabla_Z Q)\xi)\xi = 0. \tag{4.37}$$

Taking the inner product of (4.37) with the vector field U and using $g((\nabla_Z Q)\xi, U) = (\nabla_Z S)(\xi, U)$ we have

$$(\nabla_Z S)(\xi, U) - \eta((\nabla_Z Q)\xi)\eta(U) = 0, \tag{4.38}$$

for any vector fields $U, Z \in T_pM$. This implies

$$\nabla_Z S(\xi, U) - S(\nabla_Z \xi, U) - S(\xi, \nabla_Z U) - \eta((\nabla_Z Q)\xi)\eta(U) = 0. \tag{4.39}$$

Applying (3.3), (2.3) and (4.33) into (4.39) we obtain

$$\begin{aligned} &2nZ(k)\eta(U) + 2nk\nabla_Z \eta(U) - S(Z, U) + 2nk\eta(Z)\eta(U) \\ &+S(\phi hZ, U) - 2nk\eta(\nabla_Z U) - 2nZ(k)\eta(U) = 0, \end{aligned} \tag{4.40}$$

from which it follows that

$$S(U, Z) - S(\phi hZ, U) - 2nkg(U, Z) + 2nkg(\phi hZ, U) = 0, \tag{4.41}$$

for any vector fields $U, Z \in T_pM$. Now replacing Z by ϕhZ in (4.41) and noticing the fact $h^2 = (k+1)\phi^2$ we have

$$(k + 1)S(U, Z) + S(\phi hZ, U) - 2nk(k + 1)g(U, Z) - 2nkg(\phi hZ, U) = 0, \tag{4.42}$$

for any vector fields $U, Z \in T_pM$. Adding (4.41) and (4.42) yields

$$(k + 2)[S(U, Z) - 2nkg(U, Z)] = 0, \tag{4.43}$$

for any vector fields $U, Z \in T_pM$. Clearly, it follows from (4.43) that either $k = -2$ or, $S(U, Z) = 2nkg(U, Z)$, for any vector fields $U, Z \in T_pM$. Now we prove that the former case can not occur. Indeed, if we assume that the former case is true, that is, $k = -2$, a constant, then $\xi(k) = 0$. Here we recall a result due to Pastore and Saltarelli [14]. They proved that in an almost Kenmotsu manifold with generalized (k, μ)-nullity distribution and $h \neq 0$, the relation $\xi(k) = -4(k + 1)$ holds. Therefore substituting $k = -2$ in this relation we have $\xi(k) = 4$. Thus, we have $\xi(k) = 0$ and $\xi(k) = 4$, which is absurd. Hence, we get the desired result. □

5. Example of a 5-dimensional almost Kenmotsu manifold

In this section, we construct an example of an almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$-nullity distribution. We consider 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. Let ξ, e_2, e_3, e_4, e_5 are five vector fields in $\mathbb{R}^5$ which satisfies [10]

$$\begin{aligned} &[\xi, e_2] = -2e_2,\ [\xi, e_3] = -2e_3,\ [\xi, e_4] = 0,\ [\xi, e_5] = 0, \\ &[e_i, e_j] = 0,\ \text{where } i, j = 2, 3, 4, 5. \end{aligned}$$

Let g be the Riemannian metric defined by

$$g(\xi,\xi)=g(e_2,e_2)=g(e_3,e_3)=g(e_4,e_4)=g(e_5,e_5)=1$$
$$\text{and } g(\xi,e_i)=g(e_i,e_j)=0 \text{ for } i\neq j;\, i,j=2,3,4,5.$$

Let η be the 1-form defined by

$$\eta(Z)=g(Z,\xi),$$

for any $Z\in\chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(\xi)=0,\ \phi(e_2)=e_4,\ \phi(e_3)=e_5,\ \phi(e_4)=-e_2,\ \phi(e_5)=-e_3.$$

Using the linearity of ϕ and g we have

$$\eta(\xi)=1,\ \phi^2Z=-Z+\eta(Z)\xi$$

and

$$g(\phi Z,\phi U)=g(Z,U)-\eta(Z)\eta(U),$$

for any $U,Z\in\chi(M)$. Moreover,

$$h'\xi=0,\ h'e_2=e_2,\ h'e_3=e_3,\ h'e_4=-e_4,\ h'e_5=-e_5.$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned}2g(\nabla_XY,Z) &= Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)\\ &\quad -g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]).\end{aligned}$$

Using Koszul's formula we get the following:

$$\begin{aligned}
&\nabla_\xi\xi=0,\ \nabla_\xi e_2=0,\ \nabla_\xi e_3=0,\ \nabla_\xi e_4=0,\ \nabla_\xi e_5=\xi,\\
&\nabla_{e_2}\xi=2e_2,\ \nabla_{e_2}e_2=-2\xi,\ \nabla_{e_2}e_3=0,\ \nabla_{e_2}e_4=0,\ \nabla_{e_2}e_5=0,\\
&\nabla_{e_3}\xi=2e_3,\ \nabla_{e_3}e_2=0,\ \nabla_{e_3}e_3=-2\xi,\ \nabla_{e_3}e_4=0,\ \nabla_{e_3}e_5=0,\\
&\nabla_{e_4}\xi=0,\ \nabla_{e_4}e_2=0,\ \nabla_{e_4}e_3=0,\ \nabla_{e_4}e_4=0,\ \nabla_{e_4}e_5=0,\\
&\nabla_{e_5}\xi=0,\ \nabla_{e_5}e_2=0,\ \nabla_{e_5}e_3=0,\ \nabla_{e_5}e_4=0,\ \nabla_{e_5}e_5=0.
\end{aligned}$$

In view of the above relations we have

$$\nabla_X\xi=-\phi^2X+h'X,$$

for any $X\in\chi(M)$. Therefore, the structure (ϕ,ξ,η,g) is an almost contact metric structure such that $d\eta=0$ and $d\Phi=2\eta\wedge\Phi$, so that M is an almost Kenmotsu manifold.

