ON PLANE VISCOUS MAGNETOHYDRODYNAMIC FLOWS*

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1. Introduction. Martin [1] derived a new form for the basic equations governing the plane flow of viscous, incompressible, non-conducting fluids. He used this new form of equations to prove the following:

(i) If the streamlines are straight lines, the straight lines must be concurrent or parallel.

(ii) The streamlines can be involutes of a curve only if the curve reduces to a point and the streamlines are circles concentric at this point.

Following Martin's approach, we show that when streamlines Ψ = constant and magnetic lines ϕ = constant of plane, non-aligned flow of a viscous incompressible fluid of infinite electrical conductivity are taken as the curvilinear coordinate system ϕ, Ψ in the physical plane the fundamental equations governing the flow can be replaced by a new system of equations. In these equations ϕ, Ψ are the independent variables.

In case of orthogonal flows we prove the following:

(i) If the streamlines are straight lines but not parallel, then they must be concurrent.

(ii) If the streamlines are involutes of a curve, then the streamlines are concentric circles.

Finally, we find solutions to vortex and source flow problems.

2. Flow equations. The steady flow of an incompressible fluid of infinite electrical conductivity, in the absence of heat conduction, is governed by the system of five non-linear partial differential equations

$$(\partial v_1/\partial x) + (\partial v_2/\partial y) = 0, \tag{2.1}$$

$$\rho\left(v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y}\right) + \frac{\partial p}{\partial x} = \eta\left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2}\right) - \mu H_2\left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right), \tag{2.2}$$

$$\rho\left(v_1 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y}\right) + \frac{\partial p}{\partial y} = \eta\left(\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2}\right) + \mu H_1\left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right), \tag{2.3}$$

$$v_1 H_2 - v_2 H_1 = k, \tag{2.4}$$

$$(\partial H_1/\partial x) + (\partial H_2/\partial y) = 0, \tag{2.5}$$

where v_1, v_2 are the velocity components, H_1 and H_2 the components of the magnetic field vector $\mathbf{H}$, p the pressure, ρ the constant density, η the constant coefficient of viscosity, μ the constant magnetic permeability and k an arbitrary constant.

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Throughout this paper, we assume that the streamlines are nowhere parallel to the magnetic lines i.e. $k \neq 0$.

On introducing the functions

$$\omega = (\partial v_2/\partial x) - (\partial v_1/\partial y),$$
$$\Omega = (\partial H_2/\partial x) - (\partial H_1/\partial y), \tag{2.6}$$
$$h = (\rho/2)V^2 + p,$$

where

$$V^2 = v_1^{\ 2} + v_2^{\ 2}, \tag{2.7}$$

Eq. (2.2) can be written as

$$\rho\left(v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y}\right) + \frac{\partial h}{\partial x} - \rho\left(v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_2}{\partial x}\right) = \eta\left\{\frac{\partial^2 v_1}{\partial x^2} + \left(\frac{\partial^2 v_2}{\partial x\, \partial y} - \frac{\partial \omega}{\partial y}\right)\right\} - \mu\Omega H_2\,,$$

or

$$-\rho v_2\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) + \frac{\partial h}{\partial x} = -\eta \frac{\partial \omega}{\partial y} + \eta \frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\right) - \mu\Omega H_2\,.$$

Using (2.1) and the first equation of (2.6), we get

$$\eta(\partial\omega/\partial y) - \rho\omega v_2 + \mu\Omega H_2 = -(\partial h/\partial x).$$

Similiarly, (2.3) gives us

$$\eta(\partial\omega/\partial x) - \rho\omega v_1 + \mu\Omega H_1 = \partial h/\partial y.$$

The system of five partial differential equations (2.1)–(2.5) may be replaced by the following seven partial differential equations:

$$(\partial v_1/\partial x) + (\partial v_2/\partial y) = 0, \tag{2.8}$$
$$\eta(\partial\omega/\partial y) - \rho\omega v_2 + \mu\Omega H_2 = -(\partial h/\partial x), \tag{2.9}$$
$$\eta(\partial\omega/\partial x) - \rho\omega v_1 + \mu\Omega H_1 = \partial h/\partial y, \tag{2.10}$$
$$v_1 H_2 - v_2 H_1 = k \neq 0, \tag{2.11}$$
$$(\partial H_1/\partial x) + (\partial H_2/\partial y) = 0, \tag{2.12}$$
$$(\partial v_2/\partial x) - (\partial v_1/\partial y) = \omega, \tag{2.13}$$
$$(\partial H_2/\partial x) - (\partial H_1/\partial y) = \Omega. \tag{2.14}$$

The set of equations (2.8)–(2.14) is a system of non-linear partial differential equations in seven dependent variables v_1, v_2, H_1, H_2, ω, Ω and h. Although the number of equations and dependent variables has increased by two, the order has decreased from two to one.

