On polynomial differential equations of Duffing type

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Abstract

The exact and explicit general periodic solution of polynomial differential equations of Duffing type is calculated as a power law of the cosine function. In doing so the solution of all Duffing equations of three terms like the cubic, quintic and heptic equations may be easily expressed in a straightforward fashion.

Keywords: Polynomial differential equations, Duffing type equation, exact periodic solution, trigonometric functions.

Theory

Let us consider the second order nonlinear differential equation [1]

$$\ddot{x} + \frac{1}{2}(\alpha - q)a\,x^{\alpha-q-1} + \frac{qb}{2}x^{-q-1} = 0 \tag{1}$$

where a, b, α and q are arbitrary parameters, and overdot means derivative with respect to time. In [1] the problem to secure exact and sinusoidal periodic solution to (1) was solved under the conditions that $q > -2$, $\alpha = q + 2$, and $b = \frac{a(q+2)}{4}$. In such conditions the equation (1) reduces to

$$\ddot{x} + a\,x + \frac{aq(q+2)}{8}x^{-q-1} = 0 \tag{2}$$

and all the solutions of (2) are periodic and expressed as a power law of a single sine function of time t as

$$x(t) = \left[\frac{\sqrt{q+2}}{2}\sin\left(\pm\frac{q+2}{2}\sqrt{a}\,(t+K)\right)\right]^{\frac{2}{q+2}} \tag{3}$$

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where $a>0$. It was the first time such a feat has been reached for a Lienard equation with strong and high order nonlinearity in the world of mathematics. The equation (2) is of the general form

$$\ddot{x}+a_1x+a_2\,x^m=0 \tag{4}$$

so one may see that to obtain the solution (3) it was necessary that the coefficients a_1 and a_2 are related in a precise relationship. Such a relation between a_1 and a_2 can be a restriction for the usefulness of equation (2). Now for $m>0$, that is a positive integer $m=n>0$, the equation (4) reduces to Duffing type equation. As examples the cubic Duffing equation

$$\ddot{x}+a_1x+a_2\,x^3=0 \tag{5}$$

is obtained for $n=3$. The quintic Duffing equation

$$\ddot{x}+a_1x+a_2\,x^5=0 \tag{6}$$

is ensured for $n=5$. In the perspective of the polynomial differential equation of Duffing type (4) where $m=n>0$, the problem to solve is to integrate (4) explicitly under the condition that a_1 and a_2 are general parameters. To do so, consider the equation (1). Let $q=-2$ and $\alpha=n$. Then the equation (1) becomes

$$\ddot{x}+\frac{1}{2}(n+2)a\,x^{n+1}-bx=0 \tag{7}$$

The equation (7) is of the form (4) where

$$a_1=-b\,,\; a_2=\frac{(n+2)a}{2} \tag{8}$$

With these values the coefficients a_1 and a_2 are always general parameters. The corresponding first order differential equation may be written as [1]

$$\dot{x}^2x^{-2}+a\,x^n=b \tag{9}$$

from which one may get

$$\frac{dx}{x\sqrt{b-a\,x^n}}=\pm dt \tag{10}$$

The integration of (10) is immediate and yields the exact and explicit general solution of (7) in the form

$$x(t)=\left[\frac{b}{a\cos^2\left[\frac{n\sqrt{-b}}{2}(t+K)\right]}\right]^{1/n} \tag{11}$$

where $n\neq 0$, and K an arbitrary constant. One may observe that all solutions of (7) are periodic with $a<0$, and $b<0$. For $n=2$, the solution of the cubic Duffing equation

$$\ddot{x}+2a\,x^3-bx=0 \tag{12}$$

takes the form

$$x(t)=\frac{\sqrt{\frac{b}{a}}}{\cos\left[\sqrt{-b}(t+K)\right]} \tag{13}$$

The solution of the quintic Duffing equation

$$\ddot{x}+3a\,x^5-bx=0 \tag{14}$$

has the expression

$$x(t)=\left(\frac{b}{a}\right)^{1/4}\frac{1}{\cos\left[2\sqrt{-b}(t+K)\right]^{1/2}} \tag{15}$$

In conclusion the exact and explicit general solution of all polynomial differential equations of Duffing type with three terms may be obtained from the equation (11) easily.

Reference

[1] M. D. Monsia, On a nonlinear differential equation of Lienard type, Math.Phys.,viXra.org/2011.0050v3.pdf (2020).