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On pluricanonical maps for threefolds of general type

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§ 1. Introduction

Let X be a nonsingular projective threefold of general type over the complex number field $\boldsymbol{C}$. It remains open whether there exists an absolute number $m(3)$ such that $\Phi_{|mK_X|}$ is a birational map onto its image when $m \geq m(3)$ for any X. Restricting interest to objects of nonsingular minimal threefolds of general type, Benveniste [**1**] got $m(3) = 9$ and then Matsuki [**9**] obtained $m(3) = 7$. In this paper, we want to show $m(3) = 6$.

MAIN THEOREM. *Let X be a nonsingular projective threefold with nef and big canonical divisor K_X, then the 6-canonical map $\Phi_{|6K_X|}$ is a birational map onto its image.*

Throughout this paper, most our notations and terminologies are standard except the following which we are in favour of:
$:=$ —definition;
$\sim_{lin}$—linear equivalence;
$\sim_{num}$—numerical equivalence.

§ 2. Proof of the Main Theorem

2.1 Kawamata-Viehweg's vanishing theorem. We will use the vanishing theorem in the following form.

PROPOSITION 2.1 (Theorem 1.2 of [**5**]). *Let X be a nonsingular complete variety, $D \in \mathrm{Div}(X) \otimes \boldsymbol{Q}$. Assume the following two conditions:*
(1) *D is nef and big;*
(2) *the fractional part of D has the support with only normal crossings.*
Then $H^i(X, \mathcal{O}_X(\lceil D \rceil + K_X)) = 0$ for $i > 0$, where $\lceil D \rceil$ is the minimum integral divisor with $\lceil D \rceil - D \geq 0$.

2.2 Basic formula. Let X be a nonsingular projective threefold. For a divisor $D \in \mathrm{Div}(X)$, we have

$$\chi(\mathcal{O}_X(D)) = D^3/6 - K_X \cdot D^2/4 + D \cdot (K_X^2 + c_2)/12 + \chi(\mathcal{O}_X)$$

by Riemann-Roch theorem. The calculation shows that

$$\chi(\mathcal{O}_X(D)) + \chi(\mathcal{O}_X(-D)) = -K_X \cdot D^2/2 + 2\chi(\mathcal{O}_X) \in \boldsymbol{Z},$$

therefore $K_X \cdot D^2$ is an even integer, especially K_X^3 is even.

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If K_X is nef and big, then we obtain by Kawamata-Viehweg's vanishing theorem that

$$p(n) := h^0(X, \mathcal{O}_X(nK_X)) = (2n-1)[n(n-1)K_X^3/12 - \chi(\mathcal{O}_X)],$$

for $n \geq 2$.

Let X be a nonsingular projective threefold, $f : X \to C$ be a fibration onto a nonsingular curve C. From the spectral sequence:

$$E_2^{p,q} := H^p(C, R^q f_*\omega_X) \Rightarrow E^n := H^n(X, \omega_X),$$

we get by direct calculation that

$$h^2(\mathcal{O}_X) = h^1(C, f_*\omega_X) + h^0(C, R^1 f_*\omega_X),$$

$$q(X) := h^1(\mathcal{O}_X) = b + h^1(C, R^1 f_*\omega_X).$$

Therefore we obtain

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_F)\chi(\mathcal{O}_C) + \Delta_2 - \Delta_1,$$

where we set $\Delta_1 := \deg f_*\omega_{X/C}$ and $\Delta_2 := \deg R^1 f_*\omega_{X/C}$. We can also refer to corollary 3.2 of [**8**] for the above formula.

For a nonsingular threefold X with nef and big canonical divisor K_X, Miyaoka showed that $3c_2 - c_1^2$ is pseudo-effective, therefore we get $K_X^3 \leq -72\chi(\mathcal{O}_X)$ by the Riemann-Roch equality

$$\chi(\mathcal{O}_X) = -c_2 \cdot K_X/24.$$

In particular, $\chi(\mathcal{O}_X) < 0$.