3. Some results from differential geometry. Let

$$x = x(\phi, \Psi), \quad y = y(\phi, \Psi) \tag{3.1}$$

define a system of curvilinear coordinates in the (x, y)-plane. In the curvilinear coordinate system (ϕ, Ψ) the squared element of arc length is given by

$$ds^2 = E d\phi^2 + 2F\, d\phi\, d\Psi + G d\Psi^2 \tag{3.2}$$

where

$$E = (\partial x/\partial\phi)^2 + (\partial y/\partial\phi)^2,$$
$$F = (\partial x/\partial\phi)(\partial x/\partial\Psi) + (\partial y/\partial\phi)(\partial y/\partial\Psi), \tag{3.3}$$
$$G = (\partial x/\partial\Psi)^2 + (\partial y/\partial\Psi)^2.$$

Eq. (3.1) can be used to obtain $\phi = \phi(x, y)$, $\Psi = \Psi(x, y)$ such that

$$\frac{\partial x}{\partial\phi} = J\frac{\partial\Psi}{\partial y}, \qquad \frac{\partial y}{\partial\phi} = -J\frac{\partial\Psi}{\partial x}, \qquad \frac{\partial x}{\partial\Psi} = -J\frac{\partial\phi}{\partial y}, \qquad \frac{\partial y}{\partial\Psi} = J\frac{\partial\phi}{\partial x} \tag{3.4}$$

provided that $0 < |J| < \infty$, where J denotes the Jacobian

$$J = (\partial x/\partial\phi)(\partial y/\partial\Psi) - (\partial x/\partial\Psi)(\partial y/\partial\phi). \tag{3.5}$$

From (3.3) and (3.5), we have

$$J = \pm W \tag{3.6}$$

where $W = (EG - F^2)^{1/2}$. Let β be the angle made by the tangent to the coordinate line ϕ = constant, directed in the sense of increasing Ψ, with the x-axis. From the third equation of (3.3), we write

$$(\partial x/\partial\Psi) = \sqrt{G}\cos\beta, \qquad (\partial y/\partial\Psi) = \sqrt{G}\sin\beta. \tag{3.7}$$

Substitution of $\partial x/\partial\Psi$ and $\partial y/\partial\Psi$ from (3.7) in the second equation of (3.3) yields

$$F = \sqrt{G}\left[\frac{\partial x}{\partial\phi}\cos\beta + \frac{\partial y}{\partial\phi}\sin\beta\right]. \tag{3.8}$$

Eliminating $\partial y/\partial\phi$ between (3.8) and the first equation of (3.3) and solving for $\partial x/\partial\phi$, we obtain

$$\partial x/\partial\phi = (F/\sqrt{G})\cos\beta + (J/\sqrt{G})\sin\beta. \tag{3.9}$$

The first equation of (3.3) and (3.9) require

$$\partial y/\partial\phi = (F/\sqrt{G})\sin\beta - (J/\sqrt{G})\cos\beta. \tag{3.10}$$

From (3.7), (3.9), (3.10) and the conditions that the second-order mixed derivatives of x and y with respect to ϕ and Ψ are independent of the order of differentiation, we find that

$$\partial\beta/\partial\phi = (J/G){\gamma_{12}}^2, \quad \partial\beta/\partial\Psi = (J/G){\gamma_{11}}^2, \tag{3.11}$$

where

$${\gamma_{11}}^2 = \frac{1}{2W^2}\left[F\frac{\partial G}{\partial\Psi} - 2G\frac{\partial F}{\partial\Psi} + G\frac{\partial G}{\partial\phi}\right],$$
$${\gamma_{12}}^2 = \frac{1}{2W^2}\left[F\frac{\partial G}{\partial\phi} - G\frac{\partial E}{\partial\Psi}\right]. \tag{3.12}$$

Proceeding exactly as Martin [1], it is found that if E, F, G are given functions of ϕ and Ψ, then (3.1) will serve as planar curvilinear co-ordinate system if and only if

$$\partial/\partial\Psi((J/G){\gamma_{12}}^2) - \partial/\partial\phi((J/G){\gamma_{11}}^2) = 0 \tag{3.13}$$

where ${\gamma_{11}}^2$ and ${\gamma_{12}}^2$ are given by (3.12).

