Archives

Ref. Th.968

ON POSITIVE SPECTRAL FUNCTIONS IN THE MANDELSTAM REPRESENTATION

André Martin
CERN - Geneva

A B S T R A C T

We study the restrictions imposed on a neutral pseudoscalar scattering amplitude by the assumption that the double spectral function is positive. It turns out that the total cross-section is bounded by const/log s, and that the Froissart-Gribov formula holds for $\ell > 1$. An example, nearly saturating these limitations, and such that the imaginary part of each partial wave amplitude is within the unitarity limits, is given.

25 November 1968
Ref. Th.968

Ref. TH. 968

ON POSITIVE SPECTRAL FUNCTIONS IN THE MANDELSTAM REPRESENTATION

André Martin
CERN - Geneva

A D D E N D U M

The result of Section II has also been derived by Professor Ch. Goebel in :

Proceedings of the International Conference on Theoretical Aspects of Very High Energy Phenomena CERN 1961. (CERN Yellow Report !).

I. INTRODUCTION

We know that a simple, but important [1], requirement on a scattering amplitude is the positivity property, namely that each partial wave amplitude has a positive imaginary part.

If the Mandelstam representation is valid, we know that we can represent the imaginary part of each partial wave amplitude as an integral over the double spectral function, with positive weights, usually called the Froissart-Gribov formula [2]. This representation holds for sufficiently large angular momenta. Now the simplest way to ensure positivity, for the $\pi_0 \pi_0$ scattering amplitude, is to postulate that the double spectral function is positive. As a matter of fact, there is a rather extended region of the s, t plane where it has been shown [3] that the double spectral function is necessarily positive.

However, it must be realized that the assumption that the double spectral function is positive everywhere is very strong, so strong that some of the results presented here were obtained by the author in 1963 and never published, due to lack of interest. The reason why we re-investigate this question is linked with the considerable progress made by Atkinson [4] in the construction of unitary crossing symmetric scattering amplitudes. Among other things, Atkinson wants to satisfy the positivity condition and chooses to have an essentially positive double spectral function. So far, he has only investigated the case without subtractions, which corresponds to a total cross-section decreasing faster than $1/s$, where s is the square of the c.m. energy. The question is whether one can build a more realistic amplitude which still has positivity. But the only simple way to have positivity is to have positive double spectral functions.

Therefore we think that it is worth while investigating how far one can go. First of all, the leading singularity in the complex angular momentum plane cannot be a Regge pole. There is not, however, the faintest evidence that the leading singularity of an elastic amplitude (Pomeranchon) is a Regge pole. So this should not worry us too much. We shall discover however something else, i.e., that the total cross-section must decrease like const/log s. Since the bound can be computed explicitly even for finite energies, one can see whether this is disturbing or not, and it does not seem to be yet too disturbing. We shall also discover that for any energy the Froissart-Gribov formula holds for $\ell > 1$. Finally, we discuss the question whether one can saturate these restrictions or not.

We do not take into account all aspects of unitarity (as Atkinson would do), but we show at least a double spectral function such that the unitarity restrictions on the imaginary part of each partial wave are fulfilled, that the first Gribov catastrophy [5] in the elastic strip is avoided, and that the total cross-section behaves like $(1/(\log s)^{1+\epsilon})$.

II. THE BOUND ON THE TOTAL CROSS-SECTION

We postulate a Mandelstam representation for the $\pi_0\pi_0 \to \pi_0\pi_0$ amplitude (this amplitude can be either the realistic $\pi_0\pi_0$ amplitude or that of a neutral pseudoscalar theory). We use the standard variables :

$$s + t + u = 4 \,,$$

the pion mass being taken equal to unity.

