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ON PICARD'S THEOREM FOR ENTIRE QUASIREGULAR MAPPINGS

MATTI VUORINEN

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ABSTRACT. Several refinements of Picard's theorem for entire functions in the complex plane have been proved by many authors in connection with the theory of Picard sets. We prove a result of this type for entire quasiregular mappings in euclidean n-space in the case when the "Picard set" consists of a sequence a_k on a ray emanating from 0 with $|a_k| = 2^k$.

1. INTRODUCTION

Several results of geometric function theory have their multidimensional analogues for quasiregular mappings in R^n ([V3], [Ri2]). According to recent results of S. Rickman [Ri1], [Ri3], the Picard and Schottky theorems in particular have their multidimensional counterparts in this context.

Extending the classical Picard theorem, O. Lehto introduced in 1958 [L] the notion of a Picard set of an entire or meromorphic function of the complex plane. Some delicate properties of these sets were found by L. Carleson [C] and K. Matsumoto [M1], [M2] soon after the publication of [L]. Later contributions to the theory of Picard sets include works by many authors, most notably S. Toppila ([T1]–[T5]) and J. Winkler ([W1], [W2]) (for further information see the bibliography in [T4]).

The goal of this paper is to prove the following theorem, which can be formulated in terms of Picard sets in the special case of entire functions. (We shall not need the notion of a Picard set in this paper.) Let $e_1 = (1, 0, \dots, 0) \in R^n$ and let $p(n, K)$ be the integer in the multidimensional analogue of the Picard theorem ([Ri1]).

1.1. **Theorem.** *Let $f\colon R^n \to R^n$ be a K-quasiregular mapping and $W = \{a_1, \dots, a_p\}$, $a_p = \infty$, a set of distinct points in $\overline{R}^n$ with $p \geq p(n, K)$. If $f^{-1}W \subset \{2^k e_1 : k = 1, 2, \dots\}$ then the limit $\lim_{x\to\infty} f(x)$ exists.*

In the plane case, results stronger than Theorem 1.1 are known for entire functions. However, the previously known methods of studying Picard sets (such as Cauchy's intergral formula) are not applicable to the present context.

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Instead our method relies on the uniform continuity properties of quasiregular mappings with respect to the quasihyperbolic metric. The key idea is contained in a lemma about removable isolated singularities of uniformly continuous functions, which yields a quantitative upper bound for the modulus of continuity of the extended mapping. This lemma is apparently new even for $n = 2$. Its proof relies essentially on the Harnack inequality, which in the present context follows from the results of J. Serrin [S] (cf. Yu. G. Reshetnyak [R]) and also uses S. Rickman's result in [Ri3].

2. Preliminary results

We shall adopt the relatively standard notation and terminology of [V1]. the coordinate unit vectors in R^n are $e_1, \ldots, e_n$. For $x, y \in R^n$, we denote $[x, y] = \{(1-t)x + ty : 0 \le t \le 1\}$ and similarly for half-open or open segments. For $x \in R^n \setminus \{0\}$ let $[x, \infty) = \{tx : t \ge 1\}$. For $x \in R^n$ and $r > 0$ let $B^n(x, r) = \{z \in R^n : |x - z| < r\}$, $S^{n-1}(x, r) = \partial B^n(x, r)$. $B^n(r) = B^n(0, r)$, $S^{n-1}(r) = \partial B^n(r)$, $B^n = B^n(1)$, and $S^{n-1} = \partial B^n$.

For the definition and the basic properties of the modulus $M(\Gamma)$ of a curve family Γ in $\overline{R}^n = R^n \cup \{\infty\}$, the reader is referred to [V1]. The definitions of quasiconformal (qc), K-quasiconformal $(K\text{-}qc)$, quasiregular (qr), K-quasiregular, quasimeromorphic (qm), and K-quasimeromorphic $(K\text{-}qm)$ mappings can be found in [MRV1], [MRV2], [Vu3].

We shall require several metrics. The spherical metric σ on $\overline{R}^n$ is defined by the element of length

$$d\sigma = \frac{|dx|}{1 + |x|^2}. \tag{2.1}$$

The spherical chordal metric q is defined by

$$\begin{aligned} q(x, y) &= |x - y|((1 + |x|^2)(1 + |y|^2))^{-\frac{1}{2}}; \qquad x \ne \infty \ne y \\ q(x, \infty) &= (1 + |x|^2)^{-\frac{1}{2}}. \end{aligned} \tag{2.2}$$

These two metric are equivalent, in fact $q(x, y) = \sin \sigma(x, y)$, $1 \le \sigma(x, y)/q(x, y) \le \pi/2$ for distinct $x, y \in \overline{R}^n$ and $q(0, \infty) = 1 = 2\sigma(0, \infty)/\pi$. Both σ and q are invariant under a subgroup of the group $GM(\overline{R}^n)$ of Möbius transformations, called spherical isometries. If $X \subset \overline{R}^n$, then $GM(X) = \{f \in GM(\overline{R}^n) : fX = X\}$.

