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NOTES

ON PIECEWISE REFERENCE TECHNOLOGIES*

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In this paper we introduce a piecewise parametric production model that may be used as a reference technology for efficiency gauging. We show that other piecewise reference technologies used in efficiency measuring are obtained as special cases of our model.
(PIECEWISE LINEAR MODEL; CONSTANT ELASTICITY OF TRANSFORMATION AND SUBSTITUTION)

1. Introduction

Piecewise linear models of production technology are enjoying a revival due in large part to the useful role they are playing in efficiency measurement. In fact there is now a wide range of such technologies—several of which were compared in a recent paper by Grosskopf (1986) with respect to returns to scale and input disposability.

The purpose of this short note is to extend the work by Grosskopf to piecewise reference technologies that are not necessarily linear. In particular we will show that the piecewise linear model and the piecewise loglinear model, the latter due to Banker and Maindiratta (1986), are both special cases of our general piecewise model. In addition, we derive restrictions on our specification which yield input and output sets that are convex. The interest in convexity of the input and output sets is founded in part in tradition. The original piecewise linear reference technologies possess these convexity properties. Moreover, economists rely on them to formulate the duality theory. Finally, we also address the question of returns to scale within our piecewise reference technology.

2. The Reference Technologies

A production technology T is considered as the pair of input vectors $x \in R_+^N$ and output vectors $u \in R_+^M$ such that (x, u) is feasible, i.e.,

$$T = \{(x, u): u \text{ can be produced by } x\}. \tag{2.1}$$

Two set-valued mappings associated with T are of interest, namely the output correspondence

$$P(x) = \{u: (x, u) \in T\} \tag{2.2}$$

and the input correspondence

$$L(u) = \{x: (x, u) \in T\}. \tag{2.3}$$

Clearly,

$$T = \{(x, u): u \in P(x)\} = \{(x, u): x \in L(u)\}; \tag{2.4}$$

thus the expressions T, $P(x)$ and $L(u)$ are equivalent representations of a production technology.

* All Notes are refereed.

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Suppose next that there are $k = 1, \ldots, K$ observations of inputs and outputs, i.e., (x^k, u^k), $k = 1, \ldots, K$. We assume here that each x_n^k and u_m^k, $k = 1, \ldots, K$; $n = 1, \ldots, N$; $m = 1, \ldots, M$; are strictly positive. (This assumption may be relaxed for some of the following models, see e.g. Färe, Grosskopf and Lovell 1985.) To construct the piecewise reference technology introduce $z \in R_+^K$, $\sum_{k=1}^K z^k = 1$, as the intensity vector. Components of this vector serve as "weights" for each activity, constructing the piecewise segments which "connect" efficient points and serve as the frontier.

The most general of our reference technologies used here is written as

$$T = \{(x, u)\colon u_m \leqq (\sum_{k=1}^{K} z^k (u_m^k)^\delta)^{1/\delta},\ m = 1, \ldots, M,$$
$$x_n \geqq (\sum_{k=1}^{K} z^k (x_n^k)^\gamma)^{1/\gamma},\ n = 1, \ldots, N,$$
$$z \in R_+^K,\ \sum_{k=1}^{K} z^k = 1,\ \delta \text{ and } \gamma \in R\}. \tag{2.5}$$

Note that if $\delta = \gamma = 1$, (2.5) becomes

$$T^A = \{(x, u)\colon u_m \leqq \sum_{k=1}^{K} z^k u_m^k,\ m = 1, \ldots, M,$$
$$x_n \geqq \sum_{k=1}^{K} z^k x_n^k,\ n = 1, \ldots, N,$$
$$z \in R_+^K,\ \sum_{k=1}^{K} z^k = 1\}. \tag{2.6}$$

This model was (in the single output case) introduced by Afriat (1972) and it has been frequently employed as a reference technology in efficiency gauging, see e.g. Banker, Charnes and Cooper (1984) or Färe, Grosskopf and Lovell (1985), and references therein.

