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ON PRACTICAL CONDITIONS FOR THE EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE GENERAL EQUALITY QUADRATIC PROGRAMMING PROBLEM

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We present practical conditions under which the existence and uniqueness of a finite solution to a given equality quadratic program may be examined. Different sets of conditions allow this examination to take place when null-space, range-space or Lagrangian methods are used to find stationary points for the quadratic program.

Key words: Equality Quadratic Program, Existence and Uniqueness of Solutions, Null-Space Methods, Range-Space Methods, Lagrangian Methods.

1. Introduction

Many methods for the solution of the nonlinear programming problem

$$\text{NLP:} \quad \text{minimize} \quad F(x), \quad \times x \in \mathbb{R}^n, \quad \text{subject to} \quad c(x) \geqslant 0$$

proceed by finding the solution p and associated vector of Lagrange multipliers λ of a sequence of equality quadratic programming problems of the form

$$\text{EQP:} \quad \text{minimize} \quad \tfrac{1}{2}p^{\mathrm{T}}Hp + g^{\mathrm{T}}p, \quad p \in \mathbb{R}^n, \quad \text{subject to} \quad Ap = -d$$

for appropriate matrices H ($n \times n$ symmetric) and A ($t \times n$, $t \leqslant n$). (See, for example, Gill, Murray and Wright, 1981, Section 6.7). In this paper we examine conditions under which EQP can be shown to have a finite solution. Although it is possible to derive analogous conditions when A is rank deficient, this considerably complicates any computational implementation of the results and adds little to the theory. Consequently, throughout this paper, we shall assume that A is of full row rank.

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Most methods for solving EQP fall into one of three classes; null-space, Lagrangian or range-space methods. Null-space methods have the desirable property that is possible to deduce that EQP has a unique solution while attempting to find it. To date, practical range-space and Lagrangian methods have not been attributed with this property.

It is the purpose of this paper to show that it is also possible to demonstrate that EQP has a unique solution in an efficient manner when either Lagrangian or range-space methods are used. The characterization is easy to obtain from the matrix operators required by such methods.

We briefly describe the three direct methods for EQP mentioned above:

(i) Null-space methods: these methods obtain p and λ by constructing a matrix Z such that $AZ=0$ and $\operatorname{rank}(A^{\mathrm{T}} \vdots Z)=n$ and solving the unconstrained problem

$$\underset{p_Z \in \mathbb{R}^{n-t}}{\text{minimize}} \quad \tfrac{1}{2} p_Z^{\mathrm{T}} Z^{\mathrm{T}} H Z p_Z + p_Z^{\mathrm{T}} Z^{\mathrm{T}} (g + H A^{\mathrm{T}} p_a), \tag{1.1}$$

where $AA^{\mathrm{T}} p_a = -d$. Then

$$p = Z p_Z + A^{\mathrm{T}} p_a \quad \text{and} \quad \lambda = (AA^{\mathrm{T}})^{-1} A (Hp + g).$$

The calculation of p and λ may theefore be achieved by solving the linear equations

$$AA^{\mathrm{T}} p_a = -d, \tag{1.2a}$$

$$Z^{\mathrm{T}} H Z p_Z = -Z^{\mathrm{T}} (g + H A^{\mathrm{T}} p_a), \tag{1.2b}$$

and

$$AA^{\mathrm{T}} \lambda = A(Hp + g). \tag{1.2c}$$

(ii) Lagrangian methods: these methods find p and λ directly from the Kuhn-Tucker equations (Kuhn and Tucker, 1951) for the problem, viz:

$$\begin{pmatrix} H & A^{\mathrm{T}} \\ A & 0 \end{pmatrix} \begin{pmatrix} p \\ -\lambda \end{pmatrix} = \begin{pmatrix} -g \\ -d \end{pmatrix}. \tag{1.3}$$

(iii) Range-space methods: if H is nonsingular, range-space methods find p and λ by a natural decomposition of the Kuhn–Tucker equations given by

$$AH^{-1}A^{\mathrm{T}} \lambda = AH^{-1} g - d \tag{1.4a}$$

and

$$Hp = A^{\mathrm{T}} \lambda - g. \tag{1.4b}$$

If H is singular, alternative partitions of the Kuhn–Tucker equations may be used to find p and λ (see Dembo, 1982; Gould, 1983). However these techniques are relatively cumbersome and in these circumstances Lagrangian methods may sometimes be preferred.