The hyperbolic (or Poincaré) metric ρ in B^n is defined by the formula

$$\tanh^2(\rho(x, y)/2) = |x - y|^2/(|x - y|^2 + (1 - |x|^2)(1 - |y|^2)) \tag{2.3}$$

for all $x, y \in B^n$. A basic fact is that ρ is $GM(B^n)$-invariant.

Throughout the paper p will be a positive integer with $p \ge 2$. In most cases $p \ge p(n, K)$, where $p(n, K)$ is the integer in the Picard and Schottky theorem for $K\text{-}qm$ mappings in R^n. Next we shall give a construction of a metric τ following closely S. Rickman's exposition [Ri3]. A difference is that we introduce a constant A in formula (2.6) below to ensure a monotone property of τ which will be convenient later on. Therefore our metric τ differs slightly from S. Rickman's metric τ. It is easy to show, however, that these two metrics are equivalent.

For distinct points $a_1, \dots, a_p$ in $\overline{R}^n$ let $Y = \overline{R}^n \setminus \{a_1, \dots, a_p\}$, $\partial Y = \{a_1, \dots, a_p\}$, and let

$$\beta = \tfrac{1}{4}\min\{\sigma(a_i, a_j) \colon 1 \le i,\ j \le p,\ i \ne j\}. \tag{2.4}$$

It follows from (2.4) that

$$\sigma(x) = \sigma(x, \partial Y) \le \frac{\pi}{2} - 2\beta \tag{2.5}$$

for all $x \in Y$. Denote $B_\sigma(z, r) = \{w \in \overline{R}^n \colon \sigma(z, w) < r\}$ for $z \in \overline{R}^n$ and $r \in (0, \pi/2]$. Let $U_j = B_\sigma(a_j, \beta) \setminus \{a_j\}$, $j = 1, \dots, p$. $U = \bigcup_{j=1}^p U_j$. A metric $\tau_Y = \tau$ on Y is defined by the element of length

$$\begin{aligned} d\tau(x) &= \frac{d\sigma}{\sigma(x, a_j)\log(A/\sigma(x, a_j))}; \qquad x \in U_j, \qquad j = 1, \dots, p \\ d\tau(x) &= \frac{d\sigma}{\beta\log(A/\beta)}; \qquad x \in \overline{R}^n \setminus U. \end{aligned} \tag{2.6}$$

The constant A is chosen so that $d\tau(x)/d\sigma$ is monotonically decreasing as a function of $\sigma(x) = \sigma(x, \partial Y)$. Since $\sigma(x, a_j) \le \pi$ we may choose $A = \pi e$ (this choice is fixed throughout the paper). Explicitly, $\tau(x, y)$ is defined by the formula

$$\tau(x, y) = \inf_\gamma \int_\gamma d\tau(z) \tag{2.6$'$}$$

where the infimum is taken over all curves γ with $\sigma(\gamma) < \infty$ and $x, y \in \gamma$. It follows from the above monotone property of $d\tau(x)$ that for all $x, y \in Y$

$$Q_1 \sigma(x, y) \le \tau(x, y) \tag{2.7}$$

where $Q_1 = 1/(\beta\log(A/\beta))$. By integrating along circular arcs not intersecting U, we see that for all $x, y \in \overline{R}^n \setminus U$

$$\tau(x, y) \le Q_2 \sigma(x, y) \tag{2.8}$$

where $Q_2 = \pi Q_1/2$. Similarly, one can show that for $r \in (0, \beta)$ and $j = 1, \dots, p$

$$\tau(\partial B_\sigma(a_j, r)) \le \pi/\log(A/r).$$

It follows from (2.6) that for $1 \le j \le p$ and all $x, y \in \overline{U}_j \setminus \{a_j\}$

$$\tau(x, y) \ge \left|\log\frac{\log(A/\sigma(x, a_j))}{\log(A/\sigma(y, a_j))}\right|. \tag{2.9}$$

Let t_z be a sense-preserving spherical isometry with $t_z(z) = 0$, $z \in \overline{R}^n$. For convenient reference, we point out here that for $\alpha \in (0, \pi/2)$, $x, y \in \overline{R}^n$

$$\sigma(x, y) = \alpha \Leftrightarrow |t_x(y)| = \tan\alpha. \tag{2.10}$$

By (2.10) we get

$$B_\sigma(0,\alpha) = B^n(\tan\alpha). \tag{2.11}$$

It is also clear that for $\sigma(x,y)=\pi/2$ and $0<\alpha<\pi/2$

$$B_\sigma(x,\alpha) = \overline{R}^n \setminus \overline{B}_\sigma(y,\pi/2-\alpha). \tag{2.12}$$

In what follows, $n\geq 2$, $K\geq 1$, $p\geq 2$, $Y,\beta, A=\pi e$ will be fixed as defined above. Note that $\tau_{hY}(h(x),h(y))=\tau_Y(x,y)$ for spherical isometries h.

