the proposed algorithm is as good or better than Takefuji/Lee's method in terms of the solution quality for every tested graph.

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On Positive Realness of Descriptor Systems

Liqian Zhang, James Lam, and Shengyuan Xu

***Abstract*—In this brief, the positive realness of descriptor systems is studied. For the continuous-time case, two positive real lemmas are given, based on a generalized algebraic Riccati equation and inequality respectively. For the discrete-time case, the positive real lemma is given in terms of a generalized algebraic Riccati inequality.**

***Index Terms*—Continuous, descriptor systems, discrete, generalized Riccati equation, generalized Riccati inequality, positive real.**

I. INTRODUCTION

It is well known that the descriptor form has higher capability to describe a physical system. Descriptor-system models are often more convenient and natural than normal (state-space) models in the description of interconnected large-scale systems [5], economic systems [14], electrical network analysis [16], power systems [17], chemical processes [11], and so on [13]. This is the reason why descriptor systems have attracted much interest in recent years. There are many research works aimed at generalizing existing theories, especially in the time domain, from normal systems to descriptor systems. These include controllability and observability [4], feedback control [3], [12], [18], bounded real lemma and H_∞ control [8], [15], [22], [25], H_2 control [21] and Lyapunov equations [20], [23], [27].

An essential property in linear circuit and system theory is positive realness. It is well known that it has found application in the analysis of the properties of immittance or hybrid matrices of various classes of networks, inverse problem of linear optimal control, circle criterion, Popov criterion and spectral factorization by algebra [2]. Recently, positive realness has also been related to the flexible space structure [10] and the stability of 2-D systems [1]. Moreover, positive realness plays an important role to study positive real control, which is a problem to construct an internally stabilizing controller such that the given closed-loop transfer function is positive real. The main motivation of this problem comes from robust and nonlinear control. When a strictly-positive-real controller is connected to a positive real plant in a negative-feedback configuration, the closed-loop system is guaranteed to be stable for arbitrary plant variations as long as the plant remains to be positive real [19].

Positive realness for normal linear systems has been studied by many researchers [6], [7], [9], [19]. For continuous-time descriptor systems, the strict positive realness is studied in [24] under the prior conditions that the systems are impulse-free and there are no finite dynamic modes on the imaginary axis. In this brief, we will study the positive realness for both continuous- and discrete-time descriptor systems. Positive real lemmas, which give the necessary and sufficient conditions for positive real descriptor systems, are given in terms of generalized algebraic Riccati equations and inequalities.

II. PRELIMINARIES

Throughout the paper, if not explicitly stated, all matrices are assumed to have compatible dimensions. We use $M > 0$ (resp. $M \geq 0$)

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to denote a real symmetric positive definite (resp. semidefinite) matrix M.

Consider a linear time-invariant continuous-time descriptor system

$$\Sigma_c\colon E\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) + Du(t)$$

or discrete-time descriptor system

$$\Sigma_d\colon Ex(k+1) = Ax(k) + Bu(k) \qquad y(k) = Cx(k) + Du(k)$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$ are known constant matrices. The transfer function of Σ_c (resp. Σ_d) is

$$G(s) = C(sE - A)^{-1}B + D$$

(resp.

$$G(z) = C(zE - A)^{-1}B + D).$$

The following terminology may be found in [5]. (E, A) is *regular* if $\det(sE - A)$ (resp. $\det(zE - A)$) is not identically zero. If (E, A) is regular, there exist two square invertible matrices U and V such that $\Sigma_c = (E, A, B, C, D)$ (resp. $\Sigma_d = (E, A, B, C, D)$) is transformed to the Weierstrass canonical form

$$\overline{\Sigma}_c = \left(\overline{E}, \overline{A}, \overline{B}, \overline{C}, \overline{D}\right) \equiv (UEV, UAV, UB, CV, D)$$

(respectively,

$$\overline{\Sigma}_d = \left(\overline{E}, \overline{A}, \overline{B}, \overline{C}, \overline{D}\right) \equiv (UEV, UAV, UB, CV, D))$$

with

$$\overline{E} = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \qquad \overline{A} = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}$$
$$\overline{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad \overline{C} = [\, C_1 \quad C_2 \,] \tag{1}$$

where N is nilpotent.

