Tôhoku Math. Journ.
25 (1973), 213-218.

ON POSITIVELY CURVED RIEMANNIAN MANIFOLDS WITH BOUNDED VOLUME

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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(Received August 17, 1972)

It is very interesting and important to investigate the relations among curvatures, volumes and topological structures on Riemannian manifolds of positive curvature. The following theorems are well known.

THEOREM A. (Bishop-Crittenden [5]) *Let M be an n-dimensional complete Riemannian manifold with sectional curvature $K \geqq 1$. Then we have*

$$\mathrm{vol}\, M \leqq \mathrm{vol}\, S^n ,$$

and equality holds only if M is isometric to a sphere S^n with constant curvature 1, *where we denote the volume of M by* $\mathrm{vol}\, M$.

THEOREM B. (Heim [7]) *Let M be an n-dimensional complete Riemannian manifold with sectional curvature $K \geqq 1$ and* $\mathrm{vol}\, M > (1/2) \mathrm{vol}\, S^n$. *Then M is a homotopical sphere.*

In this paper we give a simple proof of Theorem B and prove the following theorem.

THEOREM C. *Let M be an n-dimensional complete Riemannian manifold with sectional curvature $K \geqq 1$ and* $\mathrm{vol}\, M \geqq (1/2) \mathrm{vol}\, S^n$. *Then M is a homotopical sphere or isometric to the real projective space with constant curvature* 1.

1. Preliminaries.

(a) **Volumes** (cf. Berger-Gauduchon-Mazet [4]). Let M be a compact Riemannian manifold, g be its Riemannian metric and v_g be the canonical measure on M. For a point $m \in M$ let v_m be the volume element of the tangent space M_m to M at m, $\exp_m$ be the exponential mapping of M_m onto M and let U_m be the maximal open neighborhood around the origin of M_m which $\exp_m$ maps diffeomorphically onto its image. We call

* The second author was supported by the Sonderforschungsbereich "Theoretische Mathematik" of the university of Bonn.

U_m the *injective neighborhood* of m. Let $\theta(x)$ be the Jacobian determinant of $\exp_x$ for any point $x \in U_m$. Then we have

$$\text{vol}\, M = \int_{U_m} \theta \cdot v_m \,. \tag{1}$$

We use Jacobi fields to calculate volume. Let $U_\varepsilon(m)$ be a normal neighborhood around m with radius $\varepsilon > 0$. For $0 < r < \varepsilon$ and a unit vector u on M we give the expression of $\theta(ru)$. Let c_u be the geodesic starting from m, with the initial direction u. Let $\{y_2, \cdots, y_n\}$ be an orthonormal basis of orthogonal complement $u^\perp$ of u. Let $Y_i(s)$ be the Jacobi field along c_u such that $Y_i(0) = 0$, $Y_i'(0) = y_i/r$. Then we have

$$\theta(ru) = \det\,(\langle Y_i(r),\ Y_j(r)\rangle)^{1/2},\ i, j = 2, \cdots, n \tag{2}$$

where $\langle\, ,\rangle$ is the inner product on M.

(b) **Rauch's comparison theorem** (cf. [6]). For a point $m \in M$ let G_m be the set of all 2-dimensional linear subspaces of M_m and we put $G_M = \bigcup_{m\in M} G_m$. If c: $[0, l] \to M$ is a differentiable curve, we denote by G_c the set of all 2-dimensional linear subspaces of $M_{c(t)}$ each of which contains a tangent vector to c. For any $\sigma \in G_M$ let (u, v) be a basis of σ. Then we denote the sectional curvature by $K_\sigma = K(u, v)$.

