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# On $\pi$-uniform Vector Bundles

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In this paper, we define a notion of "$\pi$-uniform vector bundle" over a $P^1$-bundle $\pi: V \to W$, where $V$, $W$ are algebraic varieties. First, generalizing a result of E. Sato ([5], Proposition 3), we give a necessary and sufficient condition in order that a vector bundle over a $P^1$-bundle is $\pi$-uniform (Lemma). By virtue of the Lemma, we give a cohomological condition in order that a vector bundle over the trivial ruled surface $P^1 \times P^1$ is decomposable (Theorem 1). Also we generalize a result of S. Shatz in [6] (Corollary, p. 106) (Theorem 2).

In [2], Schwarzenberger defined the notion of 'uniform vector bundle' on a projective space $P^n$. Our '$\pi$-uniform vector bundle' is an analogue of his, and is suitable for our situation of $P^1$-bundle $\pi: V \to W$. In his paper on uniform vector bundles [5], E. Sato developed some methods for treating such bundles. This paper is inspired by [5].

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## §1. A criterion for $\pi$-uniform vector bundles.

Let $k$ be an algebraically closed field of arbitrary characteristic and $\pi: V \to W$ a $P^1$-bundle, where $V$, $W$ are algebraic varieties over $k$. By a vector bundle $E$ on $V$, we mean a locally free $\mathcal{O}_V$-sheaf module of finite rank, where $\mathcal{O}_V$ is the structure sheaf of $V$. We use the following notation; $h^i(V, E) := \dim_k H^i(V, E)$.

DEFINITION 1. We say that a vector bundle $E$ on $V$ is $\pi$-uniform, if the restriction $E|_{\pi^{-1}(p)}$ of $E$ to $\pi^{-1}(p)$ is mutually isomorphic for any point $p$ of $W$.

First the following proposition is an immediate consequence of Definition 1.

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PROPOSITION 1. (1) *Any line bundle on $V$ is $\pi$-uniform.*

(2) *A direct sum of line bundles is $\pi$-uniform.*

(3) *The dual of $\pi$-uniform vector bundle is $\pi$-uniform.*

(4) *If $E$ is a $\pi$-uniform vector bundle on $V$, then so is $E\otimes L$ for any line bundle $L$ on $V$.*

Now, the following is a key lemma.

LEMMA. *Let $E$ be a vector bundle of rank $r$ on $V$. Then $E$ is $\pi$-uniform if and only if one of the following conditions* (A), (B) *holds.*

(A) *$E$ is an extension of $\pi$-uniform vector bundles,*

$$0\longrightarrow E_1\longrightarrow E\longrightarrow E_2\longrightarrow 0\ ,$$

*where $E_1$, $E_2$ satisfy $h^1(\boldsymbol{P}^1, (E_2|_{\pi^{-1}(p)})^*\otimes(E_1|_{\pi^{-1}(p)}))=0$ for any point $p$ of $W$. Here $(E_2|_{\pi^{-1}(p)})^*$ is the dual of $E_2|_{\pi^{-1}(p)}$.*

(B) *$E\cong\pi^*(F)\otimes L$, where $F$ is a vector bundle on $W$, and $L$ is a line bundle on $V$.*

The proof of Lemma is essentially due to E. Sato ([5], where Sato treats the case of $\pi\colon \mathrm{Proj}\,(\mathcal{O}_{\boldsymbol{P}^n}\oplus\mathcal{O}_{\boldsymbol{P}^n}(1))\rightarrow \boldsymbol{P}^n$; the general case can be handled similarly).

REMARK 1. If $W\cong\boldsymbol{P}^1$, then a $\pi$-uniform vector bundle which satisfies the condition (B) is a direct sum of line bundles.

REMARK 2. If $W$ is an affine variety and $V\cong\boldsymbol{P}^1_W$, then a $\pi$-uniform vector bundle of rank 2 which satisfies the condition (A) on $V$ is decomposable.