The double spectral function is designated by $\rho(s,t)$ and, the number of subtractions being finite, we have the Froissart-Gribov representation

$$\frac{\sqrt{s}}{2k} f_\ell(s) = \frac{1}{k^2} \int Q_\ell\left(1 + \frac{t}{2k^2}\right) A_t(s,t)\, dt \qquad (1)$$
$$\ell \geq L$$

where A_t represents the absorptive part in the t channel, $k^2 = (s-4)/4$, and

$$\frac{\sqrt{s}}{k}\, \mathrm{Im}\, f_\ell(s) = \frac{1}{k^2}\int_{t_0(s)}^{\infty} Q_\ell\left(1+\frac{t}{2k^2}\right)\rho(s,t)\,dt, \qquad \ell \geqslant L \tag{2}$$

with the unitarity condition

$$1 \geqslant \mathrm{Im}\, f_\ell(s) \geqslant 0 \tag{3}$$

The positivity of ρ has a simple consequence : since $Q_\ell(x)$, for $x > 1$ is a decreasing function of ℓ , we have

$$\mathrm{Im}\, f_{\ell'} < \mathrm{Im}\, f_\ell \leqslant 1 \tag{4}$$

for $\ell' > \ell \geqslant L$.

A somewhat more refined result is obtained if one notices (see Appendix A) that

$$Q_{\ell'}(x) / Q_\ell(x) \tag{5}$$

for $\ell' > \ell$ is a <u>decreasing</u> function of x.

Therefore

$$\frac{\mathrm{Im}\, f_\ell(s)}{\mathrm{Im}\, f_L(s)} \leqslant \frac{Q_\ell\left(1+\frac{t_0(s)}{2k^2}\right)}{Q_L\left(1+\frac{t_0(s)}{2k^2}\right)} \tag{6}$$

where $t_0(s)$ is the lower limit of integration in (1) :

$$t_0(s) = \min \begin{cases} 4 + \dfrac{64}{s-16} \\ 16 + \dfrac{64}{s-4} \end{cases} \tag{7}$$

So we get [6)]

$$\mathrm{Im}\, F(s,t=0) \leqslant \frac{\sqrt{s}}{2k}\left[\sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{L-2}(2\ell+1) + \sum_{L}^{\infty}(2\ell+1)\frac{Q_\ell(z_0)}{Q_L(z_0)}\right]$$

$$= \frac{\sqrt{s}}{2k}\left[\frac{L(L-1)}{2} + L\,\frac{z_0 Q_{L-1}(z_0) - Q_L(z_0)}{(z_0^2-1)\,Q_L(z_0)}\right] \tag{8}$$

with $z_0 = 1 + t_0(s)/2k^2$. The total cross-section is therefore bounded by :

$$\sigma_{total} \leqslant \frac{4\pi}{k^2}\left[\frac{L(L-1)}{2} + L\,\frac{z_0 Q_{L-1}(z_0) - Q_L(z_0)}{(z_0^2-1)\,Q_L(z_0)}\right] \tag{9}$$

Notice that the only unknown quantity in this formula is L. If $s \to \infty$, $t_0 \to 4$ and $z_0 \to 1$. It is well known that [6)] $Q_L(z_0) - \frac{1}{2}\log[(z_0+1)/(z_0-1)]$ remains bounded as $z_0 \to 1$. Thus the numerator of the second term in (9) remains bounded, while the denominator behaves like $\log(1/(z_0-1))$, i.e., like $\log s$. So we get

$$\sigma_{total} \lesssim \text{const.} \times \left[\log(1+k^2)\right]^{-1} \tag{10}$$

We shall not try to make this formula more accurate at the moment as we shall determine L in the next Section.

We can extend the bound obtained in the forward direction $(t=0)$ to arbitrary t. We shall write

$$A_s(s,t) = \frac{\sqrt{s}}{2k} \sum_{0}^{L-2} (2\ell+1) \operatorname{Im} f_\ell(s) P_\ell(z)$$

$$+ 2L \int_{z_0}^{\infty} \frac{z' Q_{L-1}(z') P_L(z) - z P_{L-1}(z) Q_L(z')}{z'^2 - z^2} \rho(z',s) dz'$$

with $z = 1 + (t/2k^2)$, $z' = 1 + (t'/2k^2)$.

