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ON PLANE VISCOUS MHD FLOWS: II*

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1. Introduction. A new form for the fundamental equations of non-aligned flows has been obtained [1]. In the present paper steady, plane, aligned *MHD* flows of a viscous, incompressible fluid of infinite electrical conductivity are considered. When the streamlines (η = constant) and their orthogonal trajectories (ξ = constant) are taken as the curvilinear coordinate system ξ, η in the physical plane the fundamental equations governing the flow are replaced by a new system of equations in which ξ and η are the independent variables.

The new form of the basic equations thus obtained is used to generalize some of the well-known results existing for the flows of non-conducting fluids [2].

Two familiar flow patterns of plane flows with straight streamlines are the source flows and the straight parallel flows. Are these the only plane flows with straight streamlines? With the help of the new equations, this question is answered in affirmative. In Sec. 3b, it is proven that if the streamlines are involutes of a curve, they are concentric circles.

2. Flow equations. The steady, plane, aligned *MHD* flow of an incompressible fluid of infinite electrical conductivity in the absence of heat conduction is governed by the system of six equations:

$$\text{div}\,\mathbf{V} = 0, \tag{2.1}$$

$$(\mathbf{V}\cdot\text{grad})\mathbf{V} + \rho^{-1}\,\text{grad}\,p = \nu\,\text{div grad}\,\mathbf{V} + \mu\rho^{-1}\,(\text{curl}\,\mathbf{H}) \times \mathbf{H}, \tag{2.2}$$

$$\mathbf{H} = \alpha\mathbf{V}, \tag{2.3}$$

$$\text{div}\,\mathbf{H} = 0, \tag{2.4}$$

where $\mathbf{V}$ is the velocity, $\mathbf{H}$ the magnetic field vector, p the pressure, ν the constant kinematic viscosity, μ the constant magnetic permeability and α an arbitrary scalar function.

Substituting (2.3) in (2.4) and using (2.1), we find that α is constant along each individual streamline, i.e.

$$\mathbf{V}\cdot\text{grad}\,\alpha = 0. \tag{2.5}$$

Elimination of $(\mathbf{V}\cdot\text{grad})\mathbf{V}$ and div grad $\mathbf{V}$ between (2.2) and the vector identities

$$(\text{curl}\,\mathbf{V}) \times \mathbf{V} = (\mathbf{V}\cdot\text{grad})\,\mathbf{V} - \text{grad}\,(V^2/2)$$

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and

$$\text{curl}\,(\text{curl}\,\mathbf{V}) = \text{grad}\,(\text{div}\,\mathbf{V}) - \text{div grad}\,\mathbf{V}$$

yields

$$(\text{curl}\,\mathbf{V}) \times \mathbf{V} + \text{grad}\,\{p\rho^{-1} + (V^2/2)\} + \nu\,\text{curl}\,(\text{curl}\,\mathbf{V}) = \mu\rho^{-1}\,(\text{curl}\,\mathbf{H}) \times \mathbf{H}, \tag{2.6}$$

where (2.1) has been used. With the introduction of the vorticity vector

$$\boldsymbol{\omega} = \text{curl}\,\mathbf{V} \tag{2.7}$$

the momentum equations (2.6) can be written as

$$(\text{curl}\,\mathbf{V}) \times \mathbf{V} + \text{grad}\,\{p\rho^{-1} + (V^2/2)\} + \nu\,\text{curl}\,\boldsymbol{\omega} = \mu\rho^{-1}\,(\text{curl}\,\mathbf{H}) \times \mathbf{H}. \tag{2.8}$$

In breaking Eqs. (2.6) into (2.7) and (2.8), we have reduced the order of equations from two to one.

In natural i.e. streamline coordinates with $g_1(\xi, \eta)\, d\xi$ and $g_2(\xi, \eta)\, d\eta$ as the components of a vector element of arc length, we have

$$\text{div}\,\mathbf{V} = \frac{1}{g_1 g_2}\frac{\partial}{\partial \xi}(g_2 V), \tag{2.9}$$

$$\text{curl}\,\mathbf{V} = -\frac{1}{g_1 g_2}\frac{\partial}{\partial \eta}(g_1 V)\mathbf{e}_3\,, \tag{2.10}$$

$$(\text{curl}\,\mathbf{V}) \times \mathbf{V} = -\frac{V}{g_1 g_2}\frac{\partial}{\partial \eta}(g_1 V)\mathbf{e}_2\,, \tag{2.11}$$

$$\text{grad}\left(\frac{p}{\rho} + \frac{V^2}{2}\right) = \frac{1}{g_1}\frac{\partial}{\partial \xi}\left(\frac{p}{\rho} + \frac{V^2}{2}\right)\mathbf{e}_1 + \frac{1}{g_2}\frac{\partial}{\partial \eta}\left(\frac{p}{\rho} + \frac{V^2}{2}\right)\mathbf{e}_2\,, \tag{2.12}$$