2.3 A lemma.

LEMMA 2.1 (Theorem 1 of [**6**]). *Let X, C be nonsingular projective varieties and C is a curve, $f : X \to C$ be an algebraic fiber space, then $f_*[\omega_{X/C}^{\otimes m}]$ is semi-positive for $m \geq 1$.*

2.4 Proof of the first part. From 2.2, we have $p(2) = 3[K_X^3/6 - \chi(\mathcal{O}_X)] \geq 4$, therefore $\dim \Phi_{|2K_X|}(X) \geq 1$, i.e., the bicanonical map is well-defined. We would like to formulate a proof through two steps: (1) $\dim \Phi_{|2K_X|}(X) \geq 2$ and (2) $\dim \Phi_{|2K_X|}(X) = 1$.

DEFINITION 2.1. Let X be a nonsingular projective threefold. Suppose that $|2K_X|$ is not composed of pencils, i.e., $\dim \Phi_{|2K_X|}(X) \geq 2$. Set $2K_X \sim_{lin} M_2 + Z_2$, where M_2 is the moving part of $|2K_X|$ and Z_2 is the fixed part. We define $\delta_2(X) := K_X^2 \cdot M_2$, $\delta_2(X)$ is intrinsic relating to X.

THEOREM 2.1 (Theorem 6 of [**3**]). *Let X be a nonsingular projective threefold with nef and big canonical divisor K_X, suppose $|2K_X|$ be not composed of pencils, i.e., $\dim \Phi_{|2K_X|}(X) \geq 2$, and suppose $\delta_2(X) \geq 2$, then $\Phi_{|6K_X|}$ is a birational map onto its image.*

PROPOSITION 2.2. *Let X be a nonsingular projective threefold whose canonical divisor K_X is nef and big. Suppose that $|2K_X|$ is not composed of pencils, then $\delta_2(X) \geq 2$.*

PROOF. Obviously, we have $\delta_2(X) \geq 1$ under the assumption of the theorem. Suppose $\delta_2(X) = 1$, we shall derive a contradiction.

Let $f_2 : X' \to X$ be a succession of blowing-ups with nonsingular centers such that $g_2 = \Phi_{|2K_X|} \circ f_2$ is a morphism. Let $g_2 : X' \xrightarrow{h_2} W_2' \xrightarrow{s_2} W_2 \subset \boldsymbol{P}^{p(2)-1}$ be the Stein factorization of g_2. Let H_2 be a hyperplane section of $W_2 = \overline{\Phi_{|2K_X|}(X)}$ in $\boldsymbol{P}^{p(2)-1}$ and S_2 be a general member of $|g_2^*(H_2)|$. Since $\dim W_2 \geq 2$, S_2 is a nonsingular irreducible projective surface. We set $2K_X \sim_{lin} M_2 + Z_2$, where Z_2 is the fixed part of $|2K_X|$, and M_2 the moving part. Set $f_2^*(M_2) \sim_{lin} S_2 + E_2'$, $K_{X'} \sim_{lin} f_2^*(K_X) + E_2$, where E_2 is the ramification divisor for f_2, E_2' is the exceptional divisor for f_2.

We have $\delta_2(X) = K_X^2 \cdot M_2 = f_2^*(K_X)^2 \cdot S_2 = 1$. Multiplying $2K_X \sim_{lin} M_2 + Z_2$ by $K_X \cdot M_2$, we have

$$2 = 2K_X^2 \cdot M_2 = K_X \cdot M_2^2 + K_X \cdot M_2 \cdot Z_2.$$

Since $|S_2|$ is not composed of pencils, $f_2^*(K_X)$ is nef and big and since S_2 is nef, we have

$$\begin{aligned} K_X \cdot M_2^2 &= f_2^*(K_X) \cdot f_2^*(M_2)^2 = f_2^*(K_X) \cdot f_2^*(M_2) \cdot S_2 \\ &= f_2^*(K_X) \cdot S_2^2 + f_2^*(K_X) \cdot S_2 \cdot E_2' \geq 1. \end{aligned}$$

Whereas, $K_X \cdot M_2^2$ is even by 2.2 and $K_X \cdot M_2 \cdot Z_2 \geq 0$ because $M_2 \cdot Z_2 \geq 0$ as a 1-cycle. Thus we have $K_X \cdot M_2^2 = 2$ and $K_X \cdot M_2 \cdot Z_2 = 0$.

