Progress of Theoretical Physics, Vol. 46, No. 6, December 1971

On Phase-Space Description of Quantum Mechanics

G. J. RUGGERI

Departamento de Física, Facultad de Ciencias
Universidad Central de Venezuela, Caracas, Venezuela

(Received June 16, 1971)

Phase-space distribution functions for a non-relativistic quantum particle are defined as the mean value of certain operators, whose most general forms are determined by the requirements of Galilean, parity and time-reversal invariances. The rule of associating phase-space functions to quantum mechanical operators, induced by the general distribution function, is considered. The resulting scheme unifies previous work on this subject.

§ 1. Introduction

In the last years, the phase-space description of quantum mechanical systems, initiated by the well-known works of Wigner,[1] Moyal[2] and others[3)~6)] has been substantially developed,[7)~11)] mainly in connection with two different kinds of problems: a) the study of the statistical properties of coherent optical fields[14)~18)] and b) the search and analysis of the essential features common to the existing statistical and dynamical theories, i.e., classical mechanics and classical statistical mechanics from one side and quantum theory from the other.[7),9),17),18)]

The main aspects of the phase-space description of a quantum system are dictated by the desire of treating the ordinary quantum theory in a framework similar to that of the classical statistical mechanics. We can briefly summarize them as follows:

a) The states of the system are described by means of the so-called distribution function $F_\Omega(\boldsymbol{x},\boldsymbol{p};t)$ which is defined on the (classical) phase space and depends parametrically on time. This function is expected to be parallel, as much as possible, to a true probability distribution.

b) A c-number function $A^\Omega(\boldsymbol{x},\boldsymbol{p})$, defined on the phase space is associated to the quantum operator $\hat{A}$, in such a way that the usual quantum expectation value could be computed in the classically-looking manner:

$$\langle\hat{A}\rangle=h^{-3}\int A^\Omega(\boldsymbol{x},\boldsymbol{p})F_\Omega(\boldsymbol{x},\boldsymbol{p};t)\,d^3x d^3p. \qquad (1\cdot1)$$

Two general works on this subject have been recently published: that of Cahill and Glauber,[11] who have determined a set of common features which must be shared by all solutions of the above sketched program and that of Agarwal and Wolf[10] who have developed a general technique to give a unified treatment to the different mapping rules $\hat{A}\rightarrow A^\Omega$ associated to the different ways in which the basic operators of the theory can be ordered in the expansion of $\hat{A}$.

Due to the wide class of phase-space description schemes which are compatible with the general requisites found in the work of Cahill and Glauber, the question of how to select a definite one in a *physically meaningful way* immediately arises. The purpose of this work is to show how this can be achieved by demanding that the phase-space description conforms to Galilean, parity and time-reversal invariances. As it usually happens these symmetry requirements are strong enough to limit severely the formal aspects of the theory, keeping at the same time a convenient degree of generality. The scheme so generated will be shown to be in formal correspondence with the work of Agarwal and Wolf, the main properties of which are then a natural consequence of the invariance of the theory under the improper Galilei group of transformations.

After stating notations and conventions, our plan will proceed as follows: In § 2 the phase-space distribution function is constructed. In § 3 we consider the phase-space representation of quantum operators, as induced by the general F_Ω. Dynamical equations are written in § 4.

Notations and Conventions. We only consider the non-relativistic system of one particle, of mass m, in three dimensions. Extension of the main results to N-particles systems is straightforward. Operators are denoted by a caret; so, for example, $\boldsymbol{x}$ is the vector position of a space point, but $\hat{\boldsymbol{x}}$ is the vector position operator.

§ 2. Distribution functions

As mentioned above, we are interested in describing the quantum states of our system by means of a real phase-space distribution function $F_\Omega(\boldsymbol{x}, \boldsymbol{p}; t)$ whose most general form we proceed now to determine. First of all, we want to recover the well-known classical expression in the limit $h \to 0$; accordingly the usual correspondence principle suggests to construct F_Ω as the mean value of certain hermitian operator $\Omega(\boldsymbol{x}, \boldsymbol{p})$ which depends parametrically on the phase-space coordinates, i.e.,

$$F_\Omega(\boldsymbol{x}, \boldsymbol{p}; t) = \langle \Psi, t | \hat{\Omega}(\boldsymbol{x}, \boldsymbol{p}) | \Psi, t \rangle. \qquad (2\cdot1)$$

Here $|\Psi, t\rangle$ is a general state vector. Note that this form limits us ab initio to F's bilinear in the wave function.

Let us now search for the restrictions on the form of $\hat{\Omega}(\boldsymbol{x}, \boldsymbol{p})$ imposed by the requirement of invariance of the phase-space description under Galilean, parity and time-reversal transformations. Consider first the Galilean transformation*)

$$\begin{aligned} \boldsymbol{x} &\to \boldsymbol{x}' = \mathcal{R}\boldsymbol{x} + \boldsymbol{a} + \boldsymbol{V}t, \\ \boldsymbol{p} &\to \boldsymbol{p}' = \mathcal{R}\boldsymbol{p} + m\boldsymbol{V}, \\ t &\to t' = t, \end{aligned} \qquad (2\cdot2)$$

*) We adhere to the active point of view.