The relative virtues of these different techniques are expounded by various authors (see, e.g., Djang, 1979; Fletcher, 1981; Gill et al., 1982). Briefly, null-space methods are most useful when t is large relative to n and the other methods are more suitable when t is small relative to n. Range-space methods may be preferred to Lagrangian methods when H is so structured that (1.4b) is trivial to solve. Lagrangian methods are sometimes preferred to either of the other techniques as they are simpler to use and may have advantages when H and A are large and sparse. Null-space methods have a significant advantage in that formulation (1.1) may also be used to say when p is a second order point for EQP. For completeness, we quote

Theorem 1.1. *Suppose* EQP *is as given with A of full row rank t and Z is constructed so that* $AZ = 0$ *and* $\operatorname{rank}(A^{\mathrm{T}} \vdots Z) = n$. *Then*

(i) EQP *has a strong minimizer if and only if $Z^{\mathrm{T}}HZ$ is positive definite.*

(ii) EQP *has weak minimizers if $Z^{\mathrm{T}}HZ$ is positive semi-definite with $Z^{\mathrm{T}}HZ$ singular and* (1.2b) *compatible.*

(iii) EQP *has no finite solution in other cases* (*i.e. when* (1.2b) *is incompatible or $Z^{\mathrm{T}}HZ$ is indefinite*).

Proof. The proof follows directly by applying analogous results for the unconstrained case (for example, Gill, Murray and Wright, 1981, pp. 65–66) to (1.1). (See also Orden, 1964). □

The projected Hessian matrix, $Z^{\mathrm{T}}HZ$, is thus seen to play a double rôle in null-space methods. As EQP has a strong minimizer when $Z^{\mathrm{T}}HZ$ is positive definite, it is common that (1.2b) is solved by forming a Cholesky factorization of the projected Hessian. Provided that $Z^{\mathrm{T}}HZ$ is 'sufficiently' positive definite (see Gill and Murray, 1978) the factorization exists and possesses excellent numerical properties. A failure of the factorization process indicates only that $Z^{\mathrm{T}}HZ$ is not sufficiently positive definite (i.e.; numerically indefinite). When $Z^{\mathrm{T}}HZ$ is 'sufficiently' indefinite, a failure of the factorization process indicates that EQP has no finite minimizer.

Now consider Lagrangian methods. We would ideally like to classify any stationary point found by solving (1.3) in the process of finding such a point. As the Kuhn-Tucker matrix

$$K = \begin{pmatrix} H & A^{\mathrm{T}} \\ A & 0 \end{pmatrix}$$

is symmetric, we should like to solve (1.3) using one of the symmetric (indefinite) matrix factorizations (see, for example, Bunch and Parlett (1971), Fletcher (1976), Bunch and Kaufman (1977), Dax (1982)) which maintain symmetry at all stages and require roughly half the storage and effort needed by good methods for non-symmetric systems.

There have been, however, a number of previous classifications of the type of stationary point encountered when solving (1.3) (Mann 1943), Afriat (1951), Debrau

(1952), Väliaho (1982) - all of these characterisations have been in terms of the signs of the principal minors of K. All practical methods for solving (1.3) which obtain the required principal minors (for instance, triangulation by elementary stabilized matrices and by plane rotations—see Wilkinson, 1965, pp. 236-240) ignore the symmetry of K. In Section 2, we give a characterisation of the stationary points of (1.3) in terms of the inertia (the number of positive negative and zero eigenvalues) of the coefficient matrix K (Theorem 2.1). This information is readily available when any of the previously mentioned symmetric factorizations are used to solve (1.3). The importance of this characterisation is that, whenever EQP has no finite solution, it may be used constructively to find feasible directions along which the objective function may be decreased without bound—an importance ingredient in many nonlinear programming algorithms (see Section 4).

Finally, consider range-space methods. When H is nonsingular, we should again like to classify any stationary point found when solving (1.4) as we find such a point. Both of the coefficient matrices H and $AH^{-1}A^{\mathrm{T}}$ are symmetric; we should therefore like to solve (1.4) by forming symmetric factorizations of both matrices. A characterization of the nature of the solution to (1.4) is possible knowing just the inertia of H and $AH^{-1}A^{\mathrm{T}}$ (Corollary 2.3) and these inertia are obtained trivially during the factorization. When H is singular, it is also possible to construct range-space methods to find stationary points for EQP (see Gould, 1983). The matrices defining the process are once again symmetric and a characterization of the stationary points obtained is again possible knowing the inertia of these matrices (Theorem 2.2).

In Section 2, we state our crucial results. The proofs follow in Section 3 and a discussion is given in Section 4.