Since $f_2^*(K_X)$ is nef and big, there exists a positive integer m such that

$$Bs|mf_2^*(K_X)| = \emptyset$$

and a general member $T \in |mf_2^*(K_X)|$ is a nonsingular projective surface of general type. $S_2|_T$ is a nef divisor on the surface T, because S_2 is nef on X'. $(S_2|_T)_T^2 = mf_2^*(K_X) \cdot S_2^2 > 0$, i.e., $S_2|_T$ is big. We have

$$(S_2|_T \cdot f_2^*(Z_2)|_T)_T = mf_2^*(K_X) \cdot S_2 \cdot f_2^*(Z_2) = mK_X \cdot M_2 \cdot Z_2 = 0,$$

therefore we should have $mK_X \cdot Z_2^2 = (f_2^*(Z_2)|_T)_T^2 \leq 0$ by Hodge's index theorem on T.

On the other hand, $4K_X^3 = K_X \cdot (M_2 + Z_2)^2 = K_X \cdot M_2^2 + K_X \cdot Z_2^2$, therefore $K_X \cdot Z_2^2 = 4K_X^3 - 2 > 0$. We obtain a contradiction. □

THEOREM 2.2. *Let X be a nonsingular projetive threefold with nef and big canonical divisor K_X, suppose $|2K_X|$ be not composed of pencils, then $\Phi_{|6K_X|}$ is a birational map onto its image.*

PROOF. This is a direct result of theorem 2.1 and proposition 2.2. □

2.5 Proof of the second part. Suppose $|2K_X|$ be composed of pencils, again take $f_2 : X' \to X$ be a succession of blowing-ups with nonsingular centers such that $g_2 := \Phi_{|2K_X|} \circ f_2$ is a morphism. Let $g_2 : X' \xrightarrow{h_2} W_2' \xrightarrow{s_2} W_2$ be the stein factorization of g_2. Because $\dim W_2 = 1$, we know that a general fiber F of the fibration h_2 is a nonsingular projective surface of general type. We denote $b := g(W_2')$.

PROPOSITION 2.3 (Claim 9.1 of [9]). *Let X be a nonsingular projective threefold with nef and big canoical divisor K_X. Suppose $|2K_X|$ be composed of pencils, then we have*

$$\mathcal{O}_F(f_2^*(K_X)|_F) \cong \mathcal{O}_F(\pi^*(K_{F_0})),$$

where $\pi : F \to F_0$ is the contraction to minimal model.

THEOREM 2.3 (Theorem 10 of [3]). *Let X be a nonsingular projective threefold with nef and big canoical divisor K_X. Suppose $|2K_X|$ be composed of pencils and $p_g(X) \geq 2$, then $\Phi_{|6K_X|}$ is a birational map onto its image.*

THEOREM 2.4. *Let X be nonsingular projective threefold with nef and big canonical divisor K_X. Suppose $|2K_X|$ be composed of pencils, $p_g(X) \leq 1$ and a general fiber F of h_2 is not a surface with $K_{F_0}^2 = 1$ and $p_g(F) = 2$, then $\Phi_{|6K_X|}$ is a birational map onto its image.*

PROOF. Let $b_2 := \deg(s_2)$ and H_2 be a hyperplane section of W_2 in $\boldsymbol{P}^{p(2)-1}$, and let a_2 be the degree of W_2 in $\boldsymbol{P}^{p(2)-1}$. Then

$$f_2^*(2K_X) \sim_{lin} g_2^*(H_2) + Z_2,$$
$$f_2^*(2K_X) \sim_{num} a_2 b_2 F + Z_2,$$

where Z_2 is the fixed part of $|f_2^*(2K_X)|$.