$\mathcal{R}$ is a rotation, $\boldsymbol{a}$ a translation and $\boldsymbol{V}$ an arbitrary velocity. The state vector changes according to

$$|\Psi, t\rangle \to |\Psi', t\rangle = \hat{U}(\boldsymbol{a}, \boldsymbol{V}, \mathcal{R})|\Psi, t\rangle. \tag{2·3}$$

Here $\hat{U}(\boldsymbol{a}, \boldsymbol{V}, \mathcal{R})$ is the unitary projective representation of the Galilei group.[19] It is given explicitly by

$$\begin{aligned}\hat{U}(\boldsymbol{a}, \boldsymbol{V}, \mathcal{R}) &= \hat{D}(\boldsymbol{a}+\boldsymbol{V}t, m\boldsymbol{V})\hat{R}\\ &\equiv \{\exp(i/\hbar)[m\boldsymbol{V}\cdot\hat{\boldsymbol{x}} - (\boldsymbol{a}+\boldsymbol{V}t)\cdot\hat{\boldsymbol{p}}]\}\hat{R}.\end{aligned} \tag{2·4}$$

Now, the physical equivalence of inertial frames of reference requires

$$F'_{\Omega}(\boldsymbol{x}', \boldsymbol{p}'; t') = F_{\Omega}(\boldsymbol{x}, \boldsymbol{p}; t)$$

which, due to the arbitraryness of $|\Psi, t\rangle$, leads to

$$\hat{\Omega}(\boldsymbol{x}', \boldsymbol{p}') = \hat{D}(\boldsymbol{a}+\boldsymbol{V}t, m\boldsymbol{V})\hat{R}\hat{\Omega}(\boldsymbol{x}, \boldsymbol{p})\hat{R}^{\dagger}\hat{D}^{\dagger}(\boldsymbol{a}+\boldsymbol{V}t, m\boldsymbol{V}). \tag{2·5}$$

As a first conclusion we see that $\hat{\Omega} \equiv \hat{\Omega}(\boldsymbol{0}, \boldsymbol{0})$ must be rotationally invariant: $\hat{\Omega} = \hat{R}\hat{\Omega}\hat{R}^{\dagger}$. This, of course, means that $F_{\Omega}(\boldsymbol{0}, \boldsymbol{0})$ must be independent of the coordinates axis orientation.

In addition, take $x=p=t=0$ and note that varying $\boldsymbol{a}$ and $\boldsymbol{V}$ the set $(\boldsymbol{a}, m\boldsymbol{V})$ can be made to cover the whole phase space. We arrive then at*)

$$\hat{\Omega}(\boldsymbol{x}, \boldsymbol{p}) = \hat{D}(\boldsymbol{x}, \boldsymbol{p})\hat{\Omega}\hat{D}^{\dagger}(\boldsymbol{x}, \boldsymbol{p}). \tag{2·6}$$

We see then that the effect of Galilean invariance is to extract the parametric dependence of the "density in phase-space operator" $\hat{\Omega}(\boldsymbol{x}, \boldsymbol{p})$. This turns out to be the result of displacing in phase space an operator $\hat{\Omega}$ associated to the origin of coordinates.

Further restrictions on $\hat{\Omega}$ are obtained by the other symmetry requirements. By a procedure analogous to that followed above, it is easy to show that by (2·6)
a) Parity invariance requires

$$\hat{P}\hat{\Omega}\hat{P} = \hat{\Omega}. \tag{2·7}$$

b) Time-reversal invariance requires

$$\hat{K}^{\dagger}\hat{\Omega}^{\dagger}\hat{K} = \hat{\Omega}. \tag{2·8}$$

$\hat{P}$ and $\hat{K}$ are the usual parity and time-reversal operators. Other desired properties of $\hat{\Omega}$, e.g., its positive-definiteness, will be considered later. For the sake of mathematical flexibility, however, we may suppose in what follows that $\hat{\Omega}$ is a sufficiently well behaved operator.