The unitarity restriction

$$2 \int_{z_0}^{\infty} Q_L(z') \rho(z',s) dz' < \frac{\sqrt{s}}{2k}$$

gives us :

$$|A_s(s,t)| < \frac{\sqrt{s}}{2k} \sum_{0}^{L-2} (2\ell+1) |P_\ell(z)|$$

$$+ L \frac{\sqrt{s}}{2k} \operatorname*{Max}_{z_0 < z' < \infty} \left| \frac{z' Q_{L-1}(z') P_L(z) - z P_{L-1}(z) Q_L(z')}{(z'^2 - z^2) Q_L(z')} \right| .$$

All we need is a crude estimate. We have

$$\operatorname{Max} \left| \frac{z' Q_{L-1}(z') P_L(z) - z P_{L-1}(z) Q_L(z')}{(z'^2 - z^2) Q_L(z')} \right|$$

$$\leqslant \operatorname{Max} \left| \frac{z'^2 - 1}{z'^2 - z^2} \right| \times \left\{ \operatorname{Max} \left| \frac{z' Q_{L-1}(z') - Q_L(z')}{(z'^2 - 1) Q_L(z')} \right| + \operatorname{Max} \left| \frac{z' Q_{L-1}(z')(P_L(z) - 1) - Q_L(z')(P_{L-1}(z) - 1)}{(z'^2 - 1) Q_L(z')} \right| \right\}$$

As $s \to \infty$ for t <u>fixed</u>, $z \to 1$ and

$$|P_L(z) - 1| < c_L |z - 1| \simeq \frac{c'_L}{s}$$

It is not very difficult to see that the first term dominates in the inequality and we get, for t fixed, $s \to \infty$

$$|A_s(s,t)| \lesssim \text{Const.} + \frac{\text{Const.} \times s}{(\log s)\,\Delta t} \tag{11}$$

where Δt is the distance of t to the right-hand cut at the energy s, i.e.,

$$\Delta(t) = |t - t_0(s)| \quad \text{if } \operatorname{Re} t \leqslant t_0(s)$$

$$\Delta(t) = |\operatorname{Im} t| \quad \text{if } \operatorname{Re} t \geqslant t_0(s)$$

The bound becomes infinite on the cut, a fact which could be expected since $\rho(s,t)$ can contain delta functions.

III. LIMITS ON THE NUMBER OF SUBTRACTIONS AND ABSOLUTE BOUNDS ON THE AMPLITUDE

Inequality (11), valid for any t, shows that in the t channel, the Froissart-Gribov representation

$$\sqrt{\frac{t}{t-4}}\, f_\ell(t) = \frac{4}{t-4} \int Q_\ell\left(1 + \frac{2s}{t-4}\right) A_s(s,t)\,dt \tag{12}$$

can be continued analytically from $\mathrm{Re}\,\ell = L$ to $\mathrm{Re}\,\ell = 1+\varepsilon$, for any t outside the cut from $t=4$ to ∞. On the cut one can still define $A_s(s,t)$ as a distribution and, in fact, it would be possible, following standard techniques [7], to get an upper bound on

$$\int A_s(s,t-t')\,w(t')\,dt'$$

where $w(t')$ is a once differentiable function with compact support. Then one could show the existence of a Froissart-Gribov representation on the cut from a smoothed $f_\ell(t)$.

Now the identification of $f_\ell(t)$, as given by the Froissart-Gribov formula, with the physical $f_\ell(t)$ for $\ell = 2, 4$, etc., is made by noticing that for $|t| < 4$, the Froissart-Gribov formula is valid for $\ell \geqslant 2$ [8]. There is no need to use the Oehme argument [9].