$$\text{curl}\,\omega = \frac{1}{g_2}\frac{\partial \omega}{\partial \eta}\,\mathbf{e}_1 - \frac{1}{g_1}\frac{\partial \omega}{\partial \xi}\,\mathbf{e}_2\,, \tag{2.13}$$

$$(\text{curl}\,\mathbf{H}) \times \mathbf{H} = \alpha(\text{curl}\,\alpha\mathbf{V}) \times \mathbf{V} = -\frac{\alpha V}{g_1 g_2}\frac{\partial}{\partial \eta}(g_1 \alpha V)\mathbf{e}_2\,, \tag{2.14}$$

$$\mathbf{V}\cdot\text{grad}\,\alpha = \frac{V}{g_1}\frac{\partial \alpha}{\partial \xi}\,, \tag{2.15}$$

where V is the magnitude of the velocity, $\mathbf{e}_1$ a unit vector in the direction of the velocity, $\mathbf{e}_2$ a unit vector in the direction perpendicular to the velocity but in the plane of flow, $\mathbf{e}_3$ a unit vector in the direction of normal to the plane of flow such that $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ form a right-handed coordinate system. The metric of this (ξ, η) net is of the form

$$ds^2 = g_1^2(\xi, \eta)\, d\xi^2 + g_2^2(\xi, \eta)\, d\eta^2$$

where g_1 and g_2 satisfy the Gauss equation

$$\frac{\partial}{\partial \xi}\left[\frac{1}{g_1}\frac{\partial g_2}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[\frac{1}{g_2}\frac{\partial g_1}{\partial \eta}\right] = 0. \tag{2.16}$$

On using Eqs. (2.9) to (2.15) in (2.1, 5, 7, 8), we find that the flow equations in natural coordinates are:

$$\frac{\partial}{\partial \xi}(g_2 V) = 0, \tag{2.17}$$

$$\partial \alpha / \partial \xi = 0, \tag{2.18}$$

$$\omega = -\frac{1}{g_1 g_2} \frac{\partial}{\partial \eta}(g_1 V), \tag{2.19}$$

$$V \frac{\partial V}{\partial \xi} + \frac{1}{\rho} \frac{\partial p}{\partial \xi} + \nu \frac{g_1}{g_2} \frac{\partial \omega}{\partial \eta} = 0, \tag{2.20}$$

$$\frac{V^2}{g_1} \frac{\partial g_1}{\partial \eta} - \frac{1}{\rho} \frac{\partial p}{\partial \eta} + \nu \frac{g_2}{g_1} \frac{\partial \omega}{\partial \xi} = \mu \frac{\alpha V}{\rho g_1} \frac{\partial}{\partial \eta}(g_1 \alpha V), \tag{2.21}$$

where $\omega = |\boldsymbol{\omega}|$.

Eqs. (2.16) to (2.21) constitute a set of six equations for six dependent variables V, g_1, g_2, p, α and ω. Eqs. (2.17), (2.18), (2.19) can be used to eliminate V, α, ω and we are left with three equations in three unknowns p, g_1, g_2. Upon eliminating p from (2.20), (2.21) a system of two equations results for g_1, g_2.

3a. Straight streamlines. We prescribe the streamlines to be straight lines. We assume that they are not parallel but envelop a curve Γ. We now take the tangent lines to the curve Γ, and their orthogonal trajectories, the involutes of Γ as the system of orthogonal curvilinear coordinates. The square of the element of arc length ds in this orthogonal curvilinear coordinate system is given by [1]:

$$ds^2 = d\xi^2 + (\xi - \sigma)^2 \, d\eta^2 \tag{3.1}$$

where $\sigma = \sigma(\eta)$ denotes the arc length of the curve Γ, η is the angle subtended by the tangent line with x-axis and ξ is a parameter constant along each involute. In this coordinate system, the coordinate curves ξ = constant (the orthogonal trajectories to streamlines) are the involutes of the curve Γ and η = constant (the streamlines) are its tangent lines.

From (3.1), we get

$$\begin{aligned} g_1 &= 1, \\ g_2 &= \xi - \sigma(\eta). \end{aligned} \tag{3.2}$$

For the flows considered in this paper, we shall now show that these forms of g_1 and g_2 imply that the streamlines are radial and hence the orthogonal trajectories are concentric circles.

Substituting for g_1 and g_2 from (3.2) in (2.16), we find that it is automatically satisfied. From Eq. (2.17), we have

$$g_2 V = A(\eta) \tag{3.3}$$

where $A(\eta)$ is an arbitrary function of η. Using (3.2) and (3.3) in (2.19), we obtain

$$\omega = -\frac{1}{[(\xi - \sigma(\eta)]^2} \left[A'(\eta) + \frac{A(\eta)\sigma'(\eta)}{\xi - \sigma(\eta)} \right] \tag{3.4}$$