THEOREM. (Rauch) *Let S^n be an n-dimensional sphere with constant curvature* 1 *and M be an n-dimensional Riemannian manifold* ($n \geqq 2$). *Let c: $[0, l] \to M$ and $\tilde{c}$; $[0, l] \to S^n$ be geodesics. Let Y (resp. $\tilde{Y}$) be the Jacobi field along c (resp. $\tilde{c}$) such that*

$$Y(0) = \tilde{Y}(0) = 0\,,$$
$$\langle Y', \dot{c}\rangle(0) = \langle Y', \dot{\tilde{c}}\rangle(0) = 0\,,$$
$$\|Y'(0)\| = \|\tilde{Y}'(0)\|\,,$$

where Y' is the covariant derivative with respect to the direction $\dot{c}$ and $\|\ \|$ is the norm. Furthermore we assume $K_\sigma \geqq 1$ for all $t \in [0, l]$ and $\sigma \in G_{c(t)}$, and c has no conjugate points on the interval $(0, l)$. *Under these conditions we have the inequality*

$$\|Y(t)\| \leqq \|\tilde{Y}(t)\|$$

for all $t \in [0, l]$.

If we have $\|Y(t_0)\| = \|\tilde{Y}(t_0)\| \neq 0$ for some $t_0 \in (0, l]$, the equality $K(Y(t), \dot{c}(t)) = 1$ holds good for all $t \in [0, t_0]$.

(c) **Diameter.** Let $d(M)$ be the diameter of M. Then the following two theorems are well known.

THEOREM. (Myers) *Let M be a complete Riemannian manifold with sectional curvature $K \geqq k > 0$, k = constant. Then M is compact and $d(M) \leqq \pi/\sqrt{k}$ holds good.*

THEOREM. (Berger [3]) *Let M be a complete Riemannian manifold with sectional curvature $K \geqq k > 0$, k = constant. If $d(M)$ is larger than $\pi/2\sqrt{k}$, then M is a homotopical sphere.*

We give a new proof of Berger's theorem in the appendix.

2. **Proof of Theorem B.** We assume that M is not a homotopical sphere. By Berger's theorem we have $d(M) \leqq \pi/2$. Let m be an arbitrary point of M and U_m be the injective neighborhood of m. Then U_m is contained in the open ball with center 0 in M_m and radius $\pi/2$. By using (1), (2), Rauch's comparison theorem and a property of Gramian determinant we have

$$\mathrm{vol}\, M = \int_{Um} \theta \cdot v_m \leqq \frac{1}{2} \mathrm{vol}\, S^n .$$

This contradicts our assumption: $\mathrm{vol}\, M > (1/2)\, \mathrm{vol}\, S^n$. Hence M is a homotopical sphere.

3. **Proof of Theorem C.** We assume that M is not a homotopical sphere. By Theorem B and Berger's theorem we have $\mathrm{vol}\, M = (1/2)\, \mathrm{vol}\, S^n$ and $d(M) \leqq \pi/2$. Let m be an arbitrary point of M and U_m be the injective neighborhood of m. Then U_m is contained in the open ball with center 0 in M_m and radius $\pi/2$. By using (1), (2), Rauch's comparison theorem and a property of Gramian determinant we have

$$\mathrm{vol}\, M \leqq \frac{1}{2} \mathrm{vol}\, S^n .$$

On the other hand we have $\mathrm{vol}\, M = (1/2)\, \mathrm{vol}\, S^n$. Hence it follows from (1), (2) and Rauch's comparison theorem that U_m coincides with the open ball with center 0 and radius $\pi/2$, and for any geodesic arc c: $[0, \pi/2] \rightarrow M$ starting from m we have $K_\sigma = 1$ for all $\sigma \in G_c$. In particular we have $K_\sigma = 1$ for all $\sigma \in G_m$. So M is a space of constant curvature 1. Since $\mathrm{vol}\, M$ is equal to $(1/2)\, \mathrm{vol}\, S^n$, M is isometric to a real projective space with constant curvature 1.

4. **Appendix: Proof of Berger's theorem.** The second author gave previously a new proof of Berger's theorem in Japanese [9], which we reproduce here. We divide the proof into several steps.