2. The crucial theorems

Theorem 2.1. *Let K be the Kuhn-Tucker matrix*

$$K = \begin{pmatrix} H & A^{\mathrm{T}} \\ A & 0 \end{pmatrix},$$

suppose A is of full row rank and let k_- and k_0 be the number of negative and zero eigenvalues of K. Then

(i) EQP *has a strong minimizer if and only if $k_- = t$, $k_0 = 0$;*

(ii) EQP *has weak minimizers if and only if $k_- = t$, $k_0 > 0$ and (1.3) is consistent;*

(iii) EQP *has no finite minimizers in other cases (i.e., when $k_- > t$ or (1.3) is inconsistent).*

Theorem 2.2. *Let H be of rank $r \leq n$ and A be of full row rank. Suppose the rows and columns of H have been ordered so that the principal $r \times r$ submatrix H_1 of H is* *nonsingular. Suppose furthermore that this induces partitions*

$$H=\left(\begin{array}{c|c} H_1 & H_2^{\mathrm{T}} \\ \hline H_2 & H_3 \end{array}\right), \quad A=(A_1 \,\vdots\, A_2) \quad \text{and} \quad g=\left(\begin{array}{c} g_1 \\ \hline g_2 \end{array}\right),$$

where A_1 *is* $t\times r$ *and* g_1 *is* $t\times 1$. *Define matrices C, E and G and the vector c by*

$$C=\left(\begin{array}{c} H_2 \\ \hline A_1 \end{array}\right), \quad E=\left(\begin{array}{c|c} H_3 & A_2^{\mathrm{T}} \\ \hline A_2 & 0 \end{array}\right), \quad G=CH_1^{-1}C^{\mathrm{T}}-E \quad \text{and} \quad c=\begin{pmatrix} g_2 \\ d \end{pmatrix}.$$

Then

(i) EQP *has a strong minimizer if and only if the sum of the number of positive eigenvalues of* H_1 *and the number of netative eigenvalues of* G *is* n;

(ii) EQP *has weak minimizers if and only if* G *is singular, the sum of the number of positive eigenvalues of* H_1 *and the number of nonpositive eigenvalues of* G *is* n *and*

$$Gz=CH_1^{-1}g_1-c \tag{2.1}$$

is consistent;

(iii) EQP *has no finite minimizers in other cases* (*i.e., when* (2.1) *is inconsistent or when the sum of the number of positive eigenvalues of* H_1 *and the number of nonpositive eigenvalues of* G *is less than* n).

When H is nonsingular, $C=A$, $E=0$, $G=AH^{-1}A^{\mathrm{T}}$, $c=d$ and $n=r$. Furthermore, in this case, the sum of the number of negative eigenvalues of H and the number of positive eigenvalues of H is n, and we obtain the immediate

Corollary 2.3. *Let H be nonsingular and A of full row rank. Then*

(i) EQP *has a strong minimizer if and only if the number of negative eigenvalues of* H *is equal to the number of negative eigenvalues of* $AH^{-1}A^{\mathrm{T}}$;

(ii) EQP *has weak minimizers if and only if* $AH^{-1}A^{\mathrm{T}}$ *is singular, the number of negative eigenvalues of* H *is equal to the number of nonpositive eigenvalues of* $AH^{-1}A^{\mathrm{T}}$ *and* (1.4a) *is consistent*;

(iii) EQP *has no finite minimizers in other cases* (*i.e., when* $AH^{-1}A^{\mathrm{T}}$ *is singular and* (1.4a) *is not consistent or when the number of nonpositive eigenvalues of* $AH^{-1}A^{\mathrm{T}}$ *is less than the number of negative eigenvalues of* H).

3. Proofs of Theorems 2.1 and 2.2

Definition. The inertia of the symmetric $n\times n$ matrix N is the triple

$$\mathrm{In}(N)=(n_+, n_-, n_0)$$

where n_+, n_- and n_0 are respectively the number of positive, negative and zero eigenvalues of N.

We shall use the fact that $n_+ + n_- + n_0 = n$ and that, for any nonsingular matrix R, the inertia of RNR^{T} and N are identical (Sulvesters law of inertia). We establish theorems 2.1 and 2.2 from the following lemmas.

Lemma 3.1. *Let I_t be the $t \times t$ identity matrix and let the matrix B have the form*

$$B = \begin{pmatrix} 0 & 0 & I_t \\ 0 & S & 0 \\ I_t & 0 & 0 \end{pmatrix}$$

for any $p \times p$ matrix S. Then B has t eigenvalues -1, t eigenvalues 1 and its remaining eigenvalues the same as those of S.

