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ON POLYNOMIAL APPROXIMATION IN $A_q(D)$

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ABSTRACT. Let D be a bounded Jordan domain with rectifiable boundary and define $A_q(D)$, the Bers space, as the space of holomorphic functions f, such that

$$\iint_D |f| \lambda_D^{2-q} \, dx \, dy$$

is finite, where λ_D is the Poincaré metric for D. It is shown that the polynomials are dense in $A_q(D)$ for $q>3/2$.

1. **Introduction.** Let D be a bounded Jordan domain. Define $A_q(D)$ $(q>1)$, the Bers space, as the Banach space of holomorphic functions $f(z)$, such that

$$\|f\|_q = \iint_D |f(z)| \, \lambda_D^{2-q}(z) \, dx \, dy < \infty, \tag{1.1}$$

when $\lambda_D(z)$ is the Poincaré metric for D. If ϕ is a Riemann mapping function from D onto U, the unit disk, and $\psi=\phi^{-1}$, then

$$\iint_D \lambda_D^{2-q}(z) \, dx \, dy = \iint_U |\psi'(\zeta)|^q \, (1 - |\zeta|^2)^{q-2} \, d\zeta \, dn. \tag{1.2}$$

Since for Jordan domains the rectifiability of the boundary is equivalent to $\psi' \in H^1(U)$, the Hardy class (cf. [3, p. 44]), it follows immediately by a theorem of Carleson [3, p. 157] that if the boundary of D is rectifiable then (1.2) is finite for all $q>1$. Hence D bounded implies that the polynomials belong to $A_q(D)$, for all $q>1$.

The question of polynomial density in $A_q(D)$ has been considered by various authors. In the case $q\geqq 2$, Bers [2] and Knopp [4] proved that the polynomials were dense in $A_q(D)$ without any assumptions on the mapping functions ψ or ϕ. Later Sheingorn [6] proved that the polynomials are dense in $A_q(U^*)$, for all $q>1$, where U^* is a particular Jordan region which appears in the proof of the "Main Lemma" of Knopp [4]. Finally Metzger

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and Sheingorn [5] proved that if either $\psi' \in H^p$, for some $p>1$, or D is a Smirnov domain, then the polynomials are dense in $A_q(D)$, for all $q>1$.

In this paper we show

THEOREM 1. *If D is a bounded Jordan domain with a rectifiable boundary, then the polynomials are dense in $A_q(D)$, for $q>3/2$.*

2. **Proof of Theorem 1.** Clearly we can assume that the origin lies in D and we choose $\psi(z)$, such that $\psi(0)=0$, $\psi'(0)>0$. We now note

LEMMA 1. *Suppose D is a bounded Jordan domain with rectifiable boundary. The polynomials are dense in $A_q(D)$ if and only if ϕ'^q can be approximated by polynomials in $A_q(D)$.*

The necessity is clear since $\phi'^q \in A_q(D)$ for all $q>1$ and the sufficiency follows easily from Lemma 1 of [5].

We define $\mathscr{H}_p(D)=\{f: f \text{ is holomorphic on } D \text{ and } \|f\|_p^*<\infty\}$, where

$$\|f\|_p^* = \left(\iint_D |f|^p \, dx\, dy\right)^{1/p}. \tag{2.1}$$

An application of Hölder's inequality yields

$$\begin{aligned} \|F\|_q &= \iint_U |F \circ \psi| \, |\psi'^q| \, (1-|z|^2)^{q-2} \, dx\, dy \\ &\leqq \left(\iint_U |F\circ\psi|^p \, |\psi'|^2 \, dx\, dy\right)^{1/p} \\ &\quad \cdot \left(\iint_U |\psi'|^{(qp-2)/(p-1)} (1-|z|^2)^{(qp-2p)/(p-1)} \, dx\, dy\right)^{(p-1)/p} \\ &= \|F\|_p^* \left(\iint_D \lambda_D^{(qp-2)/(p-1)} \, dx\, dy\right)^{(p-1)/p} < \infty, \end{aligned}$$

if $(qp-2)/(p-1)>1$, i.e. if $p>1/(q-1)$, by (1.2) and the fact that $(qp-2)/(p-1)-2=(qp-2p)/(p-1)$. Thus polynomial approximation in $\mathscr{H}_p(D)$ implies polynomial approximation in $A_q(D)$ and, by Lemma 1, it follows that we need only show that $\phi' \in \mathscr{H}_p(D)$ for $p>q/(q-1)$.

To see that this holds, we first note that $q<2$ implies $p>2$ and by changing variables we get

$$\iint_D |\phi'|^p \, dx\, dy = \iint_U |\psi'|^{2-p} \, dx\, dy = I.$$

Since ψ is a bounded schlicht function with $\psi(0)=0$, it follows that there exists an M (>0) such that $|\psi'(z)| \geqq M(1-|z|^2)$ for all z in U. Hence I is finite if $p<3$, i.e., if $q>3/2$ and this completes the proof.

REMARK. If one lets G be a discrete group of conformal transformations on D and defines $A_q(D, G)$ as in Bers [1] then Theorem 2 of that paper yields the density of the Poincaré theta series of the polynomials in $A_q(D, G)$. (Cf. [1] and [5] for a precise formulation of the result plus all definitions and notations.)

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