## §2. $\pi$-uniform vector bundles on rational ruled surfaces.

Let $\pi\colon \boldsymbol{F}_n=\mathrm{Proj}\,(\mathcal{O}_{\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1}(n))\rightarrow\boldsymbol{P}^1$ be a rational ruled surface over $k$, where $n$ is a non-negative integer. We summarize some well-known facts on $\boldsymbol{F}_n$ from M. Maruyama in ([3], Chapter IV, 3). There is a minimal section $M$ on $\boldsymbol{F}_n$ with $(M, M)=-n$. Let $F$ be a fibre of $\boldsymbol{F}_n$. Every divisor $D$ on $\boldsymbol{F}_n$ is linearly equivalent to $aM+bN$, where $a=(D, N)$ and $b=(D, M)+an$. Also we use the notions of decomposable vector bundle and of simple vector bundle in the usual sense (cf. [3]). Note that every simple vector bundle is indecomposable.

**2.1.** Let $E(a, b)$ be the set of vector bundles which are obtained by the following extension;

$$0 \longrightarrow \mathcal{O}_{F_n} \longrightarrow E \longrightarrow \mathcal{O}_{F_n}(aM+bN) \longrightarrow 0\ .$$

Note that, by Proposition 1, if a vector bundle $E$ of rank 2 is decomposable, then $E$ is $\pi$-uniform. We show here the existence of elements of $E(a, b)$, which are $\pi$-uniform but indecomposable. More precisely, let $U(a, b)$ be the subset of $\pi$-uniform vector bundles in $E(a, b)$. By Proposition 1, $\mathcal{O}_{F_n} \oplus \mathcal{O}_{F_n}(aM+bN)$ is contained in $U(a, b)$ and we put $N(a, b)=U(a, b)-\{\mathcal{O}_{F_n} \oplus \mathcal{O}_{F_n}(aM+bN)\}$. By virtue of the Oda's lemma [4], then we have;

PROPOSITION 2. (1) *The case* $n>0$;

(i) *If* $-an+b-2 \geqq 0$ *and* $b \leqq 0$, *then* $E(a, b)=U(a, b)$ *and every element of* $N(a, b)$ *is indecomposable and not simple.*

(ii) *If* $a \leqq -1$ *and* $b \geqq 1$, *then* $E(a, b)=U(a, b)$ *and every element of* $N(a, b)$ *is simple.*

(2) *The case* $n=0$: *If* $a \leqq -1$ *and* $b \geqq 2$, *then* $E(a, b)=U(a, b)$ *and every element of* $N(a, b)$ *is simple.*

COROLLARY. *There are indecomposable and* $\pi$*-uniform vector bundles on* $F_n (n \geqq 0)$.

EXAMPLE. The tangent bundle $T_{F_n}$ of $F_n$ has the following exact sequence;

$$0 \longrightarrow \mathcal{O}_{F_n}(2M+nN) \longrightarrow T_{F_n} \longrightarrow \mathcal{O}_{F_n}(2N) \longrightarrow 0\ .$$

By Lemma, we see that $T_{F_n}$ is $\pi$-uniform. When $n=0$, $T_{F_0}=\mathcal{O}_{F_0}(2M) \oplus \mathcal{O}_{F_0}(2N)$. But when $n>0$, the above sequence does not split. Therefore by Proposition 2, we see that $T_{F_1}$ is simple, and $T_{F_n} (n \geqq 2)$ is indecomposable and not simple.

**2.2.** Here, we give a cohomological criterion in order that a vector bundle of rank 2 on $P^1 \times P^1$ is decomposable. Let $E$ be a vector bundle of rank $r$ on $F_n$, and let $E(\ell)$ denote $E \otimes \mathcal{O}_{F_n}(\ell M+\ell(n+1)N)$, where $\ell$ is an integer.