(i) *M is simply connected.*

PROOF OF (i). We assume that M is not simply connected. Let $\tilde{M} \xrightarrow{\pi} M$ be the universal covering manifold and $\tilde{M}$ have the natural Riemannian metric induced by the Riemannian metric on M. Then we have the inequalities $K \geqq k > 0$ for all $\sigma \in G_{\tilde{M}}$. Let p and q be two points on M such that $d(p, q) = d(M)$. Since M is not simply connected, the set $\pi^{-1}(p)$ contains at least two different points $\tilde{p}_1$ and $\tilde{p}_2$. Let $\tilde{G}$ be a shortest geodesic arc joining $\tilde{p}_1$ to $\tilde{p}_2$. By Myer's theorem we have the inequality $L(\tilde{G}) \leqq \pi/\sqrt{k}$, where $L(\tilde{G})$ is the length of $\tilde{G}$. Then the curve $G = \pi(\tilde{G})$ is a geodesic loop starting at p and satisfies the inequalities $L(G) = L(\tilde{G}) \leqq \pi/\sqrt{k}$. We denote G by $c: [0, l] \rightarrow M$, $l = L(G)$, $c(0) = c(l) = p$. From now on we mean the parameter of geodesic by the arc length measured from its initial point. Because of $d(p, q) = d(M)$ there exists a shortest geodesic $a: [0, m] \rightarrow M$ which joins p to q and satisfies $a(0) = p$, $a(m) = q$ and $\langle \dot{c}(0), \dot{a}(0) \rangle \geqq 0$ (c.f [2], [8]), where $\dot{a}(0)$ is the unit tangent vector to the curve a at $a(0)$. Let r_0 be a nearest point on G to q, i.e, $d(q, r_0) = d(q, G)$. Then we may assume $r_0 \neq p$, because of $d(p, q) = d(M)$. We denote a shortest geodesic between r_0 and q by $b: [0, u] \rightarrow M$, $b(0) = c(s_0) = r_0$, $b(u) = q$. Then we have $\langle \dot{b}(0), \dot{c}(s_0) \rangle = 0$ and $u \leqq \pi/2\sqrt{k}$, (cf. [1]). Two points p and r_0 divide G into two subarcs. We denote the shorter one by G_1. For instance we assume that G_1 is $c: [0, s_0] \rightarrow M$. Let $0 = s_l < s_{l-1} < \cdots < s_1 < s_0$ be a subdivision such that each subarc $c|[s_i, s_{i-1}]$ is a shortest geodesic. We put $c(s_i) = r_i$ $(i = 0, 1, \cdots, l)$. In particular we have $r_l = p$. Now we construct geodesic triangles $\Delta\hat{q}\hat{r}_0\hat{r}_1, \Delta\hat{q}\hat{r}_1\hat{r}_2, \cdots, \Delta\hat{q}\hat{r}_{l-1}\hat{r}_l$ on a 2-dimensional sphere $S^2(k)$ with constant curvature k, each of which is isometric to geodesic triangles $\Delta qr_0r_1, \Delta qr_1r_2, \cdots, \Delta qr_{l-1}r_l$ on M: the corresponding sides of the corresponding triangles have same length respectively. And we attach geodesic triangles $\Delta\hat{q}\hat{r}_0\hat{r}_1, \Delta\hat{q}\hat{r}_1\hat{r}_2, \cdots, \Delta\hat{q}\hat{r}_{l-1}\hat{r}_l$ on $S^2(k)$ and obtain a geodesic polygon $\hat{q}\hat{r}_0\hat{r}_1 \cdots \hat{r}_{l-1}\hat{r}_l$ on $S^2(k)$. We may consider the point $\hat{q}$ as the north pole of $S^2(k)$. By using Toponogov's comparison theorem (c.f [6]) we have the following relations of the angles:

(a) $\measuredangle \hat{q}\hat{r}_0\hat{r}_1 \leqq \measuredangle qr_0r_1 = \pi/2$,