The zeros of $\det(sE - A)$ (resp. $\det(zE - A)$) are called the *finite poles* of (E, A). (E, A) is said to be *stable* if and only if all the finite poles of (E, A) lie in $\mathrm{Re}(s) < 0$ (resp. $|z| < 1$). (E, A) is called *impulse-free* (resp. *causal*) if and only if $N = 0$. (E, A) is *admissible* if it is regular, stable and impulse-free (resp. *causal*). For continuous-time system Σ_c, (E, A, B) is called *impulse controllable* if and only if $\mathrm{rank}[\begin{smallmatrix} E & 0 & 0 \\ A & E & B \end{smallmatrix}] = n + \mathrm{rank}E$; (E, A, B) is called *finite dynamics stabilizable* if and only if $\mathrm{rank}[sE - A \quad B] = n$ for any finite s with $\mathrm{Re}(s) \geq 0$; (E, A, C) is called *impulse observable* if and only if

$$\mathrm{rank}\begin{bmatrix} E & A \\ 0 & E \\ 0 & C \end{bmatrix} = n + \mathrm{rank}E$$

(E, A, C) is called *finite dynamics detectable* if and only if $\mathrm{rank}[\begin{smallmatrix} sE-A \\ C \end{smallmatrix}] = n$ for any finite s with $\mathrm{Re}(s) \geq 0$.

Proposition 1 [15], [25]: Consider the continuous-time descriptor system Σ_c. Suppose the pair (E, A) is regular and (E, A, C) is impulse observable and finite dynamics detectable. Then (E, A) is stable and impulse-free if and only if there exists $X \in \mathbb{R}^{n\times n}$ such that

$$E^TX = X^TE \geq 0$$
$$A^TX + X^TA + C^TC = 0.$$

Proposition 2 [15]: For the continuous-time descriptor system Σ_c, (E, A) is admissible if and only if there exists $X \in \mathbb{R}^{n\times n}$ such that

$$E^TX = X^TE \geq 0$$
$$A^TX + X^TA < 0.$$

Proposition [8], [26]: For the discrete-time descriptor system Σ_d, (E, A) is admissible if and only if there exists a matrix $X = X^T \in \mathbb{R}^{n\times n}$ such that

$$A^TXA < E^TXE$$
$$E^TXE \geq 0.$$

III. POSITIVE REALNESS OF CONTINUOUS-TIME DESCRIPTOR SYSTEMS

In this section, we will consider the extended strict positive realness of continuous-time descriptor systems.

Definition 1:

1) Σ_c is said to be *positive real (PR)* if its transfer function $G(s)$ is analytic in $\mathrm{Re}(s) > 0$ and satisfies $G(s) + G^*(s) \geq 0$ for $\mathrm{Re}(s) > 0$.
2) Σ_c is said to be *strictly positive real (SPR)* if its transfer function $G(s)$ is analytic in $\mathrm{Re}(s) \geq 0$ and satisfies $G(j\omega) + G^*(j\omega) > 0$ for $\omega \in [0, \infty)$.
3) Σ_c is said to be *extended strictly positive real (ESPR)* if it is strictly positive real and $G(j\infty) + G^*(j\infty) > 0$.

For a normal continuous-time linear system (I, A, B, C, D), the positive real lemma can be stated as follows.

Lemma 1 [19]: Consider system (I, A, B, C, D). The following statements are equivalent.

1) (I, A, B, C, D) is ESPR.
2) There exists a solution $X > 0$ such that
$$\begin{bmatrix} A^TX + XA & C^T - XB \\ C - B^TX & -(D + D^T) \end{bmatrix} < 0.$$
3) $D + D^T > 0$ and the algebraic Riccati equation
$$A^TX + XA + \left(C - B^TX\right)^T \left(D + D^T\right)^{-1} \left(C - B^TX\right) = 0$$
has a stabilizing solution X. That is, $A - B(D + D^T)^{-1}(C - B^TX)$ is stable.

Let

$$\Phi(s) = G(s) + G^T(-s).$$

It has the realization of $\Phi(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} + \hat{D}$, where

$$\hat{E} = \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B \\ C^T \end{bmatrix}$$
$$\hat{C} = [\, C \quad -B^T \,], \quad \hat{D} = D + D^T. \tag{2}$$

Lemma 2: If (E, A) is stable and impulse-free, and there exist X, Q, and W such that

$$E^TX = X^TE \tag{3}$$
$$A^TX + X^TA = -Q^TQ \tag{4}$$
$$B^TX + W^TQ = C \tag{5}$$
$$D + D^T = W^TW \tag{6}$$

then

$$\Phi(s) = M^T(-s)M(s)$$

with $M(s) = Q(sE - A)^{-1}B + W$. Furthermore, if $M(j\omega)$ has full column rank for all $\omega \in [0, \infty]$, then $G(s)$ is ESPR.