We see now that we can take $L=2$ in all formulas of the previous Section. We can get an explicit bound for the amplitude for any value of s, t, u outside the cuts, and for the absorptive part when the conjugate variable is outside the cuts. For instance, we get

$$\sigma_{total} \leqslant \frac{4\pi}{k^2}\left[1 + 2\,\frac{\dfrac{z_0}{z_0^2-1} - \dfrac{1}{2}\log\left(\dfrac{z_0+1}{z_0-1}\right)}{\dfrac{3z_0^2-1}{2}\log\left(\dfrac{z_0+1}{z_0-1}\right) - 3z_0}\right] \tag{13}$$

$$\simeq \frac{2\pi}{\log\left(\dfrac{1+k^2}{e^3}\right)}$$

for $k^2 \to \infty$.

Inequality (13) cannot yet be considered as inconsistent with "experiment" : for $k^2 = 200\ m_\pi^2$ we get $\sigma_{total} \leqslant 50$ mb.

On the other hand, with the help of inequality (11), we can now write the Mandelstam representation with the minimum number of subtractions. From Eqs. (2) and (3), we can get

$$\int \frac{\rho(s,t)\,ds\,dt}{s^{2+\varepsilon}\;t^{2+\varepsilon}} \quad \text{convergent} \tag{14}$$

$$\int \frac{\rho(s,t)\,ds\,dt}{s\;t\;|s+t-4|^{1+\varepsilon}} \quad \text{convergent} \tag{15}$$

$$\int \frac{\rho(s,t)\,ds\,dt}{|u|^{2+\varepsilon}} \quad \text{convergent} \tag{16}$$

The most economic way to write a Mandelstam representation is :

$$\begin{aligned} F = {} & \text{polynomial} + \frac{s^N}{\pi}\int \frac{\sigma(s')\,ds'}{s'^N(s'-s)} \\ & + \text{circular permutations} \\ & + \frac{u^2}{\pi}\int \frac{\rho(s',t')\,ds'\,dt'}{(s'-s)(t'-t)(s'+t'-4)^2} \\ & + \text{circular permutations} \end{aligned} \tag{17}$$

where the validity of the Froissart-Gribov formula for $L \geq 2$ has already been taken into account.

The absorptive part in the s channel for $t=0$ reduces to

$$\begin{aligned} A_s(s,0) = {} & \sigma(s) + \frac{(4-s)^2}{\pi}\int \frac{\rho(s,t')\,dt'}{t'(s+t'-4)^2} \\ & + \frac{s^2}{\pi}\int \frac{\rho(s,t')\,dt'}{(t'-4+s)(s+t'-4)^2} \end{aligned} \tag{18}$$

$A_s(s,0)$, as we know, is bounded by $s/\log s$; on the other hand, one can show directly, by comparison with $(\sqrt{s}/2k)\,\mathrm{Im}\, f_2$ that the double spectral function contribution is bounded by $s/\log s$. Hence $\sigma(s)$ is bounded by $s/\log s$, and in Eq. (17) we can take $N = 2$.

If we now look again at F, given by (17), we can easily show that for $t = 0$ F, except for the polynomial part, is bounded by $|s|^{1+\varepsilon}$ for $s \to i\infty$. Since F satisfies the Froissart bound [1),2)] and in fact better than that :

$$|F(s, t=0)| < \frac{s}{(\log s)^{1/2}}$$

we conclude that the polynomial, symmetric in s, t, u, reduces to a constant. So we get

$$\begin{aligned} F = C &+ \frac{s^2}{\pi}\int \frac{\sigma(s')\,ds'}{s'^2(s'-s)} + \text{permutations} \\ &+ \frac{u^2}{\pi^2}\int \frac{\rho(s',t')\,ds'\,dt'}{(s'-s)(t'-t)(s'+t'-4)^2} + \text{permutations} \end{aligned} \tag{19}$$

IV. AN EXAMPLE

We would like to know how close a unitary amplitude, with positive ρ, can be from the limits (11) and (13). Here we do not want to repeat all Atkinson's programme [4)] but only wish to concentrate on the positivity aspect and require only $0 \leqslant \mathrm{Im}\, f_\ell(s) \leqslant 1$. Only in one respect shall we do better than that : we shall avoid the Gribov catastrophy [5)] by taking an amplitude which behaves like $s(\log s)^{-1-\epsilon}$ rather than $s(\log s)^{-1}$ for fixed t in the elastic strip.