To proceed, let us quote some known simple formulas that will be used repeatedly in the following

*) Cahill and Glauber[11] have noted that the class of phase-space distribution functions they considered could be written in the form (2·6).

$$\hat{D}(\boldsymbol{x},\boldsymbol{p})|\boldsymbol{x}'\rangle=\{\exp(i/\hbar)\boldsymbol{p}\cdot[(\boldsymbol{x}/2)+\boldsymbol{x}']\}|\boldsymbol{x}'+\boldsymbol{x}\rangle, \tag{2·9a}$$

$$\hat{D}(\boldsymbol{x},\boldsymbol{p})|\boldsymbol{p}'\rangle=\{\exp(-i/\hbar)\boldsymbol{x}\cdot[(\boldsymbol{p}/2)+\boldsymbol{p}']\}|\boldsymbol{p}'+\boldsymbol{p}\rangle, \tag{2·9b}$$

$$\hat{D}(\boldsymbol{x},\boldsymbol{p})\hat{D}(\boldsymbol{x}',\boldsymbol{p}')=\{\exp(i/2\hbar)(\boldsymbol{p}\cdot\boldsymbol{x}'-\boldsymbol{p}'\cdot\boldsymbol{x})\}\hat{D}(\boldsymbol{x}+\boldsymbol{x}',\boldsymbol{p}+\boldsymbol{p}'). \tag{2·9c}$$

Consider now the operators

$$\hat{\Gamma}(\boldsymbol{x})=h^{-3}\int d^3p\hat{\Omega}(\boldsymbol{x},\boldsymbol{p}), \tag{2·10}$$

$$\hat{J}(\boldsymbol{p})=h^{-3}\int d^3x\hat{\Omega}(\boldsymbol{x},\boldsymbol{p}). \tag{2·11}$$

They are respectively diagonal in position and momentum basis. In fact, using (2·9a) and (2·9b) we find

$$\langle\boldsymbol{x}'|\hat{\Gamma}(\boldsymbol{x})|\boldsymbol{x}''\rangle=\langle\boldsymbol{x}'-\boldsymbol{x}|\hat{\Omega}|\boldsymbol{x}'-\boldsymbol{x}\rangle\delta^{(3)}(\boldsymbol{x}'-\boldsymbol{x}''), \tag{2·12}$$

$$\langle\boldsymbol{p}'|\hat{J}(\boldsymbol{p})|\boldsymbol{p}''\rangle=\langle\boldsymbol{p}'-\boldsymbol{p}|\hat{\Omega}|\boldsymbol{p}'-\boldsymbol{p}\rangle\delta^{(3)}(\boldsymbol{p}'-\boldsymbol{p}''). \tag{2·13}$$

Then

$$\hat{\Gamma}(\boldsymbol{x})=\int d^3x'|\boldsymbol{x}'\rangle\langle\boldsymbol{x}'-\boldsymbol{x}|\hat{\Omega}|\boldsymbol{x}'-\boldsymbol{x}\rangle\langle\boldsymbol{x}'|, \tag{2·14a}$$

$$\hat{J}(\boldsymbol{p})=\int d^3p'|\boldsymbol{p}'\rangle\langle\boldsymbol{p}'-\boldsymbol{p}|\hat{\Omega}|\boldsymbol{p}'-\boldsymbol{p}\rangle\langle\boldsymbol{p}'|, \tag{2·14b}$$

$$h^{-3}\int d^3xd^3p\hat{\Omega}(\boldsymbol{x},\boldsymbol{p})=(\mathrm{Tr}\,\hat{\Omega})\hat{1}. \tag{2·14c}$$

Upon taking mean values, the first two of these relations are easily recognized as generalized versions of the marginal probability distributions along the position and momentum basis. The last one expresses normalization if, as we do from now on, $\hat{\Omega}$ is chosen of unit trace.

At this stage it is worthwhile to point out some remarks concerning the interpretation of F_Ω:

a) The usual quantum probability densities are obtained if Ω is such that

$$\langle\boldsymbol{x}|\hat{\Omega}|\boldsymbol{x}\rangle=\delta^{(3)}(\boldsymbol{x}), \tag{2·15a}$$

$$\langle\boldsymbol{p}|\hat{\Omega}|\boldsymbol{p}\rangle=\delta^{(3)}(\boldsymbol{p}). \tag{2·15b}$$

In these cases, however, $\hat{\Omega}$ cannot be a positive definite operator and consequently, F_Ω may take negative values. A well-known example of this circumstance is Wigner's distribution function.

b) In the opposite cases $\hat{\Omega}$ and F_Ω are positive definite and the uncertainty principle precludes the possibility of having δ-function peaks like (2·15). This in turn means by (2·14) that the usual marginal distributions will not be obtained simply by integrating F_Ω over the corresponding phase-space axis. Apparently this definitively denies the possibility of attaching to a positive definite F_Ω a probability meaning in any sense. Nevertheless, suppose that the left-hand sides of (2·15), which in virtue of (2·7) and (2·8) are real, symmetric and normalized functions of its arguments, are chosen to be positive packet functions respectively peaked at $\boldsymbol{x}=\boldsymbol{0}$ and $\boldsymbol{p}=\boldsymbol{0}$. Then $\hat{\Gamma}$ and $\hat{J}$ are proper extensions of the position and momentum probability densities. This is so, not only because by (2·14a) and (2·14b) they give good measures of these probabilities, but because, in addition, the mean values