2.13. **Lemma.** *Let $x,y\in U_j$, $j=1,\ldots,p$. Then*

$$\left|\tau(x,y) - \left|\log\frac{\log(A/\sigma(y,a_j))}{\log(A/\sigma(x,a_j))}\right|\right| \leq \pi Q_1$$

where Q_1 is as in (2.7).

Proof. As pointed out in [Ri3, p. 138], a result of this form follows easily from the definitions.

2.14. **Lemma.** *For all $x,y\in Y$ the following inequality holds*

$$\tau(x,y)\geq Q_3\left|\log\frac{\log(A/\sigma(x))}{\log(A/\sigma(y))}\right|$$

where $\sigma(z)=\sigma(z,\partial Y)$ and Q_3 depends only on β.

Proof. By relabeling the points if necessary, we may assume $\sigma(x)>\sigma(y)$. Consider two cases.

Case A. $\sigma(y)>\beta/2$.

Let $D=1/\log(A/\sigma(x))$ and $d=D\log(A/\sigma(y))$. Then $D<1$ by (2.5). We obtain

$$d< D(\log(A/\sigma(x)) + \frac{A/\sigma(y)-A/\sigma(x)}{A/\sigma(x)} = 1 + D\left(\frac{\sigma(x)}{\sigma(y)}-1\right)$$

and hence

$$\log d < D\left(\frac{\sigma(x)}{\sigma(y)}-1\right)\leq D\sigma(x,y)/\sigma(y). \tag{2.15}$$

In Case A, the inequality $\sigma(y)>\beta/2$ holds, and therefore by (2.15) and by the proof of (2.7) we obtain

$$\log d\leq 2\tau(x,y)/(\beta\overline{Q}_1)$$

where $\overline{Q}_1=2/(\beta\log(2A/\beta))$. The desired inequality with $Q_{3A}=1/\log(2A/\beta)$ therefore holds in Case A.

Case B. $\sigma(y) \le \beta/2$.

Let j be such that $y \in U_j$. If also $x \in U_j$, then the proof of Case B with $Q'_{3B} = 1$ follows from (2.9). Otherwise $x \in Y \setminus U_j$. Let $x_1 \in \partial U_j$ with $\tau(x, y) \ge \tau(x_1, y)$. Then by (2.9)

$$\tau(x, y) \ge \tau(x_1, y) \ge \log \frac{\log(A/\sigma(y))}{\log(A/\beta)}.$$

By (2.5) we get because $A = \pi e$ and $\sigma(x) < \pi$

$$\frac{\log(A/\sigma(y))}{\log(A/\sigma(x))} < \log(A/\sigma(y)).$$

The last two inequalities show that it suffices to find a constant $E \in (0, 1]$ such that

$$\log \frac{\log(A/\sigma(y))}{\log(A/\beta)} \ge E \log(\log(A/\sigma(y))).$$

It is easy to show that we can choose $E = (\log w_0)/\log(Cw_0) < 1$ where $C = \log(A/\beta) > 2$ and $w_0 = (\log(2A/\beta))/\log(A/\beta) > 1$. Hence in Case B we may choose $Q_{3B} = E$.

In conclusion, in both Cases A and B we can choose $Q_3 = \min\{Q_{3A}, Q_{3B}\}$.

Lemma 2.14 is a modification of (2.9). Still another modification is needed.

2.16. **Lemma.** *For all* $x, y \in Y$ *and all* $z \in \partial Y$ *the following inequality holds*:

$$\tau(x, y) \ge Q_4 \left| \log \frac{\log(A/\sigma(x, z))}{\log(A/\sigma(y, z))} \right|$$

where Q_4 *depends only on* β.

Proof. The proof is similar to the proof of Lemma 2.14 and the details are omitted.