Proof: Transform $(\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{D})$ to an equivalent realization $(T\hat{E}S, T\hat{A}S, T\hat{B}, \hat{C}S, \hat{D})$ of $\Phi(s)$, where

$$T = \begin{bmatrix} I & 0 \\ X^T & I \end{bmatrix} \qquad S = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}.$$

From (3)–(6), we have

$$\begin{aligned} T\hat{E}S &= \begin{bmatrix} I & 0 \\ X^T & I \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \\ &= \begin{bmatrix} E & 0 \\ X^T E - E^T X & E^T \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \\ T\hat{A}S &= \begin{bmatrix} I & 0 \\ X^T & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ X^T A + A^T X & -A^T \end{bmatrix} = \begin{bmatrix} A & 0 \\ -Q^T Q & -A^T \end{bmatrix} \\ T\hat{B} &= \begin{bmatrix} I & 0 \\ X^T & I \end{bmatrix} \begin{bmatrix} B \\ -C^T \end{bmatrix} = \begin{bmatrix} B \\ X^T B - C^T \end{bmatrix} = \begin{bmatrix} B \\ -Q^T W \end{bmatrix} \\ \hat{C}S &= [\,C \quad B^T\,] \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} = [\,C - B^T X \quad B^T\,] \\ &= [\,W^T Q \quad B^T\,]. \end{aligned}$$

Then, as shown in the equation at the bottom of the page, where $M(s) = Q(sE - A)^{-1}B + W$. Clearly, this implies that

$$\Phi(j\omega) = M^T(-j\omega)M(j\omega) \geq 0$$

and if $M(j\omega)$ has full column rank for $\omega \in [0, \infty]$, we further have $\Phi(j\omega) > 0$ and hence $G(s)$ is ESPR. ■

Consider the following GARE described by

$$\text{GARE1:} \begin{cases} A^T X + X^T A + \left(C - B^T X\right)^T \\ \cdot \left(D + D^T\right)^{-1} \left(C - B^T X\right) = 0 \\ E^T X = X^T E \end{cases}$$

and the pair $(\mathcal{E}, \mathcal{A})$ defined by

$$\begin{aligned} \mathcal{E} &= \begin{bmatrix} E & 0 & 0 \\ 0 & E^T & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{E} & 0 \\ 0 & 0 \end{bmatrix} \\ \mathcal{A} &= \begin{bmatrix} A & 0 & -B \\ 0 & -A^T & -C^T \\ C & -B^T & -\left(D + D^T\right) \end{bmatrix} = \begin{bmatrix} \hat{A} & -\hat{B} \\ \hat{C} & -\hat{D} \end{bmatrix}. \end{aligned} \tag{7}$$

A solution X of GARE1 is called an *admissible solution* if $(E, A - B(D + D^T)^{-1}(C - B^T X))$ is admissible.

Lemma 3 [21]: Suppose that

1) (E, A, B) is finite dynamics stabilizable and impulse controllable.
2) $(\mathcal{E}, \mathcal{A})$ is regular, impulse-free and has no finite poles on the imaginary axis.

Then GARE1 has an admissible solution.

Lemma 4 [21]: Suppose (E, A, B) is finite dynamics stabilizable and impulse controllable, (E, A, C) is finite dynamics detectable and impulse observable. Then we have

$$\dim\ker G(s) = \dim\ker \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix}.$$

The main result is given in the following theorem.

Theorem 1: Let (E, A, B, C, D) be a realization of Σ_c. Suppose (E, A) is regular and $D + D^T > 0$. The following statements are equivalent.

1) (E, A) is admissible and Σ_c is ESPR.
2) GARE1 has an admissible solution X with $E^T X \geq 0$.

Proof: (2)⇒(1)Assume X_1 is an admissible solution of GARE1 with $E^T X_1 \geq 0$. Let $W = (D + D^T)^{1/2}$ and $Q = W^{-1}(C - B^T X_1)$. Then X_1 satisfies (3), (5), (6) and

$$A^T X_1 + X_1^T A = -Q^T Q. \tag{8}$$

Moreover, $(E, A - B(D + D^T)^{-1}(C - B^T X_1))$ is admissible. Thus (E, A, Q) is finite dynamics detectable and impulse observable, and (E, A, B) is impulse controllable and finite dynamics stabilizable. From (8) and Proposition 1, (E, A) is stable and impulse-free. Furthermore, from Lemma 4, it is known that

$$M(j\omega) = Q(j\omega E - A)^{-1}B + W$$

is nonsingular if and only if

$$N(j\omega) = \begin{bmatrix} A - j\omega E & B \\ Q & W \end{bmatrix}$$

is nonsingular. Since $(E, A - B(D + D^T)^{-1}(C - B^T X_1)) = (E, A - BW^{-1}Q)$ is admissible, $j\omega E - A + BW^{-1}Q$ is nonsingular, which is equivalent to the nonsingularity of $N(j\omega)$. Hence, $M(j\omega)$ is nonsingular for all $\omega \in [0, \infty]$, and then Σ_c is ESPR.