$$\langle\Psi, t|\hat{\Gamma}(\boldsymbol{x})|\Psi, t\rangle, \tag{2·16a}$$

$$\langle\Psi, t|\hat{J}(\boldsymbol{p})|\Psi, t\rangle \tag{2·16b}$$

evolve in time according to the usual coupled hydrodynamic equations. It is to be noted that due to the relations

$$\langle\boldsymbol{x}'|\hat{\Omega}(\boldsymbol{x},\boldsymbol{p})|\boldsymbol{x}'\rangle=\langle\boldsymbol{x}'-\boldsymbol{x}|\hat{\Omega}|\boldsymbol{x}'-\boldsymbol{x}\rangle, \tag{2·17a}$$

$$\langle\boldsymbol{p}'|\hat{\Omega}(\boldsymbol{x},\boldsymbol{p})|\boldsymbol{p}'\rangle=\langle\boldsymbol{p}'-\boldsymbol{p}|\hat{\Omega}|\boldsymbol{p}'-\boldsymbol{p}\rangle, \tag{2·17b}$$

the above assumption is equivalent to choice $\hat{\Omega}$ so that the localized states of the ordinary formulation, i.e.,: $|\boldsymbol{x}\rangle$ and $|\boldsymbol{p}\rangle$, be also well localized in the phase-space description.

c) Even when F_Ω is positive definite we cannot give to it the meaning of a phase-space localization probability according to the usual quantum-mechanical interpretative rules. This, of course, is a manifestation of the uncertainty principle which does not permit a simultaneous eigenstate of position and momentum operators. Nevertheless, She and Heffner[20] have shown that it is possible to extend slightly the usual body of quantum postulates so as to include the simultaneous, but imprecise, measurement of position and momentum variables. The distribution function constructed from $\hat{\Omega}_{SH}$ (see below) has in that theory the meaning of a probability associated to an act of measurement.

Let us now search for an alternative expression for F_Ω. For this, use the completeness of position and momentum eigenstates together with (2·6), (2·9a) and (2·9b), to put $\hat{\Omega}(\boldsymbol{x},\boldsymbol{p})$ in the form

$$\hat{\Omega}(\boldsymbol{x},\boldsymbol{p})=\int d^3x'd^3p'\{\langle\boldsymbol{x}'-\boldsymbol{x}|\hat{\Omega}|\boldsymbol{p}'-\boldsymbol{p}\rangle/\langle\boldsymbol{x}'-\boldsymbol{x}|\boldsymbol{p}'-\boldsymbol{p}\rangle\}|\boldsymbol{x}'\rangle\langle\boldsymbol{x}'|\boldsymbol{p}'\rangle\langle\boldsymbol{p}'|. \tag{2·18}$$

Introduce now the Fourier transformation*)

$$\begin{aligned}\Omega(\boldsymbol{u},\boldsymbol{v})&=h^{-3}\exp[(-i/2)\hbar\boldsymbol{u}\cdot\boldsymbol{v}]\int d^3x'd^3p'[\langle\boldsymbol{x}'|\hat{\Omega}|\boldsymbol{p}'\rangle/\langle\boldsymbol{x}'|\boldsymbol{p}'\rangle]\exp[-i(\boldsymbol{u}\cdot\boldsymbol{x}'+\boldsymbol{v}\cdot\boldsymbol{p}')]\\&=\mathrm{Tr}\{\hat{\Omega}\exp[-i(\boldsymbol{u}\cdot\hat{\boldsymbol{x}}+\boldsymbol{v}\cdot\hat{\boldsymbol{p}})]\}.\end{aligned} \tag{2·19}$$

*) In terms of it the rotationally, parity and time-reversal invariances are respectively equivalent to: $\Omega(\boldsymbol{u},\boldsymbol{v})=\Omega(\mathcal{R}\boldsymbol{u},\mathcal{R}\boldsymbol{v})$, $\Omega(\boldsymbol{u},\boldsymbol{v})=\Omega(-\boldsymbol{u},-\boldsymbol{v})$ and $\Omega^*(\boldsymbol{u},\boldsymbol{v})=\Omega(\boldsymbol{u},\boldsymbol{v})$. In addition to this, the Hermiticity and the normalization of $\hat{\Omega}$ require: $\Omega^*(\boldsymbol{u},\boldsymbol{v})=\Omega(-\boldsymbol{u},-\boldsymbol{v})$ and $\Omega(\boldsymbol{0},\boldsymbol{0})=1$.