2.17. **Quasihyperbolic metric.** Let G be a proper subdomain of R^n. The *quasihyperbolic distance* $k_G(a, b)$ of $a, b \in G$ is defined by [GP]

$$k_G(a, b) = \inf_{\gamma \in \Gamma_{ab}} \int_\gamma \frac{ds}{d(x, \partial G)} \tag{2.18}$$

where Γ_{ab} is the collection of all rectifiable curves γ in G with $a, b \in \gamma$. It is easy to see that k_G is a metric on G. For $a, b \in G$ set

$$j_G(a, b) = \log\left(1 + \frac{|a-b|}{\min\{d(a), d(b)\}}\right) \tag{2.19}$$

where $d(x) = d(x, \partial G)$. It follows from (2.19) that

$$\left|\log \frac{d(a)}{d(b)}\right| \le j_G(a, b) \le \left|\log \frac{d(a)}{d(b)}\right| + \log\left(1 + \frac{|a-b|}{d(a)}\right) \tag{2.20}$$

for all $a, b \in G$. It is well known that j_G is a metric on G. As shown in [GP, (2.2)] the very useful inequality

$$k_G(a, b) \ge j_G(a, b) \tag{2.21}$$

holds for all $a, b \in G$. In the opposite direction we have

$$k_G(a,b) \le \log\left(1 + \frac{|a-b|}{d(a) - |a-b|}\right) \tag{2.22}$$

for $b \in B^n(a, d(a))$ [Vu2, (2.32)]. Bernoulli's inequality

$$\log(1+as) \le a\log(1+s) \tag{2.23}$$

for $s > 0$, $a \ge 1$, will also be useful. Now (2.22) and (2.23) together yield

$$k_G(a,b) \le \frac{1}{1-s} j_G(a,b) \tag{2.24}$$

for $b \in \overline{B}^n(a, sd(a))$, $s \in (0,1)$. For some properties of k_G the reader is referred to [GP], [GO], [Vu3].

Let $G \subset R^n$ be a domain, $G \ne R^n$. A mapping $f\colon G \to \overline{R}^n$ is said to be *ω-normal* [Vu2] if

$$\sigma(f(x), f(y)) \le \omega(k_G(x,y)) \tag{2.25}$$

holds for all $x, y \in G$ where $\omega\colon [0,\infty) \to [0,\infty)$ is a homeomorphism with $\omega(0) = 0$. We call f *normal* if it is ω-normal for some ω. Sometimes the function ω is denoted by ω_f.

For $n = 2$, $G = B^2$, and f meromorphic, it is well known (and easy to prove) that this definition is equivalent to the definition of a normal meromorphic function given by Lehto and Virtanen [LV].

The next result shows that normality is a local property of the function.

2.26. **Lemma.** *Let $G \subset R^n$ be a domain and $f\colon G \to \overline{R}^n$ be continuous. Then f is normal if and only if there exists a homeomorphism $\bar\omega\colon [0,\infty) \to [0,\infty)$ with $\bar\omega(0) = 0$ such that*

$$\sigma(f(x), f(y)) \le \bar\omega(k_B(x,y))$$

for every ball $B = B^n(x,r)$ in G and for all $y \in B$.

Proof. The proof is a somewhat lengthy (although straightforward) discussion where (2.20)–(2.24) are useful. It should be noted that if ω is given, then $\bar\omega$ has a majorant depending only on ω (thus the majorant is independent of n and of the geometric and topological properties of G). The details are omitted.

Let $\{m_D\colon D \subset \overline{R}^n\}$ be a family of metrics. We say that this family is *monotone* if $D_1 \subset D_2$ implies $m_{D_1}(x,y) \ge m_{D_1}(x,y)$ for all $x, y \in D_1$. For instance, $\{k_D\colon D \subset R^n, D \ne R^n$ is a domain$\}$ is a monotone family, and the same is true if k_D is replaced by j_D or σ_D, the restriction of σ to D.

2.27. **Lemma.** *Let G_1, G_2 be domains in R^n with $G_1 \cap G_2 \ne \varnothing$, $G_1 \ne R^n \ne G_2$ and assume that there exists $c \in (0,1)$ such that*

(1) $d(x, \partial G_1) + d(x, \partial G_2) \ge c d(x, \partial(G_1 \cup G_2))$, *for all $x \in G = G_1 \cup G_2$.*

Suppose that $f\colon G \to fG$ *is continuous,* $fG \subset R^n$; *that* $\{m_D\colon D \subset R^n\}$ *is a monotone family of metrics; and that*

$$m_{fG_j}(f(x), f(y)) \le \omega_j(k_{G_j}(x, y))$$

for $x, y \in G_j$ *and* $j = 1, 2$. *Then there exists* $\omega\colon [0, \infty) \to [0, \infty)$ *such that*

$$m_{fG}(f(x), f(y)) \le \omega(k_G(x, y)) \tag{2}$$

and $\omega(t) \to 0$ *as* $t \to 0$ *provided* $\omega_j(t) \to 0$, *as* $t \to 0$, $j = 1, 2$.

Proof. We consider two cases. Let $d(x) = d(x, \partial G)$.

Case A. $k_G(x, y) \le \log(1 + c/4)$.