(1)⇒(2) From Lemma 3, if we can show that $(\mathcal{E}, \mathcal{A})$ is regular, impulse-free and has no finite poles on the imaginary axis, then GARE1 has an admissible solution. Since (E, A) is admissible and $\Phi(j\omega) > 0$ for $-\infty \leq \omega \leq \infty$, from (7), we have

$$\begin{aligned} &\det(j\omega\mathcal{E} - \mathcal{A}) \\ &\quad = \det\left(\begin{bmatrix} j\omega\hat{E} - \hat{A} & \hat{B} \\ -\hat{C} & \hat{D} \end{bmatrix}\right) \\ &\quad = \det\left(j\omega\hat{E} - \hat{A}\right)\det(\Phi(j\omega)) \\ &\quad = \det(j\omega E - A)\det\left(j\omega E^T + A^T\right)\det(\Phi(j\omega)) \neq 0. \end{aligned}$$

Hence, it follows that $(\mathcal{E}, \mathcal{A})$ has no finite poles on the imaginary axis. Furthermore, $(\mathcal{E}, \mathcal{A})$ is regular. It is noticed that $(\mathcal{E}, \mathcal{A})$ is impulse-free if and only if $s\mathcal{E} - \mathcal{A}$ is nonsingular at infinity. Consider

$$\begin{aligned} P(s) &= s\mathcal{E} - \mathcal{A} \\ &= \begin{bmatrix} sE - A & 0 & B \\ 0 & sE^T + A^T & C^T \\ -C & B^T & (D + D^T) \end{bmatrix} \\ &= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \Phi(s) &= [\,W^T Q \quad B^T\,] \begin{bmatrix} sE - A & 0 \\ Q^T Q & sE^T + A^T \end{bmatrix}^{-1} \begin{bmatrix} B \\ -Q^T W \end{bmatrix} + W^T W \\ &= [\,W^T Q \quad B^T\,] \begin{bmatrix} (sE - A)^{-1} & 0 \\ -\left(sE^T + A^T\right)^{-1} Q^T Q (sE - A)^{-1} & \left(sE^T + A^T\right)^{-1} \end{bmatrix} \begin{bmatrix} B \\ -Q^T W \end{bmatrix} + W^T W \\ &= \left(-B^T \left(sE^T + A^T\right)^{-1} Q^T + W^T\right)\left(Q(sE - A)^{-1}B + W\right) = M^T(-s)M(s) \end{aligned}$$

where

$$P_{11} = \begin{bmatrix} sE - A & 0 \\ 0 & sE^T + A^T \end{bmatrix}, \quad P_{12} = \begin{bmatrix} B \\ C^T \end{bmatrix}$$
$$P_{21} = [\,-C \quad B^T\,], \quad P_{22} = (D + D^T).$$

It is known that $(\mathcal{E}, \mathcal{A})$ has no finite poles on the imaginary axis. Then $P(s)$ has full normal rank. Notice that $P(s)$ is nonsingular if and only if $P_{22} - P_{21}P_{11}^{-1}P_{12}$ is nonsingular provided that P_{11} is nonsingular. Since (E, A) is admissible, $(sE - A)$ and $(sE^T + A^T)$ are nonsingular at infinity, which implies P_{11} is nonsingular at infinity. It can be seen that

$$\begin{aligned} &P_{22} - P_{21}P_{11}^{-1}P_{12} \\ &\quad = D + D^T - [\,-C \quad B^T\,] \begin{bmatrix} (sE - A)^{-1}B \\ \left(sE^T + A^T\right)^{-1} C^T \end{bmatrix} \\ &\quad = D + D^T + C(sE - A)^{-1}B - B^T(sE^T + A^T)^{-1}C^T \\ &\quad = \Phi(s). \end{aligned}$$

By assumption, $\Phi(j\infty) > 0$. This implies $P_{22} - P_{21}P_{11}^{-1}P_{12}$ is nonsingular at infinity, which is equivalent to the nonsingularity of $P(s)$ at infinity. Thus, $(\mathcal{E}, \mathcal{A})$ is impulse-free. Hence we have shown that GARE1 has an admissible solution X. To show that $E^T X \geq 0$, without loss of generality, suppose that (E, A, B, C, D) is in the Weierstrass canonical form $(\overline{E}, \overline{A}, \overline{B}, \overline{C}, D)$, and $\overline{X}$ is an admissible solution of

$$\begin{aligned} &\overline{A}^T\overline{X} + \overline{X}^T\overline{A} + \left(\overline{C} - \overline{B}^T\overline{X}\right)^T \left(D + D^T\right)^{-1} \\ &\quad \cdot \left(\overline{C} - \overline{B}^T\overline{X}\right) = 0 \\ &E^T\overline{X} = \overline{X}^T\overline{E}. \end{aligned} \tag{9}$$