PROPOSITION 3. *If* $h^1(F_n, E(\ell))=0$ *for any integer* $\ell$, *then* $E$ *is* $\pi$*-uniform.*

PROOF. For simplicity, we give the proof for $r=2$. We consider the following exact sequence;

$$0 \longrightarrow \mathcal{O}_{F_n}(-N) \longrightarrow \mathcal{O}_{F_n} \longrightarrow \mathcal{O}_N \longrightarrow 0\ .$$

Tensoring with $E(\ell M+(\ell n+\ell+1)N)$, we have

$$0 \longrightarrow E(\ell) \longrightarrow E(\ell M+(\ell n+\ell+1)N) \longrightarrow E(\ell M)|_N \longrightarrow 0 .$$

From the assumption, we see that

$$(1) \qquad 0 \longrightarrow H^0(F_n, E(\ell)) \longrightarrow H^0(F_n, E(\ell M+(\ell n+\ell+1)N)) \longrightarrow H^0(P^1, E(\ell M)|_N) \longrightarrow 0 .$$

Now we suppose that $E$ is not $\pi$-uniform. Then we may assume that for two fibres $N_1$, $N_2$ of $\pi$, $E|_{N_i} \cong \mathcal{O}_{P^1}(a_1^i) \oplus \mathcal{O}_{P^1}(a_2^i)$ with $a_1^i \geqq a_2^i (i=1, 2)$ and $a_1^1 \neq a_1^2$. Therefore, it is clear that $h^0(P^1, E(\ell M)|_{N_1}) \neq h^0(P^1, E(\ell M)|_{N_2})$. But this contradicts to the exact sequence (1). q.e.d.

THEOREM 1. *Let $E$ be a vector bundle of rank 2 on $F_0$. If $h^1(F_0, E(\ell))=0$ for any integer $\ell$, then $E$ is decomposable.*

PROOF. We put $c_1(E)=aM+bN$ and $c_2(E)=c$. By virtue of the Riemann-Roch Theorem,

$$\begin{aligned}\chi(E(\ell)) &= h^0(F_0, E(\ell))-h^1(F_0, E(\ell))+h^2(F_0, E(\ell)) \\ &= 2\ell^2+(a+b+4)\ell+(ab+a+b+2-c) .\end{aligned}$$

From the assumption, we have $\chi(E(\ell)) \geqq 0$ for any integer $\ell$, and $(a+b+4)^2-8(ab+a+b-c+2)=(a-b)^2+2(4c-2ab) \leqq 0$. Therefore we get $c_1^2(E)-4c_2(E)=2ab-4c \geqq 0$. From Proposition 3, $E$ is $\pi$-uniform, and we may put $E|_N \cong \mathcal{O}_{P^1}(a_1) \oplus \mathcal{O}_{P^1}(a_2)$ with $a_1 \geqq a_2$ and $a_1+a_2=a$. By Lemma and Remark 1, it is enough to prove the case of the condition (A) of Lemma. We may assume that $E$ is obtained by the extension of line bundles $L_1$, $L_2$ on $F_0$;

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0$$

where $L_i=\mathcal{O}_{F_0}(a_iM+b_iN)$. Also we may assume $h^1(P^1, (L_{2|N})^* \otimes (L_{1|N}))$ $h^0(P^1, \mathcal{O}_{P^1}(a-2a_1-2))=0$ (i.e., $a-2a_1 \leqq 1$). Now using the above, we show $h^1(F_0, L_2^* \otimes L_1)=0 \cdots (2)$. By Künneth formula we have $h^1(F_0, L_2^* \otimes L_1)= h^0(P^1, \mathcal{O}_{P^1}(\alpha)) \cdot h^1(P^1, \mathcal{O}_{P^1}(\beta))+h^1(P^1, \mathcal{O}_{P^1}(\alpha)) \cdot h^0(P^1, \mathcal{O}_{P^1}(\beta))$, where $\alpha= 2a_1-a$, and $\beta=2b_1-b$. If $\alpha=-1$, then $h^0(P^1, \mathcal{O}_{P^1}(\alpha))=h^1(P^1, \mathcal{O}_{P^1}(\alpha))=0$, and we have (2). If $\alpha=0$, then $E$ is an inverse image of vector bundle of rank 2 on $P^1$, and (2) holds. Finally if $\alpha>0$, then we have $\beta \geqq 0$ since $2ab-4c=\alpha\beta \geqq 0$. By $\alpha>0$, $\beta \geqq 0$, we have $h^1(P^1, \mathcal{O}_{P^1}(\alpha))= h^1(P^1, \mathcal{O}_{P^1}(\beta))=0$, and we have (2). But (2) implies that $E$ is decomposable. q.e.d.