Let $\pi : F \to F_0$ be the contraction onto the minimal model F_0 of F. From proposition 2.3, we have

$$\mathcal{O}_F(\pi^*(K_{F_0})) = \mathcal{O}_F(f_2^*(K_X)|_F).$$

Noting that $g_2^*(H_2)$ can be a disjoint union of F_i's $(1 \leq i \leq a_2 b_2)$ at least over a Zariski open subset of W_2', each F_i is of the same kind as F mentioned in proposition 2.3. We have

$$K_{X'} + 3f_2^*(K_X) + g_2^*(H_2) \leq 6K_{X'}.$$

From the exact sequence

$$\begin{aligned} 0 &\to \mathcal{O}_{X'}(K_{X'} + 3f_2^*(K_X)) \\ &\to \mathcal{O}_{X'}(K_{X'} + 3f_2^*(K_X) + g_2^*(H_2)) \\ &\to \bigoplus_{i=1}^{a_2 b_2} \mathcal{O}_{F_i}(K_{F_i} + 3f_2^*(K_X)|_{F_i}) \to 0 \end{aligned}$$

and because $H^1(X', K_{X'} + 3f_2^*(K_X)) = 0$ by proposition 2.1, we get the following surjective map

$$H^0(X', \mathcal{O}_{X'}(K_{X'} + 3f_2^*(K_X) + g_2^*(H_2))) \to \bigoplus_{i=1}^{a_2 b_2} H^0(F_i, \mathcal{O}_{F_i}(K_{F_i} + 3f_2^*(K_X)|_{F_i})).$$

This means that $\Phi_{|K_{X'}+3f_2^*(K_X)+g_2^*(H_2)|}$ separates the fibers of g_2 and the components on a general fiber at least on some nonempty Zariski open subset of X'. On the other hand,

$$\Phi_{|K_{X'}+3f_2^*(K_X)+g_2^*(H_2)|}|_{F_i} = \Phi_{|K_{F_i}+3f_2^*(K_X)|_{F_i}|} = \Phi_{|4K_{F_i}|}$$

by Proposition 2.3. If F is not a surface with $K_{F_0}^2 = 1$ and $p_g(F_0) = 2$, then $\Phi_{|4K_{F_i}|}$ is birational. Therefore we see that

$$\Phi_{|K_{X'}+3f_2^*(K_X)+g_2^*(H_2)|}$$

is birational. Thus $\Phi_{|6K_{X'}|}$ is a birational map onto its image. So is $\Phi_{|6K_X|}$. □

PROPOSITION 2.4. *Let X be a nonsingular projective threefold whose canonical divisor is nef and big. Suppose $p_g(X) \leq 1$ and $|2K_X|$ be composed of pencils, if F is a surface with $K_{F_0}^2 = 1$ and $p_g(F) = 2$, then we have $b = p_g(X) = q(X) = 1$ and $h^2(\mathcal{O}_X) = 0$.*

PROOF. We have

$$\chi(\mathcal{O}_X) = 1 - q(X) + h^2(\mathcal{O}_X) - p_g(X) < 0.$$

Since $p_g(X) \leq 1$, then $q(X) > 1 + h^2(\mathcal{O}_X) - p_g(X)$, i.e. $q(X) > 0$. Now we have a fibration $h_2 : X' \to W_2'$, where W_2' is a nonsingular curve. Denote by b the genus of W_2' and F a general fiber of h_2. If F_0 is a surface with $K_{F_0}^2 = 1$ and $p_g(F_0) = 2$, then we have $q(F_0) = 0$ by E. Bombieri's theorem in [**2**] and then $R^1 h_{2*}\omega_{X'} = 0$. Therefore we have $0 < q(X) = q(X') = b + h^1(R^1 h_{2*}\omega_{X'}) = b$, which says that $\Phi_{|2K_X|}$ is actually a morphism. We have $X = X'$.