When the condition (3.13) is satisfied, the functions $x(\phi, \Psi)$, $y(\phi, \Psi)$ can be obtained from the relation

$$z = x + iy = \int \frac{\exp(i\beta)}{\sqrt{G}} \{(F - iJ)\, d\phi + G\, d\Psi\} \tag{3.14}$$

where β as a function of ϕ and Ψ is given by

$$\beta = \int \frac{J}{G} \{\gamma_{12}^2\, d\phi + \gamma_{11}^2\, d\Psi\}. \tag{3.15}$$

4. New form for the fundamental equations. Eqs. (2.8) and (2.12), respectively, imply the existence of a stream function $\Psi(x, y)$ and the magnetic function $\phi(x, y)$ such that

$$v_2 = -(\partial\Psi/\partial x), \qquad v_1 = \partial\Psi/\partial y \tag{4.1}$$

and

$$H_2 = \partial\phi/\partial x, \qquad H_1 = -(\partial\phi/\partial y). \tag{4.2}$$

We assume that the curves Ψ = constant and the curves ϕ = constant form the curvilinear coordinate system discussed in Sec. 3 of this paper.

Using (4.1) and (4.2) in (2.11), we find

$$\frac{\partial\Psi}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial\phi}{\partial y} = \frac{\partial(\phi, \Psi)}{\partial(x, y)} = \frac{1}{J} = k \neq 0 \tag{4.3}$$

where J is defined by (3.5). Eq. (4.3) implies that if we know x and y as functions of ϕ and Ψ, then we can obtain ϕ and Ψ as functions of x and y.

In what follows we transform the flow equations to such a form that their solution gives us v_1, v_2, H_1, H_2, ω, Ω and h as functions of ϕ and Ψ.

Solenoidal condition on **H**. Using (3.4) in (4.2), we get

$$\partial x/\partial\Psi = JH_1, \; \partial y/\partial\Psi = JH_2. \tag{4.4}$$

Let θ be the angle made by the magnetic field **H** with x-axis. The components H_1 and H_2 of **H** can be written as

$$H_1 = H\cos\theta, \quad H_2 = H\sin\theta \tag{4.5}$$

where $H = |\mathbf{H}|$. From (4.4) and (4.5) we have

$$\partial x/\partial\Psi = JH\cos\theta, \quad \partial y/\partial\Psi = JH\sin\theta. \tag{4.6}$$

Now two cases arise:

1: $\theta = \beta$, where β is defined in Sec. 3. In this case (4.6) becomes

$$\partial x/\partial\Psi = JH\cos\beta, \quad \partial y/\partial\Psi = JH\sin\beta. \tag{4.7}$$

From (3.7) and (4.7), we get

$$JH = \sqrt{G}, \tag{4.8}$$

i.e. $J > 0$.

2: $\theta = \beta + \pi$. From (4.6), we obtain

$$\partial x/\partial \Psi = -JH \cos \beta, \quad \partial y/\partial \Psi = -JH \sin \beta. \tag{4.9}$$

Eqs. (4.9) together with (3.7) give

$$-JH = \sqrt{G}, \tag{4.10}$$

i.e. $J < 0$.

From the above two cases, we conclude that the magnetic field acts along the magnetic lines towards higher or lower parameter values Ψ accordingly as J is positive or negative. In either case (3.6) requires that

$$WH = \sqrt{G}. \tag{4.11}$$

Eqs. (3.7) and (4.4) imply that

$$H_1 + iH_2 = (\sqrt{G}/J) \exp (i\beta). \tag{4.12}$$

Equation of continuity. Martin [1] has shown that the equation of continuity implies that the fluid flows along the streamlines towards higher or low parameter values ϕ accordingly as J is positive or negative. He has also proven that

$$WV = \sqrt{E} \tag{4.13}$$

and

$$v_1 + iv_2 = (\sqrt{E}/J) \exp (i\alpha) \tag{4.14}$$

where α is the angle between the tangent to the coordinate line Ψ = constant, directed in the sense of increasing ϕ, with the x-axis.

The function Ω. By the definition of Ω and (3.4) we find that

$$J\Omega = \left(\frac{\partial H_2}{\partial \phi}\frac{\partial y}{\partial \Psi} - \frac{\partial H_2}{\partial \Psi}\frac{\partial y}{\partial \phi}\right) + \left(\frac{\partial H_1}{\partial \phi}\frac{\partial x}{\partial \Psi} - \frac{\partial H_1}{\partial \Psi}\frac{\partial x}{\partial \phi}\right)$$