If $\delta \to 0$ and $\gamma \to 0$, (2.5) becomes[1] the loglinear reference technology introduced by Banker and Maindiratta (1986) namely

$$T^{BM} = \{(x, u)\colon u_m \leqq \prod_{k=1}^{K} (u_m^k)^{z^k},\ m = 1, \ldots, M,$$
$$x_n \geqq \prod_{k=1}^{K} (x_n^k)^{z^k},\ n = 1, \ldots, N,$$
$$z \in R_+^K,\ \sum_{k=1}^{K} z^k = 1\}. \tag{2.7}$$

The motivation for the introduction of (2.7) is that this reference technology can model increasing marginal products or more precisely, the set T^{BM} need not be convex, even in the case of one input and one output. (See Figure 1 in Banker and Maindiratta 1986.) However, as the following example shows, (2.7) may have nonconvex output sets $P(x)$. Let the observed inputs and outputs be as in the table below.

[1] To take these limits, invoke l'Hôpital's rule.

Observation	Output u_1	u_2	Input x
1	1	2	1
2	2	1	1

For this example, T^{BM} becomes

$$T^{BM} = \{(x, u): u_1 \leqq 1^z \cdot 2^{1-z},\ u_2 \leqq 2^z \cdot 1^{1-z},\ 1^z \cdot 1^{1-z} \leqq x\}. \tag{2.8}$$

For $x = 1$, $(u_1, u_2) = (1, 2)$ and $(v_1, v_2) = (2, 1)$ are feasible output vectors, however $(1.5, 1.5) = \frac{1}{2}(1, 2) + \frac{1}{2}(2, 1)$ is not. This shows that (2.7) need not have convex output sets, a typical neoclassical assumption, see Hasenkamp (1976). Also, Shephard (1970, p. 180) argues that $P(x)$ convex is a suitable assumption for time divisible technologies. In addition, in order to develop the duality between the output set and the revenue function, $P(x)$ must be convex. In the same fashion, the duality between the cost function and the input set $L(u)$, requires that $L(u)$ convex. At this point, let us address the question for what values of δ and γ the output and input sets are convex. First we prove a sufficient condition.

PROPOSITION 1. *If $\delta \geqq 1$ and $\gamma \leqq 1$, then $P(x)$ and $L(u)$ are convex.*

PROOF. See Appendix A for the proof that $P(x)$ is convex. A similar argument applies to show that $L(u)$ is also convex.

Moreover, if in the proof of Proposition 1, $u \in P(x)$ and $v \in P(y)$, i.e., (x, u) and $(y, v) \in T$, then if $\delta \geqq 1$ and $\gamma \leqq 1$, it follows that $((1-\theta)x + \theta y, (1-\theta)u + \theta v) \in T$, thus T is also convex.

Next, we address the necessity for $P(x)$ and $L(u)$ to be convex.

PROPOSITION 2. *If $P(x)$ and $L(u)$ are convex and there are distinct $u, v \in P(x)$ and $x, y \in L(u)$ such that equality holds in (2.5), then $\delta \geqq 1$ and $\gamma \leqq 1$.*

PROOF. See Appendix B.

As noted above, the piecewise loglinear model specified in (2.7) was introduced to model increasing marginal products, allowing for increasing returns to scale which implies that T^{BM} be (at least locally) nonconvex. We note that this does not imply that the standard piecewise linear model (as in (2.6)) cannot model increasing returns to scale. Indeed, (2.6) can model increasing, constant and decreasing returns simultaneously, yet T^A is convex, as in Figure 1. Assuming a one output, one input technology and these observations A, B and C, (2.6) yields the reference set T^A which exhibits increasing returns up to B, constant returns at B, and decreasing returns beyond B.

Proposition 2 makes it clear that T^A is one piecewise technology that allows for variable

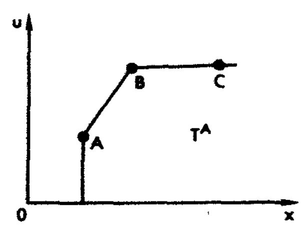

FIGURE 1

returns to scale if $L(u)$ and $P(x)$ are required to be convex. On the other hand, T^A can model increasing returns only if the origin is not included in the reference set.[2]

[2] We are grateful for suggestions of two referees whose deep insights have significantly improved our work.