Partition $\overline{X} = [\begin{smallmatrix} \overline{X}_{11} & \overline{X}_{12} \\ \overline{X}_{21} & \overline{X}_{22} \end{smallmatrix}]$ conformally to (1). From $\overline{E}^T\overline{X} = \overline{X}^T\overline{E}$, we have $\overline{X}_{11} = \overline{X}_{11}^T$ and $\overline{X}_{12} = 0$. Hence, by (9), we have

$$\begin{aligned} &A_1^T\overline{X}_{11} + \overline{X}_{11}A_1 + \left(C_1 - B_1^T\overline{X}_{11} - B_2^T\overline{X}_{21}\right)^T \\ &\qquad \cdot \left(D + D^T\right)^{-1} \left(C_1 - B_1^T\overline{X}_{11} - B_2^T\overline{X}_{21}\right) = 0. \end{aligned}$$

Notice that A_1 is stable, then

$$\begin{aligned} \overline{X}_{11} = &\int_0^\infty e^{A_1^T t} \left(C_1 - B_1^T\overline{X}_{11} - B_2^T\overline{X}_{21}\right)^T \\ &\cdot \left(D + D^T\right)^{-1} \left(C_1 - B_1^T\overline{X}_{11} - B_2^T\overline{X}_{21}\right) e^{A_1 t}\, dt \geq 0 \end{aligned}$$

and thus $\overline{E}^T\overline{X} \geq 0$ follows. ■

Theorem 1 gives a necessary and sufficient condition for the continuous-time descriptor system Σ_c to be ESPR in terms of a generalized Riccati equation when Σ_c is regular and $D + D^T > 0$. It is known that the regularity of a descriptor system can be destroyed by feedback input, which causes problems for controller synthesis. In the following theorem, we will give a necessary and sufficient condition in terms of a generalized Riccati inequality without the regularity assumption.

Theorem 2: The following statements are equivalent.

1) (E, A) is admissible and Σ_c is ESPR, and $D + D^T > 0$.
2) There exists a solution X to

$$\text{GARI: } \begin{cases} \begin{bmatrix} A^T X + X^T A & \left(C - B^T X\right)^T \\ C - B^T X & -\left(D + D^T\right) \end{bmatrix} < 0 \\ E^T X = X^T E \geq 0. \end{cases}$$

Proof: (2)⇒(1) From Proposition 2, it can be easily seen that (E, A) is admissible. Without loss of generality, we assume that (E, A, B, C, D) is in the Weierstrass canonical form $(\overline{E}, \overline{A}, \overline{B}, \overline{C}, D)$, and $\overline{X}$ is the solution to

$$\begin{bmatrix} \overline{A}^T\overline{X} + \overline{X}^T\overline{A} & (\overline{C} - \overline{B}^T\overline{X})^T \\ \overline{C} - \overline{B}^T\overline{X} & -\left(D + D^T\right) \end{bmatrix} < 0 \tag{10}$$

$$\overline{E}^T\overline{X} = \overline{X}^T\overline{E} \geq 0. \tag{11}$$

Conformally to the structures of $\overline{E}$ and $\overline{A}$, partition $\overline{X}$ as $[\begin{smallmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{smallmatrix}]$. By (11), we have $X_{11} = X_{11}^T \geq 0, X_{12} = 0$. Then (10) can be written as shown in the inequality at the bottom of the page. Postmultiplying and premultiplying this inequality by

$$\begin{bmatrix} I & 0 & 0 \\ 0 & B_2 & I \\ 0 & I & 0 \end{bmatrix}$$

and its transposition respectively, we obtain as shown in the inequality at the bottom of the page which implies $M_1 < 0$. Let $X_1 = X_{11} + \mu I > 0$, where $\mu > 0$ is small enough such that

$$\begin{aligned} &\begin{bmatrix} A_1^T X_1 + X_1 A_1 & C_1^T - X_1 B_1 \\ C_1 - B_1^T X_1 & C_2 B_2 + B_2^T C_2^T - (D + D^T) \end{bmatrix} \\ &\quad = \begin{bmatrix} A_1^T X_{11} + X_{11} A_1 & C_1^T - X_{11} B_1 \\ C_1 - B_1^T X_{11} & C_2 B_2 + B_2^T C_2 - (D + D^T) \end{bmatrix} \\ &\qquad + \mu \begin{bmatrix} A_1^T + A_1 & -B_1 \\ -B_1^T & 0 \end{bmatrix} < 0. \end{aligned}$$

Hence, Σ_c is ESPR by Lemma 1.