On substituting $H_1 = \pm H \cos \beta$, $H_2 = \pm H \sin \beta$, we find

$$\pm J\Omega = \left[\left(\frac{\partial H}{\partial \phi} \sin \beta + H \cos \beta \frac{\partial \beta}{\partial \phi}\right)\frac{\partial y}{\partial \Psi} - \left(\frac{\partial H}{\partial \Psi} \sin \beta + H \cos \beta \frac{\partial \beta}{\partial \Psi}\right)\frac{\partial y}{\partial \phi}\right]$$
$$+ \left[\left(\frac{\partial H}{\partial \phi} \cos \beta - H \sin \beta \frac{\partial \beta}{\partial \phi}\right)\frac{\partial x}{\partial \Psi} - \left(\frac{\partial H}{\partial \Psi} \cos \beta - H \sin \beta \frac{\partial \beta}{\partial \Psi}\right)\frac{\partial x}{\partial \phi}\right].$$

Using (3.7), (3.9) and (3.10), we get

$$\sqrt{G}\, W\Omega = G\frac{\partial H}{\partial \phi} - F\frac{\partial H}{\partial \Psi} + HJ\frac{\partial \beta}{\partial \Psi}. \tag{4.15}$$

Eliminating H and β between (3.11), (4.11) and (4.15), and using the identities

$$(\partial/\partial\phi)(G/2W^2) = (1/W^2)(G\gamma_{22}{}^2 - F\gamma_{12}{}^2)$$

and

$$(\partial/\partial\Psi)(G/2W^2) = (1/W^2)(G\gamma_{12}{}^2 - F\gamma_{12}{}^2),$$

where

$$\gamma_{22}{}^2 = \frac{1}{2W^2}\left[-G\frac{\partial E}{\partial \phi} + 2F\frac{\partial F}{\partial \phi} - F\frac{\partial E}{\partial \Psi}\right],$$

we find that

$$W\Omega = \frac{1}{W}\{G\gamma_{22}{}^2 - 2F\gamma_{12}{}^2 + E\gamma_{11}{}^2\}. \tag{4.16}$$

On differentiating (G/W) with respect to ϕ and (F/W) with respect to Ψ, we see that

$$\frac{\partial}{\partial \phi}\left(\frac{G}{W}\right) - \frac{\partial}{\partial \Psi}\left(\frac{F}{W}\right) = \frac{1}{W}(G\gamma_{22}{}^2 - 2F\gamma_{12}{}^2 + E\gamma_{11}{}^2). \tag{4.17}$$

From (4.16) and (4.17), we get

$$\Omega = \frac{1}{W}\left\{\frac{\partial}{\partial \phi}\left(\frac{G}{W}\right) - \frac{\partial}{\partial \Psi}\left(\frac{F}{W}\right)\right\} \tag{4.18}$$

The vorticity ω. Martin [1] has proven that

$$\omega = \frac{1}{W}\left\{\frac{\partial}{\partial \phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial \Psi}\left(\frac{E}{W}\right)\right\}.$$

Equations of momentum. Eq. (2.9) can be written as

$$\eta\left(\frac{\partial \omega}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial \omega}{\partial \Psi}\frac{\partial \Psi}{\partial y}\right) + \rho\omega\frac{\partial \Psi}{\partial x} + \mu\Omega\frac{\partial \phi}{\partial x} = -\left(\frac{\partial h}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial h}{\partial \Psi}\frac{\partial \Psi}{\partial x}\right) \tag{4.19}$$

where (4.1) and (4.2) have been used to eliminate v_2 and H_2. Eq. (4.19), on using (3.4), becomes

$$\eta\left(-\frac{\partial \omega}{\partial \phi}\frac{\partial x}{\partial \Psi} + \frac{\partial \omega}{\partial \Psi}\frac{\partial x}{\partial \phi}\right) - \rho\omega\frac{\partial y}{\partial \phi} + \mu\Omega\frac{\partial y}{\partial \Psi} = \left(-\frac{\partial h}{\partial \phi}\frac{\partial x}{\partial \Psi} + \frac{\partial h}{\partial \Psi}\frac{\partial y}{\partial \phi}\right). \tag{4.20}$$

Similarly, (2.10) gives us

$$\eta\frac{\partial \omega}{\partial \phi}\frac{\partial y}{\partial \Psi} - \frac{\partial \omega}{\partial \Psi}\frac{\partial y}{\partial \phi} - \rho\omega\frac{\partial x}{\partial \phi} + \mu\Omega\frac{\partial x}{\partial \Psi} = -\frac{\partial h}{\partial \phi}\frac{\partial x}{\partial \Psi} + \frac{\partial h}{\partial \Psi}\frac{\partial x}{\partial \phi}. \tag{4.21}$$