(b) $\measuredangle \hat{r}_0\hat{r}_1\hat{r}_2 = \measuredangle \hat{r}_0\hat{r}_1\hat{q} + \measuredangle \hat{q}\hat{r}_1\hat{r}_2 \leqq \measuredangle r_0r_1q + \measuredangle qr_1r_2 = \pi$, $\measuredangle \hat{r}_1\hat{r}_2\hat{r}_3 \leqq \pi, \cdots, \measuredangle \hat{r}_{l-2}\hat{r}_{l-1}\hat{r}_l \leqq \pi$,

(c) the length of the geodesic polygon $\hat{r}_0\hat{r}_1 \cdots \hat{r}_l = L(G_1) \leqq \pi/2\sqrt{k}$.
By the relations (a), (b) and (c) we can see that the point $\hat{r}_l$ is contained in the northern hemisphere of $S^2(k)$. Hence we have $d(p, q) = d(\hat{q}, \hat{r}_l) \leqq \pi/2\sqrt{k}$. This contradicts our assumption $d(p, q) = d(M) > \pi/2\sqrt{k}$. M is simply connected.

(ii) *For two points r and s on M let $\Omega_{r,s}$ be the set of all piecewise* *differentiable curves joining r to s. Let r and s be two points on M such that $\Omega_{r,s}$ be non-degenerate. If M is not a homotopical sphere, $\Omega_{r,s}$ contains a geodesic with length $l, \pi/\sqrt{\alpha} \leqq l \leqq \pi/\sqrt{k}$, where $\alpha = \mathrm{Max}_{\sigma \in G_M} K_\sigma$.*

PROOF OF (ii). If $\Omega_{r,s}$ contains only geodesics with length $l, l > \pi/\sqrt{k}$ or $l < \pi/\sqrt{\alpha}$, then their indices are not less than $n-1$ or equal to 0. By the fundamental theorem of Morse theory we have $\pi_i(\Omega_{r,s}) = 0, 1 \leqq i \leqq n-2$. On the other hand, by homotopy exact sequence of path space we have $\pi_i(M) = \pi_{i-1}(\Omega_{r,s})$. Hence we have $\pi_i(M) = 0, 2 \leqq i \leqq n-1$. Since M is simply connected, we have $\pi_i(M) = 0, 1 \leqq i \leqq n-1$. Hence M is a homotopical sphere. This contradicts our assumption. Hence $\Omega_{r,s}$ contains a geodesic with length $l, \pi/\sqrt{\alpha} \leqq l \leqq \pi/\sqrt{k}$.

(iii) *If M is not a homotopical sphere, we have a geodesic loop at p with length $l \leqq \pi/\sqrt{k}$ for all points $p \in M$.*

PROOF OF (iii). The set $\{q \in M \mid \Omega_{p,q}$ is non-degenerate$\}$ is dense in M. Let $\{q_i\}_{i=1,2,\dots}$ be a sequence of points on M such that each Ω_{p,q_i} is non-degenerate and the sequence $(q_i)_{i=1,2,\dots}$ converges to the point p. By (ii) we have a geodesic c_i of Ω_{p,q_i}, $i = 1, 2, \cdots$ whose length l satisfies $\pi/\sqrt{\alpha} \leqq l \leqq \pi/\sqrt{k}$. We can choose a converging subsequence of $\{c_i\}$. The limit geodesic c is a geodesic loop at p with length $l, \pi/\sqrt{\alpha} \leqq l \leqq \pi/\sqrt{k}$.

(iv) *Proof of Berger's theorem.*

We assume that M is not a homotopical sphere. Let p and q be two points on M such that $d(p, q) = d(M)$. By (iii) we have a geodesic loop at p with length $l, \pi/\sqrt{\alpha} \leqq l \leqq \pi/\sqrt{k}$. By the same argument as (i), we have $d(M) \leqq \pi/2\sqrt{k}$. This contradicts our assumption. Hence M is a homotopical sphere.

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