$$\begin{bmatrix} A_1^T X_{11} + X_{11} A_1 & X_{21}^T & C_1^T - X_{11} B_1 - X_{21}^T B_2 \\ X_{21} & X_{22} + X_{22}^T & C_2^T - X_{22}^T B_2 \\ C_1 - B_1^T X_{11} - B_2^T X_{21} & C_2 - B_2^T X_{22} & -(D + D^T) \end{bmatrix} < 0$$

$$\begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix} = \left[\begin{array}{cc|c} A_1^T X_{11} + X_{11} A_1 & C_1^T - X_{11} B_1 & X_{21}^T \\ C_1 - B_1^T X_{11} & C_2 B_2 + B_2^T C_2^T - (D + D^T) & B_2^T X_{22}^T + C_2 \\ \hline X_{21} & X_{22} B_2 + C_2^T & X_{22} + X_{22}^T \end{array}\right] < 0$$

(1)⇒(2) Let $\hat{\underline{\sigma}} := \inf_\omega \underline{\sigma}(G(j\omega) + G^T(-j\omega)) > 0$ where $\underline{\sigma}(\cdot)$ denotes the smallest singular value of a matrix and

$$\hat{\varepsilon} := \frac{2\delta\hat{\underline{\sigma}}}{\|(sE - A)^{-1}B\|_\infty} > 0$$

for some constant $\delta > 0$. We will show that if $0 < \varepsilon < \hat{\varepsilon}$, then $(E, A, \tilde{B}, \tilde{C}, \tilde{D})$ is ESPR, where

$$\tilde{B} = [B \;\; 0], \quad \tilde{C} = \begin{bmatrix} C \\ \varepsilon I \end{bmatrix} \quad \tilde{D} = \begin{bmatrix} D & 0 \\ 0 & \delta I \end{bmatrix}.$$

The transfer function of $(E, A, \tilde{B}, \tilde{C}, \tilde{D})$ is

$$\tilde{G}(s) = \begin{bmatrix} G(s) & 0 \\ \varepsilon(sE - A)^{-1}B & \delta I \end{bmatrix}.$$

Since $0 < \varepsilon < \hat{\varepsilon}$, that is, $\varepsilon^2 \, \|(sE - A)^{-1}B\|_\infty^2 < 2\delta\hat{\underline{\sigma}}$, which is equivalent to

$$\varepsilon^2 B^T \left(-j\omega E^T - A^T\right)^{-1} (j\omega E - A)^{-1} B < 2\delta\hat{\underline{\sigma}} I,$$

for all $\omega \in [0, \infty)$. It is also known that

$$\hat{\underline{\sigma}} I \leq \underline{\sigma}\left(G(j\omega) + G^T(-j\omega)\right) I \leq G(j\omega) + G^T(-j\omega).$$

Hence,

$$\varepsilon^2 B^T \left(-j\omega E^T - A^T\right)^{-1} (j\omega E - A)^{-1} B < 2\delta\left(G(j\omega) + G^T(-j\omega)\right)$$

which implies

$$\tilde{G}(j\omega) + \tilde{G}^T(-j\omega) > 0$$

for $\omega \in [0, \infty)$. Furthermore,

$$\tilde{G}(j\infty) + \tilde{G}^T(-j\infty) = \begin{bmatrix} G(j\infty) + G^T(-j\infty) & 0 \\ 0 & 2\delta I \end{bmatrix} > 0.$$

Then, $(E, A, \tilde{B}, \tilde{C}, \tilde{D})$ is ESPR.

Since $(E, A, \tilde{B}, \tilde{C}, \tilde{D})$ is admissible and ESPR, by Theorem 1, there exists X such that

$$A^T X + X^T A + \left(\tilde{C} - \tilde{B}^T X\right)^T \left(\tilde{D} + \tilde{D}^T\right)^{-1} \left(\tilde{C} - \tilde{B}^T X\right) = 0$$
$$E^T X = X^T E \geq 0.$$

This is equivalent to

$$A^T X + X^T A + \left(C - B^T X\right)^T \left(D + D^T\right)^{-1} \cdot \left(C - B^T X\right) + \frac{\varepsilon^2}{2\delta} I = 0,$$
$$E^T X = X^T E \geq 0.$$

The proof is then finished. ■

IV. POSITIVE REALNESS OF DISCRETE-TIME DESCRIPTOR SYSTEMS

The extended strict-positive realness of discrete-time descriptor systems will be considered in this section.