Multiplying (4.20) by $\partial y/\partial \phi$, (4.21) by $\partial x/\partial \phi$ and adding, we get

$$-F\left(\frac{\partial h}{\partial \phi} + \mu\Omega\right) + E\left(\frac{\partial h}{\partial \Psi} + \rho\omega\right) = \eta J\frac{\partial \omega}{\partial \phi} \tag{4.22}$$

where E, F and J are given by (3.3) and (3.5). Again, multiplying (4.20) by $\partial y/\partial \Psi$, (4.21) by $\partial x/\partial \Psi$ and adding, we obtain

$$G\left(\frac{\partial h}{\partial \phi} + \mu\Omega\right) - F\left(\frac{\partial h}{\partial \Psi} + \rho\omega\right) = -\eta J\frac{\partial \omega}{\partial \Psi}. \tag{4.23}$$

Eqs. (4.22) and (4.23) are the new forms for the momentum equations.

Eqs. (4.22) and (4.23) can be written in another form by eliminating $\partial h/\partial \Psi$ and

$\partial h/\partial \phi$ respectively between them; the resulting equations are

$$\begin{aligned}\frac{\partial h}{\partial \phi} + \mu\Omega &= \frac{\eta}{J}\left(F\frac{\partial \omega}{\partial \phi} - E\frac{\partial \omega}{\partial \Psi}\right),\\ \frac{\partial h}{\partial \Psi} + \rho\omega &= \frac{\eta}{J}\left(G\frac{\partial \omega}{\partial \phi} - F\frac{\partial \omega}{\partial \Psi}\right).\end{aligned} \tag{4.24}$$

Summing up the results obtained thus far, we have

THEOREM 1. When the streamlines Ψ = constant and the magnetic lines ϕ = constant of steady, plane flow of a viscous, infinitely conducting (electrically), incompressible fluid are taken as the curvilinear coordinate system ϕ, Ψ in the physical plane, the set of seven partial differential equations (2.8)–(2.14) for v_1, v_2, H_1, H_2, ω, Ω and h as functions of x, y may be replaced by the system

$$\begin{aligned}&-F((\partial h/\partial \phi) + \mu\Omega) + E((\partial h/\partial \Psi) + \rho\omega) = \eta J(\partial \omega/\partial \phi),\\ &G((\partial h/\partial \phi) + \mu\Omega) - F((\partial h/\partial \Psi) + \rho\omega) = -\eta J(\partial \omega/\partial \Psi),\\ &(\partial/\partial \Psi)((J/G)\gamma_{12}^2) - (\partial/\partial \phi)((J/G)\gamma_{11}^2) = 0,\\ &\Omega = \frac{1}{W}\left[\frac{\partial}{\partial \phi}\left(\frac{G}{W}\right) - \frac{\partial}{\partial \Psi}\left(\frac{F}{W}\right)\right],\\ &\omega = \frac{1}{W}\left[\frac{\partial}{\partial \phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial \Psi}\left(\frac{E}{W}\right)\right],\\ &W^2 = J^2 = EG - F^2 = 1/k^2\end{aligned} \tag{4.25}$$

of six partial differential equations for E, F, G, ω, Ω and h as functions of ϕ, Ψ. Here E, F, G are given by $ds^2 = E\,d\phi^2 + 2F\,d\phi\,d\Psi + G\,d\Psi^2$, where ds is the element of arc length in the physical plane. The Jacobian J is positive or negative as the parameter Ψ increases or decreases in the direction of the magnetic field vector $\mathbf{H}$.

Given a solution

$$\begin{aligned}&E = E(\phi, \Psi) \qquad F = F(\phi, \Omega); \qquad G = G(\phi, \Psi)\\ &\omega = \omega(\phi, \Psi); \qquad \Omega = \Omega(\phi, \Psi); \qquad h = h(\phi, \Psi)\end{aligned}$$

of the system (4.25), we can find x, y as functions of ϕ, Ψ from

$$z = x + iy = \int \frac{\exp(i\beta)}{\sqrt{G}}\{(F - iJ)\,d\phi + G\,d\Psi\}$$

where $\beta = \int J/G(\gamma_{12}^2\,d\phi + \gamma_{11}^2\,d\Psi)$, and thus obtain E, F, G, ω, Ω and h as functions of x, y, since $0 < |J| < \infty$. Once we obtain E, F, G and h as functions of x, y then H_1, H_2, v_1, v_2 and p as functions of x, y are given by

$$\begin{aligned}H_1 + iH_2 &= (\sqrt{G}/J)\exp(i\beta),\\ v_1 + iv_2 &= (\sqrt{E}/J)\exp(i\alpha),\\ p &= h - (\rho/2)(E/W^2).\end{aligned}$$