For the fibration $h_2 : X \to W_2'$, we have $\deg h_{2*}\omega_X \geq 4(b-1)$ by Lemma 2.1. From Riemann-Roch theorem, we have

$$\begin{aligned} 1 \geq p_g(X) = h^0(h_{2*}\omega_X) &= h^1(h_{2*}\omega_X) + \deg(h_{2*}\omega_X) + 2(1-b) \\ &\geq 2(b-1). \end{aligned}$$

Therefore $b = 1$ and then $q(X) = 1$. From $\chi(\mathcal{O}_X) = h^2(\mathcal{O}_X) - p_g(X) < 0$, we get $p_g(X) = 1$ and $h^2(\mathcal{O}_X) = 0$. □

THEOREM 2.5. *Let X be a nonsingular projective threefold with nef and big canonical divisor. Suppose $p_g(X) \leq 1$ and $|2K_X|$ be composed of pencils, if F is a surface with $K_{F_0}^2 = 1$ and $p_g(F) = 2$, then $\Phi_{|6K_X|}$ is a birational map onto its image.*

PROOF. Under the assumption of this theorem, we know from proposition 2.4 that $\Phi_{|2K_X|}$ is a morphism because $b = 1 > 0$. We actually have

$$X \xrightarrow{h_2} W_2' \xrightarrow{s_2} W_2.$$

We can take a modification $f : X' \to X$ according to Hironaka such that all the singular fibers of the fibration $h_2' = h_2 \circ f : X' \to W_2'$ have the support with only normal crossings. Let $g_2' := \Phi_{|2K_X|} \circ f = s_2 \circ h_2'$. From proposition 2.4, we have $p_g(X') = p_g(X) = 1$. Let $D \in |K_{X'}|$ be the unique effective divisor. Set $D = V_0 + H_0$, where V_0 is the vertical part and H_0 the horizontal one. Because $2D \sim_{lin} 2K_{X'}$, there is a hyperplane section H_2^0 of W_2 in $\boldsymbol{P}^{P(2)-1}$ such that

$$2D = g_2'^*(H_2^0) + E,$$

where E is the fixed part. Note that each component of $g_2'^*(H_2^0)$ is vertical with respect to h_2', we have $g_2'^*(H_2^0) \leq 2V_0$ as divisors. Therefore $(1/2)g_2'^*(H_2^0) \leq V_0$ as $\boldsymbol{Q}$-divisors and then $\lceil (1/2)g_2'^*(H_2^0) \rceil \leq V_0$ as divisors. Denote $D_0 := \lceil (1/2)g_2'^*(H_2^0) \rceil$.

Now we consider the system $|K_{X'} + 4f^*(K_X) + D_0|$. Obviously, we have

$$|K_{X'} + 2f^*(K_X) + g_2'^*(H_2)| \subset |K_{X'} + 4f^*(K_X) + D_0| \subset |6K_{X'}|.$$

At least over a nonempty Zariski open subset of W_2', $g_2'^*(H_2)$ can split into disjoint union of fibers of h'_2. We have the following exact sequence:

$$\begin{aligned}0 &\to \mathcal{O}_{X'}(K_{X'} + 2f^*(K_X)) \\ &\to \mathcal{O}_{X'}(K_{X'} + 2f^*(K_X) + g_2'^*(H_2)) \\ &\to \bigoplus_{i=1}^{a_2b_2} \mathcal{O}_{F_i}(K_{F_i} + 2f^*(K_X)|_{F_i}) \to 0.\end{aligned}$$

From Kawamata-Viehweg's vanishing theorem, we have $H^1(X', K_{X'} + 2f^*(K_X)) = 0$. Therefore we get the surjective map

$$H^0(X', K_{X'} + 2f^*(K_X) + g_2'^*(H_2)) \to \bigoplus_{i=1}^{a_2b_2} H^0(F_i, K_{F_i} + 2f^*(K_X)|_{F_i}).$$