In this case $|x-y| \le \frac{c}{4}\min\{d(x), d(y)\}$ by virtue of (2.21). We may assume $d(x) \le d(y)$. By the hypothesis (1) of the lemma there exists $i \in \{1, 2\}$ such that $d(x, \partial G_i) \ge \frac{c}{2} d(x)$, i.e., $B^n(x, cd(x)/2) \subset G_i$. Fix such i. By relabeling (if necessary), we may assume that $i = 1$. Then

$$y \in B^n(x, c\min\{d(x), d(y)\}/4) \subset B^n(x, \tfrac{1}{2} d(x, \partial G_1)).$$

It follows from (2.24) that

$$k_{G_1}(x, y) \le 2 j_{G_1}(x, y) = 2\log\left(1 + \frac{|x-y|}{\min\{d_1(x), d_1(y)\}}\right)$$

where $d_1(z) = d(z, \partial G_1)$. By the above calculation, $|x - y| \le \frac{1}{2} d_1(x)$ and hence $d_1(y) \ge \frac{1}{2} d_1(x)$. The last inequality, together with (2.23) and (2.21), now yields

$$\begin{aligned} k_{G_1}(x, y) &\le 2\log\left(1 + \frac{2|x-y|}{d_1(x)}\right) \\ &\le 2\log\left(1 + \frac{4|x-y|}{cd(x)}\right) < \frac{8}{c} j_G(x, y) \\ &\le \frac{8}{c} k_G(x, y). \end{aligned}$$

Conclusion: In Case A we have

$$\begin{aligned} m_{fG}(f(x), f(y)) &\le m_{fG_1}(f(x), f(y)) \le \omega_1(k_{G_1}(x, y)) \\ &\le \omega_1\left(\frac{8}{c} k_G(x, y)\right) \end{aligned} \tag{2.28}$$

where also the monotone property of the family $\{m_D\}$ was applied.

Case B. $k_G(x, y) > \log(1 + c/4)$.

Fix a geodesic segment γ of the quasihyperbolic metric [GO] with $x, y \in \gamma$ and points $z_1, \ldots, z_{p+1} \in \gamma$ with $z_1 = x$, $z_{p+1} = y$ and $k_G(z_i, z_{i+1}) = \log(1 + c/4)$ for $i = 1, \ldots, p-1$ and $k_G(z_p, z_{p+1}) \le \log(1 + c/4)$, and with

$p \le 1 + k_G(x,y)C$, $C = 1/\log(1+c/4)$. Then by Case A

$$\begin{aligned} m_{fG}(f(x), f(y)) &\le \sum_{j=1}^{p} m_{fG}(f(z_i), f(z_{i+1})) \\ &\le (1 + k_G(x,y)C)\max\{T_1, T_2\} \\ &\quad \text{where } T_i = \omega_i(\tfrac{8}{c}\log(1+c/4)). \end{aligned} \tag{2.29}$$

Hence in Case B we obtain by (2.29)

$$m_{fG}(f(x), f(y)) \le 2C\max\{T_1, T_2\}k_G(x,y).$$

Finally, the desired function ω is defined as follows with the aid of (2.28) and (2.29):

$$\begin{aligned} \omega(t) &= 2\max\left\{\omega_1\left(\frac{8t}{c}\right), \omega_2\left(\frac{8t}{c}\right)\right\}; \qquad t \in (0, \log(1+c/4)] \\ \omega(t) &= (Ct+1)\max\{T_1, T_2\}; \qquad t > \log(1+c/4). \end{aligned}$$

3. A removable singularity lemma

We begin with the following result of S. Rickman [Ri3]:

3.1. **Lemma.** *For $K \ge 1$ and each integer $n \ge 3$ there exists a number $\delta = \delta(n,K) > 0$ and a positive integer $p_0 = p(n,K) \ge 3$ such that the following holds. If $p \ge p_0$ and $a_1, \dots, a_p \in \overline{R}^n$ are distinct points and if $f\colon B^n \to Y$, $Y = \overline{R}^n \setminus \{a_1, \dots, a_p\}$, is K-qm, then*

$$\tau(f(x), f(y)) \le C_1 \max\{\rho(x,y), \delta\}$$

where τ is the metric defined in (2.6) and C_1 is a constant depending only on n, K, and β (cf. (2.4)).

Throughout this section we assume that p is as in Lemma 3.1. It is easy to show that $\rho(x,y) \le 2k_{B^n}(x,y) \le 2\rho(x,y)$ for all $x, y \in B^n$. Therefore, in the definition of a normal function of the unit ball B^n we may, and shall, replace the quasihyperbolic metric k_{B^n} by the $GM(B^n)$-invariant metric ρ.

3.2. **Corollary.** *Let $f\colon B^n \to Y$ be as in Lemma 3.1. Then f is normal and furthermore*

$$\sigma(f(x), f(y)) \le C_2\rho(x,y)^\alpha; \qquad \alpha = K^{1/(1-n)}$$

for all $x, y \in B^n$ where C_2 depends only on n, K, and β.

Proof. The proof follows from (2.7), 3.1, and [Vu2, 5.8].