Then

$$\hat{\Omega}(\boldsymbol{x},\boldsymbol{p})=(\hbar/2\pi)^3\int d^3x'd^3ud^3v\Omega(\boldsymbol{u},\boldsymbol{v})\exp\{i[\boldsymbol{u}\cdot(\boldsymbol{x}'-\boldsymbol{x})-\boldsymbol{v}\cdot\boldsymbol{p}+(\hbar/2)\boldsymbol{u}\cdot\boldsymbol{v}]\}$$
$$\times|\boldsymbol{x}'\rangle\langle\boldsymbol{x}'|\exp(i\boldsymbol{v}\cdot\boldsymbol{p}) \qquad (2\cdot20)$$

which, by (2·9a) and a simple change of variables, is equivalent to

$$\hat{\Omega}(\boldsymbol{x},\boldsymbol{p})=(\hbar/2\pi)^3\int d^3x'd^3ud^3v\Omega(\boldsymbol{u},\boldsymbol{v})\exp\{i[\boldsymbol{u}\cdot(\boldsymbol{x}'-\boldsymbol{x})-\boldsymbol{v}\cdot\boldsymbol{p}]\}$$
$$\times|\boldsymbol{x}'-\hbar\boldsymbol{v}/2\rangle\langle\boldsymbol{x}'+\hbar\boldsymbol{v}/2|. \qquad (2\cdot21)$$

Besides numerical factors and a difference in notation, the mean value of this expression is *formally* identical to the general form for the distribution function proposed by Cohen and Margenau.[8),9)] Note however that our $\Omega(\boldsymbol{u},\boldsymbol{v})$ is always independent of the state of the system. This last one is a necessary condition in order to avoid undesired consequences of the formalism.

As an illustration of the formalism developed up to now, consider the class of distribution functions which give "Gaussian localization" in phase space, i.e., which are such that

$$\langle\boldsymbol{x}|\hat{\Omega}|\boldsymbol{x}\rangle=(2\pi\sigma_1^2)^{-3/2}\exp\{-|\boldsymbol{x}|^2/2\sigma_1^2\}, \qquad (2\cdot22\text{a})$$
$$\langle\boldsymbol{p}|\hat{\Omega}|\boldsymbol{p}\rangle=(2\pi\sigma_2^2)^{-3/2}\exp\{-|\boldsymbol{p}|^2/2\sigma_2^2\}. \qquad (2\cdot22\text{b})$$

By inverting (2·19), after some algebra it is easy to show that the most general corresponding $\Omega(\boldsymbol{u},\boldsymbol{v})$ has the form

$$\Omega(\boldsymbol{u},\boldsymbol{v})=f(\boldsymbol{u},\boldsymbol{v})\exp\{(-1/2)[(|\boldsymbol{u}|^2/\sigma_1^2)+(|\boldsymbol{v}|^2/\sigma_2^2)]\}, \qquad (2\cdot23)$$

where f is rotationally, parity and time-reversal invariant, and satisfy in addition $f(\boldsymbol{u},\boldsymbol{0})=f(\boldsymbol{0},\boldsymbol{v})=1$. The further requirement of positive-definiteness imposes, as expected, the uncertainty relation; $\sigma_1\sigma_2\geq\hbar/2$. The simplest choice $f\equiv1$ produces an operator $\hat{\Omega}_{SH}$ which in the theory of She and Heffner[20)] defines the (mixed) state of the system after a simultaneous imprecise measurement of position and momentum. The corresponding $F_{SH}(\boldsymbol{x},\boldsymbol{p})$ is then the probability density of finding the couple $(\boldsymbol{x},\boldsymbol{p})$ as a consequence of such measurement.

§ 3. Phase-space representation of quantum operators

In this section we consider the second part of the program sketched in the Introduction, i.e., the association of a phase-space function to every well-behaved operator $\hat{A}$. This mapping is defined in such a way that Eq. (1·1) be satisfied irrespective of the state of the system. Then

$$\hat{A}=h^{-3}\int d^3xd^3pA^{\Omega}(\boldsymbol{x},\boldsymbol{p})\hat{D}(\boldsymbol{x},\boldsymbol{p})\hat{\Omega}\hat{D}^{\dagger}(\boldsymbol{x},\boldsymbol{p}). \qquad (3\cdot1)$$

To solve for A^{Ω} note that (3·1) has the form of a convolutive product which takes a very simple aspect in Fourier space. Define first (cf. Eq. (2·19))

$$A(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{Tr}\{\hat{A} \exp[(-i)(\boldsymbol{u}\cdot\hat{\boldsymbol{x}}+\boldsymbol{v}\cdot\hat{\boldsymbol{p}})]\}. \qquad (3\cdot2)$$

Now, by using Eqs. (2·9c), (3·1) and the cyclic invariance of the trace, we find

$$A(\boldsymbol{u}, \boldsymbol{v}) = h^{-3}\Omega(\boldsymbol{u}, \boldsymbol{v}) \int d^3x d^3p A^{\Omega}(\boldsymbol{x}, \boldsymbol{p}) \exp\{(-i)(\boldsymbol{u}\cdot\boldsymbol{x}+\boldsymbol{v}\cdot\boldsymbol{p})\} \qquad (3\cdot3)$$

or*)

$$A(\boldsymbol{u}, \boldsymbol{v}) = h^{-3}\Omega(\boldsymbol{u}, \boldsymbol{v}) A_F^{\Omega}(\boldsymbol{u}, \boldsymbol{v}) \qquad (3\cdot4)$$

with an obvious notation. That (3·4) is the only solution of (3·1) is proved easily by starting from Eq. (3·7) below.