Appendix A. Proof of Proposition 1

$$P(x) = \{u: u_m \leq (\sum_{k=1}^{K} z^k(u_m^k)^\delta)^{1/\delta},\ m = 1, \ldots, M,$$

$$x_n \geq (\sum_{k=1}^{K} z^k(x_n^k)^\gamma)^{1/\gamma},\ n = 1, \ldots, N,$$

$$z \in R_+^K,\ \sum_{k=1}^{K} z^k = 1,\ \delta \geq 1,\ \gamma \leq 1\}.$$

Let $u \in P(x)$, $v \in P(x)$ and $\theta \in [0, 1]$ then there are activity vectors z and $\bar{z}$ such that

$$u_m \leq (\sum_{k=1}^{K} z^k(u_m^k)^\delta)^{1/\delta}, \qquad m = 1, \ldots, M,$$

$$v_m \leq (\sum_{k=1}^{K} \bar{z}^k(u_m^k)^\delta)^{1/\delta}, \qquad m = 1, \ldots, M.$$

Define $\alpha = \sum_{k=1}^{K} z^k(u_m^k)^\delta$ and $\beta = \sum_{k=1}^{K} \bar{z}^k(u_m^k)^\delta$, then since $\delta \geq 1$,

$$(1 - \theta)u_m + \theta v_m \leq (1 - \theta)\alpha^{1/\delta} + \theta\beta^{1/\delta} \leq ((1 - \theta)\alpha + \theta\beta)^{1/\delta},$$

thus

$$(1 - \theta)u_m + \theta v_m \leq (\sum_{k=1}^{K} (1 - \theta)z^k + \theta\bar{z}^k)(u_m^k))^{1/\delta} \qquad \text{where}$$

$$\sum_{k=1}^{K} (1 - \theta)z^k + \theta\bar{z}^k = 1 \qquad \text{and} \qquad m = 1, \ldots, M.$$

Therefore, $(1 - \theta)u + \theta v \in P(x)$. Using arguments paralleling the above, one can show that $L(u)$ is convex if $\gamma \leq 1$. QED.

Appendix B. Proof of Proposition 2

Let $u_m = (\sum_{k=1}^{K} z^k(u_m^k)^\delta)^{1/\delta}$, $m = 1, \ldots, M$, and $v_m = \sum_{k=1}^{K} \bar{z}^k(u_m^k)^\delta)^{1/\delta}$, $m = 1, \ldots, M$. Since $P(x)$ is convex, for all $\theta \in [0, 1]$,

$$(1 - \theta)(\sum_{k=1}^{K} z^k(u_m^k)^\delta)^{1/\delta} + \theta(\sum_{k=1}^{K} \bar{z}^k(u_m^k)^\delta)^{1/\delta} \leq (\sum_{k=1}^{K} ((1 - \theta)z^k + \theta\bar{z}^k)(u_m^k)^\delta)^{1/\delta}.$$

Also

$$(\sum_{k=1}^{K} ((1 - \theta)z^k + \theta\bar{z}^k)(u_m^k)^\delta)^{1/\delta} = ((1 - \theta) \sum_{k=1}^{K} z^k(u_m^k)^\delta + \theta \sum_{k=1}^{K} \bar{z}^k(u_m^k)^\delta)^{1/\delta}.$$

Define $\alpha = \sum_{k=1}^{K} z^k(u_m^k)^\delta$ and $\beta = \sum_{k=1}^{K} \bar{z}^k(u_m^k)^\delta$, then $(1 - \theta)(\alpha)^{1/\delta} + \theta(\beta)^{1/\delta} \leq ((1 - \theta)\alpha + \theta\beta)^{1/\delta}$, thus the function $f(\cdot) = (\cdot)^{1/\delta}$ must be concave, which is true if and only if $\delta \geq 1$.

Using arguments parallelling the above, one may prove that $L(u)$ convex, with

$$x_n = (\sum_{k=1}^{K} z^k(x_n^k)^\gamma)^{1/\gamma}, \qquad n = 1, \ldots, N, \qquad \text{and}$$

$$y_n = (\sum_{k=1}^{K} \bar{z}^k(x_n^k)^\gamma)^{1/\gamma}, \qquad n = 1, \ldots, N,$$

imply that $\gamma \leq 1$. QED.

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ERRATUM: "THE THEORY OF RATIO SCALE ESTIMATION: SAATY'S ANALYTIC HIERARCHY PROCESS" (VOL. 33, NO. 11, NOVEMBER 1987)

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The statement of Axiom 3 on p. 1386 should read:

AXIOM 3 (Dependence). Let $\mathscr{H}$ be a hierarchy with levels $L_1, L_2, \ldots, L_h$. For each L_k, $k = 1, 2, \ldots, h - 1$,

(1) L_{k+1} is outer dependent on L_k,
(2) L_{k+1} is inner independent with respect to all $x \in L_k$,
(3) L_k is outer independent of L_{k+1}.

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