3.3. **Corollary.** *Let $f\colon B^n \to Y$, be as in Lemma 3.1 and $z \in \{a_1, \dots, a_p\}$. Then $v_f^z(x) = \log(A/\sigma(f(x), z))$ satisfies a Harnack condition*

$$v_f^z(x) \le C_3 v_f^z(y) \tag{3.4}$$

for all $x, y \in B^n$ *with* $|x-y| \le (1-|x|)/2$, *where* C_3 *depends only on* n, K, *and* β.

Proof. It follows easily from (2.3) that $\rho(x,y) < \log 6$ for all $x, y \in B^n$ with $|x-y| \le (1-|x|)/2$. By 2.16 and 3.1

$$Q_4 \log(v_f^z(x)/v_f^z(y)) \le \tau(f(x), f(y)) \le C_1 \max\{\delta, \log 6\}$$

for all $x, y \in B^n$ with $|x-y| \le (1-|x|)/2$ and hence (3.4) follows with $C_3 = \exp\{C_1 \max\{\delta, \log 6\}/Q_4\}$.

We next consider the problem of extending a K-qm mapping to an isolated singularity. We are mainly interested in finding a quantitative estimate for the modulus of continuity of the extended mapping.

3.4. **Lemma.** *Let* Y *be as in Lemma* 3.1 *and let* $f: B^n\backslash\{0\} \to Y$ *be* K-qm. *Then* f *has a limit at* 0. *Denote by* f^* *the extended mapping. Then*

$$\sigma(f^*(x), f^*(y)) \le C_4 \rho(x,y)^\alpha; \qquad \alpha = K^{1/(1-n)}$$

for all $x, y \in B^n$, *where* C_4 *depends only on* n, K, *and* β.

Proof. The (finite or infinite) limit

$$\lim_{x\to 0} f(x) = f^*(0)$$

exists by [Ri1, 1.2]. Consider two cases. We may assume that f is nonconstant.

Case A. $f^*(0) \in Y$.

In this case the assertion follows from 3.2 with the constant C_2.

Case B. $f^*(0) \in \partial Y$.

Since the left side of the asserted inequality is invariant under spherical isometries, we may assume that $f^*(0) = a_1$. By 3.3, the function

$$v_i(x) = \log(A/\sigma(f(x), a_i))$$

satisfies a Harnack condition in $B^n\backslash\{0\}$. That is, if $x \in B^n\backslash\{0\}$ and $B^n(x, 2r) \subset B^n\backslash\{0\}$, then

$$\max_{\overline{B}^n(x,r)} v_i(z) \le C_4 \min_{\overline{B}^n(x,r)} v_i(z); \qquad 1 \le i \le p \tag{3.5}$$

where C_4 depends only on n, K, β. Next it follows from (3.5) that [Vu1, 3.3]

$$\max_{|z|=t} v_i(z) \le C_5 \min_{|z|=t} v_i(z); \qquad t \in (0, \tfrac{1}{2}] \tag{3.6}$$

for $i = 1, \ldots, p$, where $C_5 \ge C_4$ depends only on n, K, β.

For $t \in (0, \frac{1}{2}]$ let $R_t = B^n(\frac{1}{2})\backslash\overline{B}^n(t^2)$. We shall find a quantitative upper bound for the spherical diameter $\sigma(fS^{n-1}(t))$, $t \in (0, \frac{1}{2}]$.

For each $t \in (0, \frac{1}{2}]$ choose $x_t \in S^{n-1}(t)$ and let $I(t)$ be the set of all those indices i for which $\sigma(f(x_t), a_i) \ge \beta$. Then card $I(t) \ge p-1$ and

$v_i(x_t) \le \log(A/\beta)$ for $i \in I(t)$. It follows from (3.6) that if $t \in (0, \frac{1}{2}]$, $i \in I(t)$, and $|y| = t$, then

$$\sigma(f(y), a_i) \ge A(\beta/A)^{C_5} = C_6 .$$

Because $p \ge 3$ and card $I(t) \ge p-1$ for all $0 < t \le \frac{1}{2}$, it follows that $I(\frac{1}{2}) \cap I(t^2) \ne \varnothing$ and hence there is $j_t \in I(\frac{1}{2}) \cap I(t^2)$ for each $t \in (0, \frac{1}{2}]$. Because $a_{j_t} \in \partial Y$, it follows from the maximum principle (or from the fact that f is open) that