Through Eqs. (3·1), (3·2) and (3·4) it is straightforward to show that the mapping $\hat{A}\rightleftarrows A^{\Omega}$ has the following elementary properties:

a) To Hermitian operators correspond real functions. This obviously follows from the Hermiticity of $\hat{\Omega}$.

b) To the unit operator corresponds the unit function. This follows from the normalization of $\hat{\Omega}$.

c) The Galilean, parity and time-reversal invariances imply

$$\hat{D}^{\dagger}(\boldsymbol{x}', \boldsymbol{p}')\hat{A}\hat{D}(\boldsymbol{x}', \boldsymbol{p}') \leftrightarrows A^{\Omega}(\boldsymbol{x}+\boldsymbol{x}', \boldsymbol{p}+\boldsymbol{p}'), \qquad (3\cdot5\text{a})$$

$$\hat{R}^{\dagger}\hat{A}\hat{R} \leftrightarrows A^{\Omega}(\mathcal{R}\boldsymbol{x}, \mathcal{R}\boldsymbol{p}), \qquad (3\cdot5\text{b})$$

$$\hat{P}\hat{A}\hat{P} \leftrightarrows A^{\Omega}(-\boldsymbol{x}, -\boldsymbol{p}), \qquad (3\cdot5\text{c})$$

$$\hat{K}^{\dagger}\hat{A}\hat{K} \leftrightarrows \{A^{\Omega}(\boldsymbol{x}, -\boldsymbol{p})\}^*. \qquad (3\cdot5\text{d})$$

Other interesting consequences of Eq. (3·1) can be drawn by first putting it in a different form. For this, let us call for the Weyl representation of $\hat{A}$[21)]

$$\hat{A} = (\hbar/2\pi)^3 \int d^3u d^3v A(\boldsymbol{u}, \boldsymbol{v}) \exp\{i(\boldsymbol{u}\cdot\hat{\boldsymbol{x}}+\boldsymbol{v}\cdot\hat{\boldsymbol{p}})\}. \qquad (3\cdot6)$$

It gives together with (3·4) the connection between A^{Ω} and the development of $\hat{A}$ in terms of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$

$$\hat{A} = (2\pi)^{-6} \int d^3u d^3v A_F^{\Omega}(\boldsymbol{u}, \boldsymbol{v}) \Omega(\boldsymbol{u}, \boldsymbol{v}) \exp\{i(\boldsymbol{u}\cdot\hat{\boldsymbol{x}}+\boldsymbol{v}\cdot\hat{\boldsymbol{p}})\}. \qquad (3\cdot7)$$

This last relation is a convenient starting point to establish contact with the class of mapping rules which come from the different ways we can order the non-commuting operators $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$. In fact, let us suppose we choose a definite arbitrary order, say ω, in the right-hand side of (3·7) and $\Omega(\boldsymbol{u}, \boldsymbol{v})$ is such that

$$\omega[\exp\{i(\boldsymbol{u}\cdot\hat{\boldsymbol{x}}+\boldsymbol{v}\cdot\hat{\boldsymbol{p}})\}] = \Omega(\boldsymbol{u}, \boldsymbol{v}) \exp\{i(\boldsymbol{u}\cdot\hat{\boldsymbol{x}}+\boldsymbol{v}\cdot\hat{\boldsymbol{p}})\}. \qquad (3\cdot8)$$

---

*) In order to avoid convergence problems, we restrict ourselves to $\hat{\Omega}$'s such that $\Omega(\boldsymbol{u}, \boldsymbol{v})$ has no zeros.

Here, the operator in the left-hand side is obtained from the exponential by putting $\hat{x}$ and $\hat{p}$ in the order ω without using the commutation relations. It is then obvious from (3·7) that $A^{\Omega}(x, p)$ is obtained from the ω-ordered expression of $\hat{A}$ by simply making the formal replacement $\hat{x} \to x$ and $\hat{p} \to p$. Conversely, for every suitable restricted $\Omega(u, v)$ we can *define* a generalized order, and an associated mapping rule, in such a way that (3·4) holds identically. This is essentially the procedure followed by Agarwal and Wolf.[10] It is then clear that the formal structure of the ordering techniques developed in that work follows naturally from very general invariance principles.*)

Furthermore, from Eqs. (2·7), (3·2), (3·3) and (3·7) it follows that

$$A^{\Omega}(x, p) = \mathrm{Tr}\{\hat{A}\hat{D}(x, p)\hat{\Omega}'\hat{D}^{\dagger}(x, p)\} \equiv \mathrm{Tr}\{\hat{A}\hat{\Omega}'(x, p)\}, \qquad (3\cdot 9)$$

where $\hat{\Omega}'$ is the operator such that $\Omega'(u, v) = \Omega^{-1}(u, v)$. Comparing (2·6) and (3·9) with Eqs. (10·4) and (10·8) of Ref. 11b) we conclude that the X and Y operators of that work are here just $\hat{\Omega}'(x, p)$ and $\hat{\Omega}(x, p)$.