Proof. Straightforward.

Lemma 3.2. *Let Z be any matrix which satisfies the conditions of Theorem 1.1 and K be the Kuhn–Tucker matrix from (1.3). Then there is a nonsingular matrix R such that*

$$RKR^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & I_t \\ 0 & Z^{\mathrm{T}}HZ & 0 \\ I_t & 0 & 0 \end{pmatrix}.$$

Proof. As A is of full rank t, there exist a nonsingular lower triangular matrix L and a (partitioned) orthogonal matrix $(\tilde{Y} \,\vdots\, \tilde{Z})$ such that $A(\tilde{Y} \,\vdots\, \tilde{Z}) = (L \,\vdots\, 0)$ and $\tilde{Y}$ is $n \times t$ (see, e.g. Golub and Van Loan, 1983, p. 146). The matrix $Z = \tilde{Z}$, so constructed, satisfies the conditions of theorem 1.1 and any other suitable Z is related by the equation $Z = \tilde{Z}U$ for some nonsingular $(n-t)\times(n-t)$ matrix U.

Let V be the nonsingular matrix

$$V = \left(\begin{array}{cc|c} I_t & 0 & 0 \\ 0 & U^{\mathrm{T}} & 0 \\ \hline 0 & 0 & I_t \end{array}\right) \cdot \left(\begin{array}{c|c} \tilde{Y}^{\mathrm{T}} & 0 \\ \tilde{Z}^{\mathrm{T}} & 0 \\ \hline 0 & I_t \end{array}\right).$$

Then

$$VKV^{\mathrm{T}} = \begin{pmatrix} \tilde{Y}^{\mathrm{T}}H\tilde{Y} & \tilde{Y}^{\mathrm{T}}HZ & L^{\mathrm{T}} \\ Z^{\mathrm{T}}H\tilde{Y} & Z^{\mathrm{T}}HZ & 0 \\ L & 0 & 0 \end{pmatrix}.$$

Now, letting W be the nonsingular matrix,

$$W = \begin{pmatrix} I_t & 0 & -\frac{1}{2}\tilde{Y}^{\mathrm{T}}H\tilde{Y}L^{-1} \\ 0 & I_{n-t} & -Z^{\mathrm{T}}H\tilde{Y}L^{-1} \\ 0 & 0 & L^{-1} \end{pmatrix},$$

It is straightforward to show that

$$W\begin{pmatrix} \tilde{Y}^{\mathrm{T}}H\tilde{Y} & \tilde{Y}^{\mathrm{T}}HZ & L^{\mathrm{T}} \\ Z^{\mathrm{T}}H\tilde{Y} & Z^{\mathrm{T}}HZ & 0 \\ L & 0 & 0 \end{pmatrix}W^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & I_t \\ 0 & Z^{\mathrm{T}}HZ & 0 \\ I_t & 0 & 0 \end{pmatrix}$$

and hence on setting $R = WV$ the lemma is established.

Lemma 3.3. *Let the matrices* H_1, C *and* G *be as defined in the statement of Theorem* 2.2. *Then*

$$\mathrm{In}(K) = \mathrm{In}(H_1) + \mathrm{In}(-G).$$

Proof (Haynsworth, 1968). Follows directly from the decomposition

$$K = \left(\begin{array}{c|c} I_r & 0 \\ \hline CH_1^{-1} & I_{n+t-r} \end{array}\right)\left(\begin{array}{c|c} H_1 & 0 \\ \hline 0 & -G \end{array}\right)\left(\begin{array}{c|c} I_r & H_1^{-1}C^{\mathrm{T}} \\ \hline 0 & I_{n+t-r} \end{array}\right)$$

and Sylvesters law of 'inertia.

Definition. Define the inertia of K, H_1, G and $Z^{\mathrm{T}}HZ$ as (k_+, k_-, k_0), (h_+, h_-, h_0), (g_+, g_-, g_0) and $(3_+, 3_-, 3_0)$ respectively.

Lemma 3.4.

$$(k_+, k_-, k_0) = (3_+, 3_-, 3_0) + (t, t, 0) \tag{3.1}$$

$$= (h_+, h_-, 0) + (g_-, g_+, g_0). \tag{3.2}$$

Proof. Equation (3.1) follows by applying Lemma 3.1 to the particular form of B, $B = RKR^{\mathrm{T}}$, formulated in Lemma 3.2. Equation (3.2) follows directly from Lemma 3.3, recalling that $h_0 = 0$ by assumption.