Definition 2:

1) Σ_d is said to be *positive real (PR)* if its transfer function $G(z)$ is analytic in $|z| > 1$ and satisfies $G(z) + G^*(z) \geq 0$ for $|z| > 1$.
2) Σ_d is said to be *strictly positive real (SPR)* if its transfer function $G(z)$ is analytic in $|z| \geq 1$ and satisfies $G(e^{j\theta}) + G^*(e^{j\theta}) > 0$ for $\theta \in [0, 2\pi]$.
3) Σ_d is said to be *extended strictly positive real (ESPR)* if it is strictly positive real and $G(\infty) + G^*(\infty) > 0$.

For the normal discrete-time linear system (I, A, B, C, D), the positive real lemma can be stated as follows:

Lemma 5 [7]: Consider the normal discrete-time linear system (I, A, B, C, D). The following statements are equivalent.

1) (I, A, B, C, D) is ESPR.
2) There exists a solution $X > 0$ such that

$$\begin{bmatrix} A^T X A - X & \left(C - B^T X A\right)^T \\ C - B^T X A & -\left(D + D^T - B^T X B\right) \end{bmatrix} < 0.$$

Now, we may consider the positive realness of discrete-time descriptor system Σ_d.

Theorem 3: For discrete-time descriptor system Σ_d, the following statements are equivalent.

1) (E, A) is admissible and Σ_d is ESPR.
2) There exists a solution $X = X^T$ such that

$$\begin{bmatrix} A^T X A - E^T X E & \left(C - B^T X A\right)^T \\ C - B^T X A & -\left(D + D^T - B^T X B\right) \end{bmatrix} < 0$$
$$E^T X E \geq 0.$$

Proof: (2)⇒(1) From Proposition 3, it can be easily seen that (E, A) is admissible. Without loss of generality, we assume that (E, A, B, C, D) is in the Weierstrass canonical form $(\overline{E}, \overline{A}, \overline{B}, \overline{C}, D)$, and $\overline{X}$ is the solution to

$$\begin{bmatrix} \overline{A}^T \overline{X}\,\overline{A} - \overline{E}^T \overline{X}\,\overline{E} & \left(\overline{C} - \overline{B}^T \overline{X}\,\overline{A}\right)^T \\ \overline{C} - \overline{B}^T \overline{X}\,\overline{A} & -\left(D + D^T - \overline{B}^T \overline{X}\,\overline{B}\right) \end{bmatrix} < 0 \tag{12}$$

$$\overline{E}^T \overline{X}\,\overline{E} \geq 0. \tag{13}$$

Conformally to the structures of $\overline{E}$ and $\overline{A}$, partition $\overline{X}$ as $[\begin{smallmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{smallmatrix}]$. By (13), we have $X_{11} \geq 0$. Then (12) can be written as shown in the equation at the bottom of the page. Postmultiplying and premultiplying this inequality by

$$\begin{bmatrix} I & 0 & 0 \\ 0 & B_2 & I \\ 0 & I & 0 \end{bmatrix}$$

$$\begin{bmatrix} A_1^T X_{11} A_1 - X_{11} & A_1^T X_{12} & \left(C_1 - B_1^T X_{11} A_1 - B_2^T X_{12}^T A_1\right)^T \\ X_{12}^T A_1 & X_{22} & \left(C_2 - B_1^T X_{12} - B_2^T X_{22}\right)^T \\ C_1 - B_1^T X_{11} A_1 - B_2^T X_{12}^T A_1 & C_2 - B_1^T X_{12} - B_2^T X_{22} & \begin{array}{c} -D - D^T + B_1^T \left(X_{11} B_1 + X_{12} B_2\right) \\ +B_2^T \left(X_{12}^T B_1 + X_{22} B_2\right) \end{array} \end{bmatrix} < 0$$

$$\begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix} = \left[\begin{array}{cc|c} A_1^T X_{11} A_1 - X_{11} & \left(C_1 - B_1^T X_{11} A_1\right)^T & A_1^T X_{12} \\ C_1 - B_1^T X_{11} A_1 & C_2 B_2 + B_2^T C_2^T - \left(D + D^T\right) + B_1^T X_{11} B_1 & C_2 - B_1^T X_{12} \\ \hline X_{12}^T A_1 & \left(C_2 - B_1^T X_{12}\right)^T & X_{22} \end{array}\right] < 0$$

and its transposition respectively, we obtain the equation as shown at the top of the page which implies $M_1 < 0$. Let $X_1 = X_{11} + \mu I > 0$, where $\mu > 0$ is small enough such that

$$\begin{bmatrix} A_1^T X_1 A_1 - X_1 & \left(C_1 - B_1^T X_1 A_1\right)^T \\ C_1 - B_1^T X_1 A_1 & C_2 B_2 + B_2^T C_2^T - \left(D + D^T\right) + B_1^T X_1 B_1 \end{bmatrix} = M_1 + \mu \begin{bmatrix} A_1^T A_1 - I & -A_1^T B_1 \\ -B_1^T A_1 & B_1^T B_1 \end{bmatrix} < 0.$$

Hence Σ_d is ESPR by Lemma 5.