REMARK 3. By virtue of Maruyama's Theorem ([3], Theorem 4.6), the inequality $c_1^2(E)-4c_2(E) \geqq 0$ in the proof of Theorem 1 implies that

$E$ is not simple. By the way, from Proposition 2, there are many indecomposable, not simple and $\pi$-uniform vector bundles on $F_n(n \geqq 1)$. Hence it is impossible to generalize the above result to general rational ruled surfaces.

## §3. The direct image under a finite flat covering map of ruled surfaces.

In [6], Shatz showed; let $\pi: X \rightarrow C$ be a ruled surface, $\phi: D \rightarrow C$ a finite flat covering of degree $s$. If $\theta$: $Y \rightarrow X$ is the induced covering of $X$ by base extension, and if $L$ is any line bundle on $Y$, then the direct image $\theta_* L$ is a $\pi$-uniform vector bundle of rank $s$ on $X$. Moreover, under the same hypothesis as in the above result, the conclusion is still valid when $L$ is replaced by a $\pi_1$-uniform vector bundle $V$ on $Y$ whose restriction to the fibre of $Y$ has the special form $\mathcal{O}_{P^1}(a)^{\oplus r}$, where $\pi_1$ is a morphism $\pi_1$: $Y \rightarrow D$ (cf. [6] Proposition 7 and its Corollary). But, using Lemma, these results can be generalized to arbitrary $\pi_1$-uniform vector bundles of rank 2 on $Y$.

THEOREM 2. *If $E$ is a $\pi_1$-uniform vector bundle of rank 2 on $Y$, then the direct image $\theta_* E$ is a $\pi$-uniform vector bundle on $X$.*

PROOF. We have only to prove the case (A) in Lemma, because the case of (B) is contained in Shatz's result. We consider the extension; $0 \rightarrow L_1 \rightarrow E \rightarrow L_2 \rightarrow 0$. Since $\theta$ is a finite flat covering, the direct image of the above extension is a short exact sequence;

$$0 \longrightarrow \theta_* L_1 \longrightarrow \theta_* E \longrightarrow \theta_* L_2 \longrightarrow 0 .$$

Shatz proved that for any line bundle $L$ on $Y$, the restriction of the direct image $\theta_* L$ to the fibres of $\pi$ has the form $L_{|\pi^{-1}(d)}{}^{\oplus s}$ for any $d$ of $D$ (cf. [6] Proposition 7). Hence, we have $\theta_* L_{i|\pi^{-1}(c)} \cong L_{i|\pi_1^{-1}(d)}{}^{\oplus s}$, where $c=\phi(d)$, and there is an isomorphism $H^1(\boldsymbol{P}^1, (\theta_* L_{2|\pi^{-1}(c)})^* \otimes (\theta_* L_{1|\pi^{-1}(c)})) \simeq \bigoplus H^1(\boldsymbol{P}^1, (L_{2|\pi_1^{-1}(d)})^* \otimes (L_{1|\pi_1^{-1}(d)}))$. By Lemma, $h^1(\boldsymbol{P}^1, (L_{2|\pi_1^{-1}(d)})^* \otimes (L_{1|\pi_1^{-1}(d)}))=0$, and therefore we have $h^1(\boldsymbol{P}^1, (\theta_* L_{2|\pi^{-1}(c)})^* \otimes (\theta_* L_{1|\pi^{-1}(c)}))=0$. This completes the proof.

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