5. Application of the fundamental equations in the new form. Eqs. (4.24) can be rewritten as

$$\frac{\partial h}{\partial \phi} = -\mu\Omega + \frac{\eta}{J}\left(F\frac{\partial \omega}{\partial \phi} - E\frac{\partial \omega}{\partial \Psi}\right), \tag{5.1}$$

$$\frac{\partial h}{\partial \Psi} = -\rho\omega + \frac{\eta}{J}\left(G\frac{\partial \omega}{\partial \phi} - F\frac{\partial \omega}{\partial \Psi}\right). \tag{5.2}$$

Differentiating (5.1) with respect to Ψ, (5.2) with respect to ϕ and using the condition that the second-order mixed derivative of h with respect to ϕ and Ψ is independent of the order of differentiation, we find

$$\eta J \Delta_2\omega + \mu(\partial\Omega/\partial\Psi) - \rho(\partial\omega/\partial\phi) = 0$$

where

$$\Delta_2\omega \equiv \frac{1}{J}\left[\frac{\partial}{\partial\phi}\left\{\frac{1}{J}\left(G\frac{\partial\omega}{\partial\phi} - F\frac{\partial\omega}{\partial\Psi}\right)\right\} + \frac{\partial}{\partial\Psi}\left\{\frac{1}{J}\left(E\frac{\partial\omega}{\partial\Psi} - F\frac{\partial\omega}{\partial\phi}\right)\right\}\right]. \tag{5.3}$$

Therefore, the system of equations (4.25) is reduced to five equations

$$\eta J \Delta_2\omega + \mu(\partial\Omega/\partial\Psi) - \rho(\partial\omega/\partial\phi) = 0, \tag{5.4}$$

$$\Omega = \frac{1}{J}\left[\frac{\partial}{\partial\phi}\left(\frac{G}{J}\right) - \frac{\partial}{\partial\Psi}\left(\frac{F}{J}\right)\right], \tag{5.5}$$

$$\omega = \frac{1}{J}\left[\frac{\partial}{\partial\phi}\left(\frac{F}{J}\right) - \frac{\partial}{\partial\Psi}\left(\frac{E}{J}\right)\right], \tag{5.6}$$

$$EG - F^2 = 1/k^2, \tag{5.7}$$

$$\frac{\partial}{\partial\Psi}\left(\frac{J}{G}\gamma_{12}^{\ 2}\right) - \frac{\partial}{\partial\phi}\left(\frac{J}{G}\gamma_{11}^{\ 2}\right) = 0 \tag{5.8}$$

in five dependent variables E, F, G, ω and Ω. If the solutions to these equations are given, we can find $h = h(\phi, \Psi)$ from the equations of momentum.

We shall now study two examples in which the curves Ψ = constant and the curves ϕ = constant form an orthogonal curvilinear coordinate system.

Example 1. In this example we prescribe the streamlines to be straight lines. We assume that they are not parallel but envelop a curve Γ. We now take the tangent lines to the curve Γ, and their orthogonal trajectories, the involutes of Γ as the system of orthogonal curvilinear coordinates. The square of the element of arc length ds in this orthogonal curvilinear coordinate system is given by $ds^2 = ds_1^{\ 2} + ds_2^{\ 2}$, where ds_1 and ds_2 are the elements of arc length of the involute and the tangent respectively.

The element of arc length of the involute is [2]

$$ds_1 = (\xi - \sigma)\kappa\, d\sigma$$

where σ denotes the arc length, κ the curvature of the curve Γ and ξ is a parameter constant along each involute. Therefore, we have

$$ds^2 = d\xi^2 + (\xi - \sigma)^2\kappa^2\, d\sigma^2. \tag{5.9}$$

But

$$\kappa = d\eta/d\sigma \tag{5.10}$$

where η is the angle subtended by the tangent line with x-axis. Eqs. (5.9) and (5.10) give us

$$ds^2 = d\xi^2 + (\xi - \sigma)^2 \, d\eta^2 \tag{5.11}$$

where $\sigma = \sigma(\eta)$. In this coordinate system, the coordinate curves $\xi =$ constant are the involutes of the curve Γ and the curves $\eta =$ constant its tangent lines.