In fact this comes rather naturally : we know that the Froissart-Gribov formula holds for Re $\ell > 1$. However, in the elastic strip the unitarity condition can be continued for real, unphysical ℓ, down to the lower limit of validity of the representation. Hence for ℓ arbitrarily close to 1, $0 \leqslant \mathrm{Im}\, f_\ell(s) \leqslant 1$ and since ρ is positive, the Froissart-Gribov representation holds for $\ell = 1$ because $\mathrm{Im}\, f_\ell(s)$ is a monotonous bounded function of ℓ. Therefore

$$\int \frac{\rho(s,t)\, dt}{t^2}$$

must be integrable. We propose

$$\rho(s,t) = \lambda \; \theta\left(\psi(s,t)\right) \; \theta(s-4)\, \theta(t-4)$$

$$\times \frac{1}{\left[(\log s)(\log t)\right]^{1+\varepsilon}} \left[\frac{s-4}{t} + \frac{t-4}{s}\right]$$

where

$$\psi(s,t) = \sup \begin{cases} (s-4)(t-16)-64 \\ (s-16)(t-4)-64 \end{cases}$$

then it is not difficult to check that $\mathrm{Im}\, f_1$ as given by the Froissart-Gribov formula, and hence $\mathrm{Im}\, f_\ell$, $\ell \geqslant 1$, are bounded by unity (see Appendix B). In fact, here we have

$$\mathrm{Im}\, f_1 \leqslant \frac{\lambda}{(\log s)^{\varepsilon}}$$

In this particular example one can simplify the expression of the Mandelstam representation, i.e., write F as

$$F = C + \frac{s}{\pi}\int \frac{\sigma(s')\,ds'}{s'(s'-s)} + \text{permutations}$$

$$- \frac{u}{\pi^2}\int \frac{\rho(s',t')\,ds'dt'}{(s'-s)(t'-t)(s'+t'-4)} + \text{permutations}$$

One has of course to adjust σ to make $\mathrm{Im}\, f_0$ satisfy the unitarity inequalities.

V. CONCLUDING REMARKS

We have found some necessary conditions on the scattering amplitude which follow from the postulate that the double spectral function is positive. Many other consequences could be obtained from this postulate. For instance, it is possible to calculate a bound on scattering lengths for $\ell > 0$.

It is not yet completely clear that our conditions,

$$\sigma_t < C\,(\log s)^{-1},$$

are in contradiction with experiment.

We have also shown that our necessary conditions are nearly saturated by explicit examples which satisfy $0 \leqslant \mathrm{Im}\, f_\ell \leqslant 1$. What remains to be done is to generalize these examples and find a characterization of the ρ's which is invariant under the cycle of singular and non-linear operations described by Atkinson in the construction of an amplitude satisfying elastic unitarity exactly.

ACKNOWLEDGEMENTS

The author thanks T. Kinoshita and S.W. MacDowell for discussions on this subject.

APPENDIX A

We want to show that

$$Q_{\ell+1}(x)/Q_\ell(x)$$

is a decreasing function of x which is equivalent to

$$Q_{\ell+1}(x)\,Q_\ell(X) - Q_{\ell+1}(X)\,Q_\ell(x) > 0$$

for $x > X$.

From the Darboux-Christoffel formula we have

$$(z'_1 - z'_2)\sum_0^n (2m+1)\,P_m(z'_1)\,P_m(z'_2)$$
$$= (n+1)\left[P_{n+1}(z'_1)\,P_n(z'_2) - P_n(z'_1)\,P_{n+1}(z'_2)\right]$$