(1)⇒(2) Without loss of generality, we also suppose (E, A, B, C, D) is in the Weierstrass canonical form $(\overline{E}, \overline{A}, \overline{B}, \overline{C}, D)$. Since A_1 is stable and $G(z) = C(zE - A)^{-1}B + D = C_1(zI - A_1)^{-1}B_1 - C_2 B_2 + D$ is ESPR, from Lemma 5 there exists $X_{11} = X_{11}^T > 0$ such that $M_1 < 0$. If let $X_{22} = -\alpha I$, where $\alpha > 0$ is large enough such that

$$M_1 - M_2 M_3^{-1} M_2^T < 0,$$

then $\overline{X} = \left[\begin{smallmatrix} X_{11} & 0 \\ 0 & X_{22} \end{smallmatrix}\right] = \overline{X}^T$ is the solution to (12) and (13). ■

V. ILLUSTRATIVE EXAMPLE

Due to the space limitations, we consider a continuous-time descriptor model in Weierstrass canonical form (1), where

$$E = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \quad A = \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \quad B = \left[\begin{array}{c} 1 \\ 1 \\ \hline b \end{array}\right]$$

$$C = [1 \quad 1 \quad | \quad 1], \qquad D = \tfrac{1}{2}$$

and b is a constant. It can be seen that this model is regular, stable, and impulse-free. Its transfer function is

$$G(s) = \frac{1}{s+1} + \frac{1}{s+2} - b + \frac{1}{2}.$$

From

$$G(j\omega) + G(-j\omega) = \frac{2}{\omega^2 + 1} + \frac{4}{\omega^2 + 4} - 2b + 1$$

$G(s)$ is ESPR when $b = 0$, and not ESPR when $b = 1$.

Consider the solution of GARE1 for $b = 0$ and $b = 1$ with $E^T X \geq 0$, which is supposed to have the form of

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{14}$$

partitioned conformally to (1). From $E^T X = X^T E \geq 0$, we have $X_{11} = X_{11}^T \geq 0$ and $X_{12} = 0$. Hence GARE1 becomes

$$\begin{aligned} &A_1^T X_{11} + X_{11} A_1 + \left(C_1 - B_1^T X_{11} - b X_{21}\right)^T \\ &\quad \cdot \left(C_1 - B_1^T X_{11} - b X_{21}\right) = 0 \\ &X_{21} + (1 - b X_{22}) \left(C_1 - B_1^T X_{11} - b X_{21}\right) = 0 \\ &2X_{22} + (1 - b X_{22})^2 = 0. \end{aligned} \tag{15}$$

When $b = 0$, a solution is given by

$$X = \begin{bmatrix} 0.2055 & 0.1534 & 0 \\ 0.1534 & 0.1288 & 0 \\ -0.6411 & -0.7178 & -0.5 \end{bmatrix}$$

which is an admissible solution to GARE1 since

$$\left(E, A - B\left(D + D^T\right)^{-1}\left(C - B^T X\right)\right) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1.6411 & -0.7178 & -1 \\ -0.6411 & -2.7178 & -1 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

is regular, stable (with finite poles at $s_1 = -1.3134$ and $s_2 = -3.0455$) and impulse-free. That is, an admissible solution of GARE1 has been obtained. However, when $b = 1$, (15) becomes $X_{22}^2 + 1 = 0$, which has no real solution. Hence, from Theorem 1, $G(s)$ is ESPR when $b = 0$, and not ESPR when $b = 1$.

Now consider the solution of GARI when $b = 0$ and $b = 1$ respectively. When $b = 0$, one solution of GARI is

$$X = \begin{bmatrix} 1.4926 & -0.0281 & 0 \\ -0.0281 & 0.8443 & 0 \\ 0.3333 & 0.3333 & -1.3986 \end{bmatrix}.$$

From Theorem 2, $G(s)$ is ESPR. Notice that a necessary condition for X to be a solution of GARI is

$$2X_{22} + (1 - b X_{22})^2 < 0$$

if X has the form of (14). It can be easily seen that when $b = 1$, this inequality has no solution. Hence, from Theorem 2, $G(s)$ is not ESPR.

VI. CONCLUSIONS

We have derived necessary and sufficient conditions for descriptor systems to be admissible and ESPR. The conditions are given based on generalized algebraic Riccati inequalities for continuous- and discrete-time descriptor systems. For the continuous-time case, we also give the necessary and sufficient condition based on a generalized Riccati equation under the condition of regularity.

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