$$f(R_t) \cap B_\sigma(a_{j_t}, C_6) = \varnothing .$$

Let Γ_t' be the set of all curves $\gamma \colon [0,1] \to \overline{R}^n$ with $\gamma(0) \in fS^{n-1}(t)$ and $\gamma(1) \in \partial B_\sigma(a_{j_t}, C_6)$, $t \in (0, \frac{1}{2}]$. Denote by Γ_t the set of all maximal liftings of the elements of Γ_t', starting at $S^{n-1}(t)$ (for terminology, see [MRV3, 3.11]). It follows from [MRV3, 3.12] that each $\alpha \in \Gamma_t$ intersects either $S^{n-1}(\frac{1}{2})$ or $S^{n-1}(t^2)$, $0 < t \le \frac{1}{2}$. Therefore by a standard inequality [V1, 7.5]

$$M(\Gamma_t) \le 2\omega_{n-1}(\log 1/2t)^{1-n} \tag{3.7}$$

for $t \in (0, \frac{1}{2})$. By performing an auxiliary Möbius transformation and a spherical symmetrization one can show that (cf. [Vu3, 7.32, (7.24), (2.27), 1.17])

$$M(\Gamma_t') \ge \omega_{n-1}(\log(\lambda_n C_7/\sigma(fS^{n-1}(t)))^{1-n} \tag{3.8}$$

where $\lambda_n \in [4, 2e\alpha^{n-1})$ is a constant depending only on n and $C_7 = 4(1-C_6^2)C_6^{-2}$. Because $M(\Gamma_t') \le KM(\Gamma_t)$ by [V2], it follows from (3.7) and (3.8) that

$$\sigma(fS^{n-1}(t)) \le C_7\lambda_n(2t)^\alpha ; \qquad \alpha = (2K)^{1/(1-n)} \tag{3.9}$$

for all $t \in (0, \frac{1}{2})$. Hence there exists a number $t_0 \in (0, \frac{1}{2})$ depending only on n, K, and β such that

$$f^* B^n(t) \subset B_\sigma(f^*(0), \sigma(fS^{n-1}(t))) \subset B_\sigma(a_1, \beta)$$

whenever $t \in (0, t_0]$.

It follows from 2.26 and 3.2 that $f|B^n\backslash\{0\}$ is ω_1-normal where $\omega_1(t) = C_8 t^\alpha$ and C_8 depends only on n, K, β. On the other hand, $f^*|B^n(t_0)$ is ω_2-normal with $\omega_2(t) = C_9 t^\alpha$ by [Vu2, 5.8] where C_9 depends only on n, K, β. Finally, by the proof of Lemma 2.27 we see that f^* is ω-normal in B^n with $\omega(t) = 2\max\{\omega_1(8t/t_0), \omega_2(8t/t_0)\}$ for small t.

As a corollary we obtain a result due to L. Carleson [C] (see also [M2]) in the case of meromorphic functions.

3.10. **Corollary.** *Let $K \ge 1$, Y as in Lemma 3.1, let $\theta > 0$, $R(1, e^{2\theta}) = \{z \in R^n \colon 1 < |z| < e^{2\theta}\}$, and let $f \colon R(1, \varepsilon^{2\theta}) \to Y$ be K-qm. Then*

$$\sigma(fS^{n-1}(e^\theta)) \le C_{10}e^{-\theta\alpha} ; \qquad \alpha = (2K)^{1/(1-n)} ,$$

where C_{10} depends only on n, K, β.

Proof. Apply the proof of Lemma 3.4 to a slightly smaller annulus.

4. The proof of Theorem 1.1

In this section we prove the main result of this paper.

4.1. **Theorem.** *Let* $\omega: [0, \infty) \to [0, \infty)$ *be a homeomorphism with* $\omega(0) = 0$, *let* (b_i) *be a sequence in* R^n *with* $|b_i| = 2^i$, *and let* $f: R^n \to R^n$ *be a* K*-qr mapping such that* $|f(b_i)| < 1$ *for all* $i = 1, 2, \ldots,$ *and the following holds. For each* $i = 1, 2, \ldots,$ *the mapping* $f|A_i$ *with* $A_i = B^n(2^{i+1})\backslash\overline{B}^n(2^{i-1})$, *is* ω*-normal. Then* f *is a constant.*

Proof. From the proof of [Vu2, 5.13] and [Vu1, 3.3] it follows that $|f(x)| < D_1 < \infty$ for all $x \in \bigcup_{i=1}^{\infty} S^{n-1}(2^i)$. If f is a constant there is nothing to prove. Otherwise f is open and hence $|f(x)| < D_1$ for all $x \in R^n$. But this contradicts the Liouville theorem in [MRV2, 3.9].

4.2. *Proof of Theorem* 1.1. If $f^{-1}W$ is bounded, the limit exists by [Ri1, 1.2]. Thus we may assume that $f^{-1}W$ is not bounded, and without loss of generality we may assume that $f^{-1}W = \{2^k e_1 : k = 1, 2, \ldots\}$. Let

$$A_i = B^n(2^{i+1})\backslash\overline{B}^n(2^{i-1}), \qquad A_i^0 = A_i\backslash\{2^i e_1\}.$$

From 2.26 it follows that $f|A_i^0$ is ω-normal where ω depends only on n, K, β, and from 3.4 and 2.26 we also conclude that $f|A_i$ is ω_1-normal where ω_1 depends only on n, K, and β. Thus after a rescaling the hypotheses of Theorem 4.1 are satisfied, and accordingly f is a constant and the limit exists.