A problem of interest which is conveniently studied starting from Eq. (3·7) is the structure of the mapping of quantum operators algebra onto the set of phase-space functions.[7),17),18),22)] To see the general form of this mapping which emerges here, we need an expression for the phase-space representative of a product $\hat{A}\hat{B}$. This is obtained directly from (3·7) as

$$\begin{aligned}\hat{A}\hat{B} = (2\pi)^{-12} \int & d^3u_1 d^3v_1 d^3u_2 d^3v_2 A_F^{\Omega}(u_1, v_1) B_F^{\Omega}(u_2, v_2)\Omega(u_1, v_1) \\ & \times \Omega(u_2, v_2) \exp\{i[(u_1+u_2)\cdot\hat{x} + (v_1+v_2)\cdot\hat{p} \\ & + (1/2)\hbar(u_1\cdot v_2 - u_2\cdot v_1)]\},\end{aligned} \qquad (3\cdot 10)$$

where (2·9c) has been used. Then by (3·7)

$$\begin{aligned}(\hat{A}\hat{B})^{\Omega} = (2\pi)^{-12} \int & d^3u_1 d^3u_2 d^3v_1 d^3v_2 A_F^{\Omega}(u_1, v_1) B_F^{\Omega}(u_2, v_2) \\ & \times \Omega(u_1, v_1)\Omega(u_2, v_2)\Omega^{-1}(u_1+u_2, v_1+v_2) \\ & \times \exp\{i[(u_1+u_2)\cdot x + (v_1+v_2)\cdot p \\ & + (1/2)\hbar(u_1\cdot v_2 - u_2\cdot v_1)]\}.\end{aligned} \qquad (3\cdot 11)$$

A significant feature of (3·11) is that it permits one to define, in the set of phase-space functions, a non-commutative algebra isomorphic to the quantum operators algebra. In fact, define the $*$-product by

$$A * B = \text{right-hand side of } (3\cdot 11).$$

As expected, and as it can be verified by a lengthy but straightforward calcu-

*) In view of this, many of the relations found there are equally valid here. Complete equivalence does not exist, however, because the set of restrictions imposed to $\Omega(u, v)$ by Agarwal and Wolf is not the same as ours (see last footnote in § 2).

lation, this product is associative. As it is obviously distributive, we see then that the bracket $A*B-B*A$ is a Lie product for an ample class of Ω's. In fact, a *form* identical to the right-hand side of (3·11) has been recently found by Simoni, Sudarshan and Zaccaria[22] who searched for general associative products between phase-space functions. Note, however, that the assumptions of that work are not identical to ours.

§ 4. Dynamics

For completeness sake we now consider very briefly the dynamical aspects of the theory.[8),10)] Again we shall only quote the results.

There are two evolution equations to consider: that of the Heisenberg picture of $\hat{\Omega}(\boldsymbol{x},\boldsymbol{p})$, which we denote by $\hat{\Omega}(\boldsymbol{x},\boldsymbol{p};t)$, and that of the Ω-representative $A^{\Omega}(\boldsymbol{x},\boldsymbol{p};t)$ of $\hat{A}$, considered also in the Heisenberg picture. These equations fall in the forms (cf. Eq. (5·1) in Ref. 8))

$$(\partial/\partial t)\hat{\Omega}(\boldsymbol{x},\boldsymbol{p};t)=L\hat{\Omega}(\boldsymbol{x},\boldsymbol{p};t), \tag{4·1a}$$

$$(\partial/\partial t)A^{\Omega}(\boldsymbol{x},\boldsymbol{p};t)=MA^{\Omega}(\boldsymbol{x},\boldsymbol{p};t), \tag{4·1b}$$

where

$$L=(2/\hbar)H^{\Omega}(\boldsymbol{x},\boldsymbol{p})\Omega(-i\overleftarrow{\nabla}_x,-i\overleftarrow{\nabla}_p)\Omega(i\overleftarrow{\nabla}_x+i\overrightarrow{\nabla}_x,i\overleftarrow{\nabla}_p+i\overrightarrow{\nabla}_p)$$
$$\times\Omega^{-1}(i\overrightarrow{\nabla}_x,i\overrightarrow{\nabla}_p)\sin\{(\hbar/2)(\overleftarrow{\nabla}_x\cdot\overrightarrow{\nabla}_p-\overleftarrow{\nabla}_p\cdot\overrightarrow{\nabla}_x)\}, \tag{4·2a}$$