We now investigate the flows for which

$$\phi = \phi(\xi), \quad \Psi = \Psi(\eta). \tag{5.12}$$

Using (5.12) in (3.2), we get

$$ds^2 = (\phi')^2 E \, d\xi^2 + 2F\phi'\Psi' \, d\xi \, d\eta + G(\Psi')^2 \, d\eta^2. \tag{5.13}$$

Comparing (5.11) and (5.13), we find

$$\begin{aligned} &E = (1/\phi')^2, \qquad F = 0, \qquad G = (\xi - \sigma(\eta))/\Psi')^2, \\ &J = (\xi - \sigma(\eta))/\phi'\Psi'. \end{aligned} \tag{5.14}$$

Since $F = 0$, (3.12) gives us

$$\gamma_{11}{}^2 = \frac{1}{2J^2} G \frac{\partial G}{\partial \phi}, \qquad \gamma_{12}{}^2 = -\frac{1}{2J^2} G \frac{\partial E}{\partial \Psi}.$$

Using (5.14), we get

$$\gamma_{11}{}^2 = \frac{\xi - \sigma(\eta)}{\Psi'} \phi', \qquad \gamma_{12}{}^2 = 0. \tag{5.15}$$

Substituting for G, J, $\gamma_{11}{}^2$, $\gamma_{12}{}^2$ from (5.14) and (5.15) in (5.8), we find that it is automatically satisfied. Using (5.14) in (5.5) and (5.6), we find

$$\Omega = \frac{1}{\xi - \sigma(\eta)} \frac{\partial}{\partial \xi} \{\phi'(\xi - \sigma(\eta)\} \tag{5.16}$$

and

$$\omega = -\frac{1}{\xi - \sigma(\eta)} \frac{\partial}{\partial \eta} \left(\frac{\Psi'}{\xi - \sigma(\eta)}\right). \tag{5.17}$$

From (5.4) and (5.14), we obtain

$$\eta\left[\frac{\partial}{\partial \xi}\left\{(\xi - \sigma(\eta)) \frac{\partial \omega}{\partial \xi}\right\} + \frac{\partial}{\partial \eta}\left\{\frac{1}{\xi - \sigma(\eta)} \frac{\partial \omega}{\partial \eta}\right\}\right] + \mu\phi' \frac{\partial \Omega}{\partial \eta} - \rho\Psi' \frac{\partial \omega}{\partial \xi} = 0 \tag{5.18}$$

or

$$\begin{aligned} &15\eta\Psi'(\sigma')^3 + (\xi - \sigma)[10\eta\Psi'\sigma'\sigma'' + 15\eta\Psi''(\sigma')^2] \\ &\quad + (\xi - \sigma)^2[9\eta\Psi'\sigma' + 4\eta\Psi''\sigma'' + \eta\Psi'\sigma''' + 6\eta\Psi'''\sigma' + 3\rho(\Psi')^2\sigma'] \\ &\quad + (\xi - \sigma)^3[4\eta\Psi'' + \eta\Psi^{(iv)} + 2\rho\Psi'\Psi''] - (\xi - \sigma)^4[\mu(\phi')^2\sigma'] \equiv 0. \end{aligned} \tag{5.19}$$

The curve Γ appears as the curve $\xi = \sigma(\eta)$ in the plane of variables ξ, η. For the relation (5.19) to hold identically, it must hold on the curve $\xi = \sigma(\eta)$, and therefore we have [1] $\sigma' = 0$, i.e. $\kappa \to \infty$.

THEOREM 2. If the streamlines in two dimensional flow of a viscous fluid are straight but not parallel, then they must be concurrent.

Example 2. In this example, we consider the involutes of the curve Γ as the streamlines and the tangents to the curve Γ as the magnetic lines.

As in the previous example, the square of the element of arc length in this orthogonal curvilinear coordinate system is

$$ds^2 = d\xi^2 + (\xi - \sigma)^2 \, d\eta^2. \tag{5.20}$$

For the flows under investigations, we have

$$\phi = \phi(\eta), \quad \Psi = \Psi(\xi). \tag{5.21}$$

Using (5.21) in (3.2), we get

$$ds^2 = (\phi')^2 E \, d\eta^2 + 2F\phi'\Psi' \, d\xi \, d\eta + G(\Psi')^2 \, d\xi^2. \tag{5.22}$$

Comparing (5.20) with (5.22), we obtain

$$E = \left(\frac{\xi - \sigma}{\phi'}\right)^2, \quad F = 0, \quad G = \left(\frac{1}{\Psi'}\right)^2, \quad J = \frac{\xi - \sigma}{\phi'\Psi'}. \tag{5.23}$$