By the above results, we can easily obtain the components of the curvature tensor R as follows:

$$\begin{aligned}
&R(\xi,e_2)\xi=4e_2,\ R(\xi,e_2)e_2=-4\xi,\ R(\xi,e_3)\xi=4e_3,\ R(\xi,e_3)e_3=-4\xi,\\
&R(\xi,e_4)\xi=R(\xi,e_4)e_4=R(\xi,e_5)\xi=R(\xi,e_5)e_5=0,\\
&R(e_2,e_3)e_2=4e_3,\ R(e_2,e_3)e_3=-4e_2,\ R(e_2,e_4)e_2=R(e_2,e_4)e_4=0,\\
&R(e_2,e_5)e_2=R(e_2,e_5)e_5=R(e_3,e_4)e_3=R(e_3,e_4)e_4=0,\\
&R(e_3,e_5)e_3=R(e_3,e_5)e_5=R(e_4,e_5)e_4=R(e_4,e_5)e_5=0.
\end{aligned}$$

With the help of the expressions of the curvature tensor we conclude that the characteristic vector field ξ belongs to the $(k,\mu)'$-nullity distribution, with $k=-2$ and $\mu=-2$.

References

[1] Blair, D.E., Contact manifold in Riemannian geometry. Lecture Notes on Mathematics. Springer, Berlin, 509, 1976.
[2] Blair, D.E., Riemannian geometry on contact and symplectic manifolds. *Progr. Math.* 203, Birkhäuser, 2010.
[3] Blair, D.E., Koufogiorgos, T. and Papantoniou, B.J., Contact metric manifolds satisfying a nullity condition. *Israel J. Math.* 91 (1995), 189-214.
[4] De, U.C., On ϕ-symmetric Kenmotsu manifolds. *Int. Electron. J. Geom.* 1 (2008), 33-38.
[5] De, U.C., Sarkar, A., On ϕ-Ricci symmetric Sasakian manifolds. *Proc. of the Jangjeon Math. Soc.* 11 (2008), 47-52.

[6] De, U.C., Shaikh, A.A. and Biswas, S., On ϕ-recurrent Sasakian manifolds. *Novi Sad J. Math.* 33 (2003), 43-48.
[7] De, U.C., Yildiz, A. and Yaliniz, A.F., On ϕ-recurrent Kenmotsu manifolds. *Turkish J. Math.* 33 (2009), 17-25.
[8] De, U.C., Yildiz, A. and Yaliniz, A.F., Locally ϕ-symmetric normal almost contact metric manifolds of dimension 3. *Applied Math. Letters.* 22 (2009), 723-727.
[9] Dileo, G. and Pastore, A.M., Almost Kenmotsu manifolds and local symmetry. *Bull. Belg. Math. Soc. Simon Stevin.* 14 (2007), 343-354.
[10] Dileo, G. and Pastore, A.M., Almost Kenmotsu manifolds and nullity distributions. *J. Geom.* 93 (2009), 46-61.
[11] Dileo, G. and Pastore, A.M., Almost Kenmotsu manifolds with a condition of η-parallelism. *Differential Geom. Appl.* 27 (2009), 671-679.
[12] Gray, A., Spaces of constancy of curvature operators. *Proc. Amer. Math. Soc.* 17 (1966), 897-902.
[13] Kenmotsu, K., A class of almost contact Riemannian manifolds. *Tohoku Math. J.* 24 (1972), 93-103.
[14] Pastore, A.M. and Saltarelli, V., Generalized nullity distributions on almost Kenmotsu manifolds. *Int. Electron. J. Geom.* 4 (2011), 168-183.
[15] Takahashi, T., Sasakian ϕ-symmetic spaces. *Tohoku Math. J.* 29 (1977), 91-113.
[16] Tanno, S., Some differential equations on Riemannian manifolds. *J. Math. Soc. Japan.* 30 (1978), 509-531.
[17] Wang, Y. and Liu, X., Second order parallel tensors on almost Kenmotsu manifolds satisfying the nullity distributions. *Filomat.* 28 (2014), 839-847.
[18] Wang, Y. and Liu, X., Riemannian semisymmetric almost Kenmotsu manifolds and nullity distributions. *Ann. Polon. Math.* 112 (2014), 37-46.
[19] Wang, Y. and Liu, X., On ϕ-recurrent almost Kenmotsu manifolds. *Kuwait J. Sci.* 42 (2015), 65-77.
[20] Wang, Y. and Liu, X., On a type of almost Kenmotsu manifolds with harmonic curvature tensors. *Bull. Belg. Math. Soc. Simon Stevin.* 22 (2015), 15-24.
[21] Wang, Y. and Liu, X., On almost Kenmotsu manifolds satisfying some nullity distributions. *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.* 86 (2016), 347-353.

Affiliations

U.C. De
Address: Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kol-700019, West Bengal, INDIA.
e-mail: uc_de@yahoo.com

Krishanu Mandal
Address: Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kol-700019, West Bengal, INDIA.
e-mail: krishanu.mandal013@gmail.com