multiplying by

$$\frac{1}{2(z_1 - z'_1)} \times \frac{1}{2(z_2 - z'_2)}$$

and integrating on z'_1 and z'_2 in the integral -1 $+1$ we get

$$\frac{1}{4}\int_{-1}^{+1}\int_{-1}^{+1}\frac{z'_1 - z'_2}{(z_1 - z'_1)(z_2 - z'_2)}\,dz'_1\,dz'_2$$
$$+ (z_1 - z_2)\sum_1^n (2m+1)\,Q_m(z_1)\,Q_m(z_2)$$
$$= (n+1)\left[Q_{n+1}(z_1)\,Q_n(z_2) - Q_{n+1}(z_2)\,Q_n(z_1)\right]$$

and, taking the limit $n \to \infty$, we get

$$\frac{1}{4}\int_{-1}^{+1}\int_{-1}^{+1} \frac{z_1' - z_2'}{(z_1 - z_1')(z_2 - z_2')}\, dz_1' \, dz_2'$$

$$= -(z_1 - z_2) \sum_{1}^{\infty} (2m+1)\, Q_m(z_1)\, Q_m(z_2)$$

and subtracting the two equations

$$(z_1 - z_2) \sum_{n+1}^{\infty} (2m+1)\, Q_m(z_1)\, Q_m(z_2)$$

$$= (n+1)\left[Q_n(z_1)\, Q_{n+1}(z_2) - Q_n(z_2)\, Q_{n+1}(z_1)\right]$$

for $z_1 > 1$, $z_2 > 1$, Q_m is positive; hence if $z_1 < z_2 < 1$, the right-hand side is positive, which proves our statement.

APPENDIX B

We have

$$\frac{\sqrt{s}}{k}\,\mathrm{Im}\, f_1 = 4\lambda \int_{t_0}^{\infty} \frac{dt}{t(\log t)^{1+\varepsilon}} \frac{Q_1\left(1+\frac{2t}{s-4}\right)}{(\log s)^{1+\varepsilon}}$$

$$+ \frac{4\lambda}{(\log s)^{1+\varepsilon} s(s-4)} \int_{t_0}^{\infty} \frac{dt\,(t-4)}{(\log t)^{1+\varepsilon}} Q_1\left(1+\frac{2t}{s-4}\right)$$

The first term is very easy to bound :

$$Q_1\left(1+\frac{2t}{s-4}\right) < Q_0\left(1+\frac{2t_0(s)}{s-4}\right) < \frac{1}{2}\log\left(\frac{s}{4}\right)$$

The second term can be bounded by :

$$\frac{4\lambda}{(\log s)^{1+\varepsilon} s(s-4)} \int_{t_0}^{s(\log s)^{\varepsilon/2}} \frac{dt\,(t-4)}{(\log t)^{1+\varepsilon}} \frac{1}{2}\log\left(\frac{s}{4}\right)$$

$$+ \frac{4\lambda C}{(\log s)^{1+\varepsilon} s(s-4)} \int_{s(\log s)^{\varepsilon/2}}^{\infty} dt\, \frac{(t-4)}{t^3}(s-4)^3$$

It is clear that none of these contributions exceeds the unitarity bound for λ small enough.

REFERENCES

1) See for instance, A. Martin, Nuovo Cimento 42, 930 (1966).

2) M. Froissart, Phys.Rev. 123, 2047 (1961);
V.N. Gribov, Soviet Physics JETP 41, 1962 (1961):
see also, A. Martin, Phys.Letters 1, 72 (1962).

3) G. Mahoux and A. Martin, Nuovo Cimento 33, 883 (1964).

4) D. Atkinson, NORDITA preprint (1968) - to appear in Nuclear Phys. and Rutherford Laboratory preprint (1968).

5) V.N. Gribov, Proceedings of the 1960 Annual International Conference on High Energy Physics at Rochester, p.340.

6) See for instance, W. Magnus and S. Oberettinger, "Formulas and Theorems for Mathematical Physics", Chelsea Publ.Co., New York (1954).

7) H. Epstein and V. Glaser, private communication.

8) Y.S. Jin and A. Martin, Phys.Rev. 135, B1375 (1963).

9) R. Oehme, Phys.Rev. 130, 424 (1963).