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References

[C] L. Carleson, *A remark on Picard's theorem.* Bull. Amer. Math. Soc. **67** (1961), 142–144.

[GO] F. W. Gehring and B. G. Osgood, *Uniform domains and the quasihyperbolic metric.* J. Analyse Math. **36** (1979), 50–74.

[GP] F. W. Gehring and B. P. Palka, *Quasiconformally homogeneous domains.* J. Analyse Math. **30** (1976), 172–199.

[L] O. Lehto, *A generalization of Picard's theorem.* Ark. Mat. **3** (1958), 495–500.

[LV] O. Lehto and K. I. Virtanen, *Boundary behavior and normal meromorphic functions.* Acta Math. **97** (1957), 47–65.

[MRV1] O. Martio, S. Rickman, and J. Väisälä, *Definitions for quasiregular mappings.* Ann. Acad. Sci. Fenn. Ser. A I Math. **448** (1969), 1–40.

[MRV2] ——, *Distortion and singularities of quasiregular mappings.* Ann. Acad. Sci. Fenn. Ser A I Math. **465** (1970), 1–13.

[MRV3] ——, *Topological and metric properties of quasiregular mappings.* Ann. Acad. Sci. Fenn. Ser. A I Math. **488** (1971), 1–31.

[M1] K. Matsumoto, *Some notes on exceptional values of meromorphic functions.* Nagoya Math. J. **22** (1963), 189–201.

[M2] ——, *Some remarks on Picard sets.* Ann. Acad. Sci. Fenn. Ser. A I Math. **403** (1967), 1–17.

[R] Yu. G. Reshetnyak, *Spatial mappings with bounded distortion.* Izdat. Nauka, Novosibirsk, 1982 (Russian).

[Ri1] S. Rickman, *On the number of omitted values of entire quasiregular mappings.* J. Analyse Math. **37** (1980), 100–117.

[Ri2] ——, *Value distribution of quasiregular mappings.* Lecture Notes in Math. **981**, Springer-Verlag, New York, 1983, pp. 220–245.

[Ri3] ——, *Quasiregular mappings and metrics on the n-sphere with punctures.* Comment. Math. Helv. **59** (1984), 136–148.

[S] J. Serrin, *Local behavior of solutions of quasilinear equations.* Acta Math. **111** (1964), 247–302.

[T1] S. Toppila, *Picard sets for meromorphic functions.* Ann. Acad. Sci. Fenn. Ser. A I Math. **417** (1967), 1–24.

[T2] ——, *Some remarks on the value distribution of entire functions.* Ann. Acad. Sci. Fenn. Ser. A I Math. **421** (1968), 1–11.

[T3] ——, *Some remarks on the value distribution of meromorphic functions.* Ark. Mat. **9** (1971), 1–9.

[T4] ——, *Picard sets for meromorphic functions with a deficient value.* Ann. Acad. Sci. Fenn. Ser. A I 5 Math. (1980), 179–184.

[T5] ——, *On the value distribution of meromorphic functions with a deficient value.* Ann. Acad. Sci. Fenn. Ser. A I 5 Math. (1980), 179–184.

[V1] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings.* Lecture Notes in Math., **229**, Springer-Verlag, New York, 1971.

[V2] ——, *Modulus and capacity inequalities for quasiregular mappings.* Ann. Acad. Sci. Fenn. Ser. A I Math. **509** (1972), 1–14.

[V3] ——, *A survey of quasiregular mappings in* R^n. Proc. 1978 Internat. Congr. Math. (Helsinki, Finland), Academia Scientiarum Fennica, Helsinki, 1980.

[Vu1] M. Vuorinen, *On the Harnack constant and boundary behavior of Harnack functions.* Ann. Acad. Sci. Fenn. Ser. A I 7 Math. (1982), 259–277.

[Vu2] ——, *Conformal invariants and quasiregular mappings.* J. Analyse Math. **45** (1985), 69–115.

[Vu3] ——, *Conformal geometry and quasiregular mappings.* Lecture Notes in Math., **1319**, Springer-Verlag, New York, 1988.

[W1] J. Winkler, *Zur Existenz ganzer Funktionen bei vorgegebener Menge der Nullstellen und Einsstellen.* Math. Z. **168** (1979), 77–85.

[W2] ——, *On the existence of meromorphic functions with prescribed zeros, ones and poles.* Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 563–572.

Department of Mathematics, University of Helsinki, Helsinki, Finland