Differentiating (2.20) with respect to η, (2.21) with respect to ξ and adding, we find

$$\frac{\partial}{\partial \eta}\left(V \frac{\partial V}{\partial \xi}\right) + \nu \frac{\partial}{\partial \eta}\left[\frac{1}{\xi - \sigma(\eta)} \frac{\partial \omega}{\partial \eta}\right] + \nu \frac{\partial}{\partial \xi}\left[(\xi - \sigma(\eta)) \frac{\partial \omega}{\partial \xi}\right] = \frac{\mu}{\rho} \frac{\partial}{\partial \xi}\left[\alpha(\eta) V \frac{\partial}{\partial \eta}\{\alpha(\eta) V\}\right] \tag{3.5}$$

Elimination of V and ω between (3.3), (3.4) and (3.5) yields

$$Y_0 + Y_1\{\xi - \sigma(\eta)\} + Y_2\{\xi - \sigma(\eta)\}^2 + Y_3\{\xi - \sigma(\eta)\}^3 \equiv 0 \tag{3.6}$$

where $Y_i = Y_i(\eta)$ and

$$Y_0 = 15\nu A(\eta)[\sigma'(\eta)]^3,$$

$$Y_1 = 15\nu A'(\eta)[\sigma'(\eta)]^2 + 10\nu A(\eta)\sigma'(\eta)\sigma''(\eta),$$

$$Y_2 = 3A^2(\eta)\sigma'(\eta) + 6\nu A''(\eta)\sigma'(\eta) + 4\nu A'(\eta)\sigma''(\eta) + \nu A(\eta)\sigma'''(\eta)$$
$$+ 9\nu A(\eta)\sigma'(\eta) + 3\mu\rho^{-1}\alpha^2(\eta)A^2(\eta)\sigma'(\eta),$$

$$Y_3 = 2A(\eta)A'(\eta) + \nu A'''(\eta) + 4\nu A'(\eta) + \mu\rho^{-1}[2\alpha^2(\eta)A(\eta)A'(\eta) + 2\alpha(\eta)\alpha'(\eta)A^2(\eta)].$$

Since ξ, η are independent variables, the identity (3.6) can hold only if all the Y_i vanish identically. In particular, this requires that

$$Y_0 = 15\nu A(\eta)[\sigma'(\eta)]^3 \equiv 0$$

i.e.

$$\text{either} \quad A(\eta) = 0 \quad \text{or} \quad \sigma'(\eta) = 0.$$

The first possibility together with (3.3) implies that either $V = 0$ or $g_2 = 0$, which is not true. Therefore, the radius of curvature R ($\equiv \sigma'(\eta)$) of Γ vanishes identically and hence we have the following theorem.

THEOREM. If the streamlines in steady, plane aligned *MHD* flow of a viscous fluid are straight lines, then they must be concurrent or parallel.

3b. Vortex flows. In this example, we consider the involutes of the curve Γ as the streamlines and the tangents to the curve Γ as the otrhogonal trajectories. The square of the element of arc length ds in this orthogonal curvilinear coordinate system is given by

$$ds^2 = [(\eta - \sigma(\xi)]^2 \, d\xi^2 + d\eta^2 \tag{3.7}$$

where the coordinate curves ξ = constant (the orthogonal trajectories of the streamlines) are the tangent lines to the curve Γ and η = constant (the streamlines) are the involutes of Γ. From (3.7), we obtain

$$g_1 = \eta - \sigma(\xi), \qquad g_2 = 1. \tag{3.8}$$

These metric coefficients imply that the flow field is the general vortex flow.

Eq. (2.16) is again automatically satisfied. The second equation of (3.8) together with (3.3) requires that the velocity magnitude is constant along each individual streamline. Using (3.3) and (3.8) in (2.19), we get

$$\omega = -\left[A'(\eta) + \frac{A(\eta)}{\eta - \sigma(\xi)}\right]. \tag{3.9}$$

Eliminating p between (2.20) and (2.21), we get

$$\begin{aligned} &3\nu A(\eta)[\sigma'(\xi)]^2 + [\nu A(\eta)\sigma''(\xi)]\{\eta - \sigma(\xi)\} \\ &\quad + [\nu A(\eta) - A^2(\eta)\sigma'(\xi) + \mu\rho^{-1}\alpha^2(\eta)A^2(\eta)\sigma'(\xi)]\{\eta - \sigma(\xi)\}^2 \\ &\quad - \nu A'(\eta)\{\eta - \sigma(\xi)\}^3 + 2\nu A''(\eta)\{\eta - \sigma(\xi)\}^4 + \nu A'''(\eta)\{\eta - \sigma(\xi)\}^5 \equiv 0 \end{aligned} \tag{3.10}$$

when eqs. (3.3), (3.8) and (3.9) are used. For the relation (3.10) to hold identically, it must hold on the curve $\eta = \sigma(\xi)$ and therefore, we have

$$R \equiv \sigma' = 0.$$

THEOREM. If the streamlines in steady, plane, aligned *MHD* flow of a viscous fluid are involutes of a curve Γ, then the streamlines are concentric circles.

REFERENCES

[1] V. I. Nath and O. P. Chandna, *On plane viscous MHD flows,* Quart. Appl. Math. **31**, 351–362 (1973)
[2] M. H. Martin, *The flow of a viscous fluid,* Arch, Rat. Mech. Anal. **41**, 266–286 (1971)