Condition (5.8) is again automatically satisfied. Using (5.23) in (5.4), (5.5) and (5.6), we get

$$\eta \frac{\partial}{\partial \xi}\left[(\xi - \sigma)\frac{\partial \omega}{\partial \xi}\right] + \eta \frac{\partial}{\partial \eta}\left[\frac{1}{\xi - \sigma}\frac{\partial \omega}{\partial \eta}\right] + \mu\phi' \frac{\partial \Omega}{\partial \xi} - \rho\Psi' \frac{\partial \omega}{\partial \eta} = 0, \tag{5.24}$$

$$\Omega = \frac{1}{\xi - \sigma}\frac{\partial}{\partial \eta}\left(\frac{\phi'}{\xi - \sigma}\right), \tag{5.25}$$

and

$$\omega = -\frac{1}{\xi - \sigma}\frac{\partial}{\partial \xi}[\Psi'(\xi - \sigma)] \tag{5.26}$$

Elimination of Ω and ω between (5.24), (5.25) and (5.26) gives

$$3\sigma'[\eta\Psi'\sigma' + \mu(\phi')^2] + (\xi - \sigma)[\eta\Psi'\sigma'' + 2\mu\phi'\phi'']$$
$$+ (\xi - \sigma)^2[\eta\Psi' - (\Psi')^2\sigma'] - \Psi''(\xi - \sigma)^3 + 2\Psi'''(\xi - \sigma)^4 + \Psi^{(iv)}(\xi - \sigma)^5 \equiv 0$$

By the same argument as used in example 1, we have either $\sigma' = 0$, i.e. $\kappa \to \infty$, or

$$\Psi' = -\mu \frac{(\phi')^2}{\eta\sigma'} = A,$$

a constant. If $\Psi' =$ constant, Eqs. (5.23) imply that $G =$ constant. This is not possible. Therefore, we have

THEOREM 3. If the streamlines in plane flow of a viscous fluid are involutes of a curve Γ, then the streamlines are concentric circles.

6. Radial and vortex flows. In this section we study the radial and vortex flows when the magnetic field vector **H** is orthogonal to the velocity vector **V**.

A. *Radial flows.* The square of the element of arc length in polar coordinate system is given by

$$ds^2 = dr^2 + r^2\, d\theta^2. \tag{6.1}$$

Since the flows are radial, we have

$$\phi = \phi(r), \quad \Psi = \Psi(\theta). \tag{6.2}$$

Using (6.2) in (3.2), we get

$$ds^2 = E(\phi')^2\, dr^2 + 2F\phi'\Psi'\, d\theta\, dr + G(\Psi')^2\, d\theta^2. \tag{6.3}$$

Comparing (6.1) with (6.3), we find

$$E = 1/(\phi')^2, \quad F = 0, \quad G = r^2/(\Psi')^2. \tag{6.4}$$

From (5.7) and (6.4), we have

$$\Psi' = kr/\phi' = A, \tag{6.5}$$

where A is an arbitrary constant. Using (6.4) and (6.5) in (5.5) and (5.6), we obtain

$$\Omega = 2k/A, \quad \omega = 0. \tag{6.6}$$

Eqs. (4.11) and (4.13) give

$$H = (k/A)r, \quad V = A/r \tag{6.7}$$

where A, an arbitrary constant, can be determined from the boundary conditions. From Eqs. (5.1), (5.2) and (6.6), we get

$$h = -\mu(k^2r^2/A^2) + D$$

or

$$p = -\mu\frac{k^2r^2}{A^2} - \frac{\rho}{2}\frac{A^2}{r^2} + D,$$

where D is an arbitrary constant.

B. *Vortex flows.* We investigate the flows for which

$$\Psi = \Psi(r), \quad \phi = \phi(\theta) \tag{6.8}$$

where (r, θ) are the polar coordinates of a point in the plane of flow. For this case, we have

$$E = r^2/(\phi')^2, \quad F = 0, \quad G = 1/(\Psi')^2. \tag{6.9}$$

Eq. (5.7), on using (6.9), gives

$$\phi' = rk/\Psi' = A, \tag{6.10}$$

where A is an arbitrary constant. Using (6.9) and (6.10) in (5.5) and (5.6), we obtain

$$\Omega = 0, \quad \omega = -(2k/A). \tag{6.11}$$

Substituting (6.9) and (6.10) in (4.11) and (4.13), we find

$$H = (A/r) \quad \text{and} \quad V = (k/A)r \tag{6.12}$$

From Eqs. (5.1), (5.2) and (6.12), we get

$$h = \rho \frac{k^2 r^2}{A^2} + D$$

or

$$= \frac{\rho}{2} \frac{k^2 r^2}{A^2} + D,$$

where D is an arbitrary constant.

References

[1] M. H. Martin, *The flow of a viscous fluid,* Arch. Rat. Mech. Anal. **41** (1971) 266–286
[2] C. E. Weatherburn, *Differential geometry of three dimensions,* Cambridge (1939), 30–31