$$M=(2/\hbar)H^{\Omega}(\boldsymbol{x},\boldsymbol{p})\Omega(-i\overleftarrow{\nabla}_x,-i\overleftarrow{\nabla}_p)\Omega^{-1}(-i\overleftarrow{\nabla}_x-i\overrightarrow{\nabla}_x,-i\overleftarrow{\nabla}_p-i\overrightarrow{\nabla}_p)$$
$$\times\Omega(-i\overrightarrow{\nabla}_x,-i\overrightarrow{\nabla}_p)\sin\{(\hbar/2)(\overleftarrow{\nabla}_x\cdot\overrightarrow{\nabla}_p-\overleftarrow{\nabla}_p\cdot\overrightarrow{\nabla}_x)\}. \tag{4·2b}$$

The equation of motion for F_{Ω} follows from (4·2a) by taking the mean value of (4·1a).

Due to the form of $\Omega(\boldsymbol{u},\boldsymbol{v})$ and the fact that $\mathrm{Tr}(\hat{\Omega}\hat{\boldsymbol{x}})=\mathrm{Tr}(\hat{\Omega}\hat{\boldsymbol{p}})=0$, from parity invariance, it is certain that for any positive-definite $\hat{\Omega}$

$$\lim_{\hbar\to 0}\Omega(\boldsymbol{u},\boldsymbol{v})=1 \tag{4·3}$$

and then, the operators L and M tend to the classical expression in this limit, namely, Poisson-Bracket operator.*) In this limit $F_{\Omega}(\boldsymbol{x},\boldsymbol{p};t)\to\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_c(t))\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}_c(t))$, where $\boldsymbol{x}_c(t)=\lim_{\hbar\to 0}\langle\Psi,t|\hat{\boldsymbol{x}}|\Psi,t\rangle$ and $\boldsymbol{p}_c(t)=\lim_{\hbar\to 0}\langle\Psi,t|\hat{\boldsymbol{p}}|\Psi,t\rangle$.

Conclusions. This work has been devoted to studying the general form of phasespace distribution function which emerges when one imposes, essentially, the requisite of physical equivalence of inertial frames of reference. We can say that the whole possible set of phase-space description schemes has been restricted in the more physically meaningful way by these symmetry requirements. As expected, the resulting scheme is, simultaneously, strongly determined and sufficiently broad to permit many particular solutions.

*) This, of course, is also valid for some non-positive $\hat{\Omega}$'s.

Acknowledgement

It is a pleasure to thank Professor A. J. Kálnay for his continued encouragement and for many helpful discussions which improved very much the final presentation of this work.

References

1) E. P. Wigner, Phys. Rev. **40** (1932), 749.
2) J. E. Moyal, Proc. Camb. Phil. Soc. **45** (1949), 99.
3) H. J. Groenewold, Physica **12** (1946), 405.
4) T. Takabayasi, Prog. Theor. Phys. **11** (1954), 341.
5) G. A. Baker, Jr., Phys. Rev. **109** (1958), 2198.
6) H. Margenau and R. N. Hill, Prog. Theor. Phys. **26** (1961), 722.
7) C. L. Mehta, J. Math. Phys. **5** (1964), 677.
8) L. Cohen, J. Math. Phys. **7** (1966), 781.
9) H. Margenau and L. Cohen, *Quantum Theory and Reality*, M. Bunge Ed. (Springer Verlag, Berlin, 1967), p. 71.
10) G. S. Agarwal and E. Wolf, Phys. Letters **26A** (1968), 485; Phys, Rev. Letters **21** (1968), 180; Phys. Rev. **2D** (1970), 2161, 2187, 2206.
11) K. E. Cahill and R. J. Glauber, a) Phys. Rev. **177** (1969), 1857; b) Phys. Rev. **177** (1969), 1882.
12) R. J. Glauber, Phys. Rev. **130** (1963), 2529.
13) R. J. Glauber, Phys. Rev. **131** (1963), 2766.
14) E. C. G. Sudarshan, Phys. Rev. Letters **10** (1963), 277.
15) C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138** (1965), B274.
16) Y. Kano, J. Math. Phys. **6** (1965), 1913.
17) T. F. Jordan and E. C. G. Sudarshan, Rev. Mod. Phys. **33** (1961), 515.
18) E. C. G. Sudarshan, in *Lectures in Theoretical Physics*, Brandeis University Summer Institute, 1961 (W. A. Benjamin, New York, 1962), Vol. 2.
19) J-M Levy-Leblond, J. Math. Phys. 4 (1963), 776.
20) C. Y. She and H. Heffner, Phys. Rev. **152** (1966), 1103.
21) R. Kubo, J. Phys. Soc. Japan **19** (1964), 2127.
22) A. Simoni, E. C. G. Sudarshan and F. Zaccaria, Preprint AEC-19 CPT-69, The Center for Particle Theory, The University of Texas at Austin.