Which means that $\Phi_{|K_{X'}+2f^*(K_X)+g_2'^*(H_2)|}$ can separate fibers of g'_2 and disjoint components of a general fiber of g'_2 at least over a nonempty Zariski open subset of W_2, so can $\Phi_{|K_{X'}+4f^*(K_X)+D_0|}$. In order to prove the birationality of $\Phi_{|K_{X'}+4f^*(K_X)+D_0|}$ we have to show that $\Phi_{|K_{X'}+4f^*(K_X)+D_0|}|_F$ is birational for a general fiber F of h'_2. Now let F be a general fiber of h'_2, denote

$$G := 4f^*(K_X) + \frac{1}{2}g_2'^*(H_2^0) - F.$$

Because $b = 1$, $p(2) = h^0(g_2'^*(H_2^0)) = h^0(a_2b_2F) = a_2b_2$. Noting that $p(2) \geq 4$ and $a_2 \geq p(2) - 1$, we actually have $b_2 = 1$ and $p(2) = a_2 \geq 4$. Therefore $(1/2)g_2'^*(H_2^0) - F$ is nef and then G is nef. It is easy to see that G is big. G is also an effective $\boldsymbol{Q}$-divisor because $4f^*(K_X) - F \geq 0$. Note that the fractional part $\{G\}$ of G is composed of components from singular fibers of h'_2 and at most one smooth fiber of h'_2 (one only has to consider the components of V_0), therefore $\{G\}$ has support with only normal crossings. Thus by Kawamata-Viehweg's vanishing theorem, we have

$$H^1(X', K_{X'} + 4f^*(K_X) + D_0 - F) = H^1(X', K_{X'} + \lceil G \rceil) = 0.$$

Noting that D_0 is vertical, we have $D_0|_F = 0$. By the definition of f, we see that the ramification divisor of f is contained in singular fibers of h'_2, therefore $f^*(K_X)|_F = K_{X'}|_F = K_F$. From the exact sequence

$$\begin{aligned}0 &\to \mathcal{O}_{X'}(K_{X'} + 4f^*(K_X) + D_0 - F) \\ &\to \mathcal{O}_{X'}(K_{X'} + 4f^*(K_X) + D_0) \\ &\to \mathcal{O}_F(5K_F) \to 0,\end{aligned}$$

we get the surjective map

$$H^0(X', K_{X'} + 4f^*(K_X) + D_0) \to H^0(F, 5K_F).$$

Which means $\Phi_{|K_{X'}+4f^*(K_X)+D_0|}|_F = \Phi_{|5K_F|}$ is a birational map, therefore

$$\Phi_{|K_{X'}+4f^*(K_X)+D_0|}$$

is a birational map, so is $\Phi_{|6K_{X'}|}$. □

Theorem 2.2, theorem 2.3, theorem 2.4 and theorem 2.5 imply main theorem.

References

[1] Benveniste, X.: Sur les applications pluricanoniques des variétés de type très général en dimension 3, Amer. J. Math. **108** (1986), 433–449.
[2] Bombieri, E.: Canonical models of surfaces of general type, Publ. Math. Inst. Hautes Etude. Sci. **42** (1973), 171–219.
[3] Chen, M.: An extension of Benveniste-Matsuki's method on 6-canonical maps for threefolds, Comm. in Algebra **22** (1994), 5759–5767.
[4] Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. of Math. **79** (1964), 109–326.
[5] Kawamata, Y.: Cone of curves of algebraic varieties, Ann. of Math. **119** (1984), 603–633.
[6] Kawamata, Y.: Kodaira dimension of algebraic fiber spaces over curves, Invent. Math. **66** (1982), 57–71.
[7] Kollár, J.: Higher direct images of dualizing sheaves I, Ann. of Math. **123** (1986), 11–42.
[8] Kollár, J.: Higher direct images of dualizing sheaves II, Ann. of Math. **124** (1986), 171–202.
[9] Matsuki, K.: On pluricanonical maps for 3-folds of general type, J. Math. Soc. Japan **38** (1986), 339–359.
[10] Miyaoka, Y.: The Chern classes and Kodaira dimension of a minimal variety, In: Algebraic Geometry, Sendai, 1985 (Adv. Stud, in Pure Math. 10, 1987, pp. 449–476).

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