Proof of Theorem 2.1. From (3.1),

$$k_+ = t + 3_+, \quad k_- = t + 3_- \quad \text{and} \quad k_0 = 3_0.$$

From Theorem 1.1, EQP has a strong minimizer if and only if $3_0 = 3_- = 0$, i.e., $k_- = t$, $k_0 = 0$. This proves (i). The matrix $Z^{\mathrm{T}}HZ$ is positive semidefinite and singular if and only if $3_- = 0$, $3_0 > 0$ which is true if and only if $k_0 > 0$ and $k_- = 0$. Equation (1.3) is consistent if and only if EQP has at least one stationary point which is if and only if (1.3) is consistent. This proves (ii). If (1.3) is inconsistent, EQP has no stationary points and hence no minimizers. Finally EQP has no stationary points when $3_- > 0$, i.e., when $k_- > t$. Thus the theorem is established. □

Proof of Theorem 2.2. Combining (3.1) and (3.2) and equating coefficients

$$3_+ = h_+ + g_- - t, \quad 3_- = h_- + g_+ - t \quad \text{and} \quad 3_0 = g_0.$$

Furthermore, the dimensionality of $Z^{\mathrm{T}}HZ$, H_1 and G give

$$\mathfrak{z}_+ + \mathfrak{z}_- + \mathfrak{z}_0 = n - t, \quad h_+ + h_- = r \quad \text{and} \quad g_+ + g_- + g_0 = n + t - r.$$

Thus $\mathfrak{z}_- + \mathfrak{z}_0 = n - g - h_+$. Hence, from Theorem 1.1, EQP has a strong minimizer if and only if $\mathfrak{z}_0 = \mathfrak{z}_- = 0$, i.e., if and only if $g_- + h_+ = n$. This proves (i). The matrix $Z^{\mathrm{T}}HZ$ is positive semi-definite and singular if and only if $\mathfrak{z}_- = 0$ and $\mathfrak{z}_0 > 0$ which occurs if and only if $g_0 > 0$ and $\mathfrak{z}_- = n - (g_- + g_0) - h_+ = 0$. Equation (1.2b) is consistent if and only if EQP has a stationary point. (That is, if there is a vector $\binom{p}{-\lambda}$ such that (1.3) is satisfied). Using the nonsingularity of H_1, (1.3) may be decomposed to give

$$Gz = CH^{-1}g_1 - c \tag{3.3a}$$

and

$$H_1 p_1 = -g_1 - C^{\mathrm{T}}z \tag{3.3b}$$

where $p = \binom{p_1}{p_2}$, $g = \binom{g_1}{g_2}$, $z = \binom{p_2}{-\lambda}$ and the remaining quantities are as defined in the theorem. Thus as H_1 is nonsingular, (1.3) is consistent if and only if (3.3a) is consistent. This establishes (ii). Finally, if (3.3a) is inconsistent, EQP has no finite minimizers. Furthermore if $\mathfrak{z}_- > 0$ then $g_0 + g_- + h_+ < n$. Thus the theorem is proved. □

4. Remarks

There are other proofs of Theorems 2.1 and 2.2 (see Gould, 1983) which depend upon properties of the Schur complement of a partitioned matrix (see Cottle, 1974).

As we have remarked, our classification of the nature of stationary points of EQP using properties of the inertia of the relevant coefficient matrices is imporant because this inertia is available from symmetric factorizations of these matrices. Whenever simple changes (such as might result from active set methods for quadratic programming) are made to the coefficient matrices, it is often possible to update the existing factorizations rather than compute them afresh (see, e.q. Sorensen, 1977). Furthermore, if the coefficient matrices are significantly large and sparse, frontal elimination methods may be used to find sparse indefinite factorizations (Duff et al., 1979; Duff and Reid, 1982).

If EQP has no finite solution, it is often desirable to find a vector p along which the objective function may be decreased as much as desired while satisfying the constraints (for example, in many nonlinear programming algorithms). The use of indefinite symmetric factorizations to calculate such directions have been considered by (e.g.) Fletcher and Freeman (1977), Sorensen (1977), More ans Sorensen (1979) and Goldfarb (1980) in the unconstrained (i.e., $t = 0$) case. The null-space method of Bunch and Kaufman (1980) also uses such factorizations to calculate a suitable p. The results presented here form the basis for the null-space, Lagrangian and range-space procedures for calculating such search directions presented in Conn and Gould (1983).

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