# On presentations of Brauer-type monoids

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**Abstract**

We obtain presentations for the Brauer monoid, the partial analogue of the Brauer monoid, and for the greatest factorizable inverse submonoid of the dual symmetric inverse monoid. In all three cases we apply the same approach, based on the realization of all these monoids as Brauer-type monoids.

## 1 Introduction and preliminaries

The classical Coxeter presentation of the symmetric group $S_n$ plays an important role in many branches of modern mathematics and physics. In the semigroup theory there are several "natural" analogues of the symmetric group. For example the symmetric inverse semigroup $\mathcal{IS}_n$ or the full transformation semigroup $\mathcal{T}_n$. Perhaps a "less natural" generalization of $S_n$ is the so-called Brauer semigroup $\mathfrak{B}_n$, which appeared in the context of centralizer algebras in representation theory in [Br]. The basis of this algebra can be described in a nice combinatorial way using special *diagrams* (see Section 2). This combinatorial description motivated a generalization of the Brauer algebra, the so-called *partition algebra*, which has its origins in physics, see [Mar1]. This algebra leads to another finite semigroup, the *partition semigroup*, usually denoted by $\mathfrak{C}_n$. Many classical semigroups, in particular, $S_n$, $\mathcal{IS}_n$, $\mathfrak{B}_n$ and some others (again see Section 2) are subsemigroups in $\mathfrak{C}_n$.

In the present paper we address the question of finding a presentation for some subsemigroups of $\mathfrak{C}_n$. As we have already mentioned, for $S_n$ this is a famous and very important result, where the major role is played by the so-called *braid relations*. Because of the "geometric" nature of the generators of the semigroups we consider, our initial motivation was that the additional relations for our semigroups would be some kind of "singular deformations" of the braid relations (analogous to the case of the singular braid monoid, see [Ba, Bi]). In particular, we wanted to get a complete list of "deformations" of the braid relations, which can appear in our cases. Surprizingly enough, in some cases it turned out that the variations of the braid relations are not enough. For example, already for the Brauer semigroup $\mathfrak{B}_n$ there appears the "ghost relation" (3.5), which we can not interpret as any kind of deformation of the braid relations. Analogous effect also happens for $\mathcal{P}\mathfrak{B}_n$.

As the main results of the paper we obtain a presentation for the semigroup $\mathfrak{B}_n$ (see Section 3), its partial analogue $\mathcal{P}\mathfrak{B}_n$ (which can be also called the *rook Brauer monoid*, see Section 5, and is a kind of mixture of $\mathfrak{B}_n$ and $\mathcal{IS}_n$), and a special inverse subsemigroup $\mathcal{IT}_n$ of $\mathfrak{C}_n$, which is isomorphic to the greatest factorizable inverse submonoid of the dual symmetric inverse monoid (see Section 4). The technical details in all cases are quite different, however, the general approach is the same. We first "guess" the relations and in the standard way obtain an epimorphism from the semigroup $T$, given by the corresponding presentation, onto the semigroup we are dealing with. The only problem is to show that this epimorphism is in fact a bijection. For this we have to compare the cardinalities of the semigroups. In all our cases the symmetric group $S_n$ is the group of units in $T$. The product $S_n \times S_n$ thus acts on $T$ via multiplication from the left and from the right. The idea is to show that each orbit of this action contains a very special element, for which, using the relations, one can estimate the cardinality of the stabilizer. The necessary statement then follows by comparing the cardinalities.

**Acknowledgments.** The paper was written during the visit of the first author to Uppsala University, which was supported by the Swedish Institute. The financial support of the Swedish Institute and the hospitality of Uppsala University are gratefully acknowledged. For the second author the research was partially supported by the Swedish Research Council.

# 2 Brauer type semigroups

For $n \in \mathbb{N}$ we denote by $S_n$ the *symmetric group* of all permutations on the set $\{1, 2, \dots, n\}$. We will consider the natural *right* action of $S_n$ on $\{1, 2, \dots, n\}$ and the induced action on the Boolean of $\{1, 2, \dots, n\}$. For a semigroup, $S$, we denote by $E(S)$ the set of all idempotents of $S$.

Fix $n \in \mathbb{N}$ and let $M = M_n = \{1, 2, \dots, n\}$, $M' = \{1', 2', \dots, n'\}$. We will consider $' : M \to M$ as a bijection, whose inverse we will also denote by $'$.

Consider the set $\mathfrak{C}_n$ of all decompositions of $M \cup M'$ into disjoint unions of subsets. Given $\alpha, \beta \in \mathfrak{C}_n$, $\alpha = X_1 \cup \dots \cup X_k$ and $\beta = Y_1 \cup \dots \cup Y_l$, we define their product $\gamma = \alpha\beta$ as the unique element of $\mathfrak{C}_n$ satisfying the following conditions:

(P1) For $i, j \in M$ the elements $i$ and $j$ belong to the same block of the decomposition $\gamma$ if an only if they belong to the same block of the decomposition $\alpha$ or there exists a sequence, $s_1, \dots, s_m$, where $m$ is even, of elements from $M$ such that $i$ and $s'_1$ belong to the same block of $\alpha$; $s_1$ and $s_2$ belong to the same block of $\beta$; $s'_2$ and $s'_3$ belong to the same block of $\alpha$ and so on; $s_{m-1}$ and $s_m$ belong to the same block of $\beta$; $s'_m$ and $j$ belong to the same block of $\alpha$.

(P2) For $i, j \in M$ the elements $i'$ and $j'$ belong to the same block of the decomposition $\gamma$ if an only if they belong to the same block of the decomposition $\beta$ or there exists a sequence, $s_1, \dots, s_m$, where $m$ is even, of elements from $M$ such that $i'$ and $s_1$ belong to the same block of $\beta$; $s'_1$ and $s'_2$ belong to the same block of $\alpha$; $s_2$ and $s_3$ belong to the same block of $\beta$ ans so on; $s'_{m-1}$ and $s'_m$ belong to the same block of $\alpha$; $s_m$ and $j'$ belong to the same block of $\beta$.

(P3) For $i, j \in M$ the elements $i$ and $j'$ belong to the same block of the decomposition $\gamma$ if an only if there exists a sequence, $s_1, \dots, s_m$, where $m$ is odd, of elements from $M$ such that $i$ and $s'_1$ belong to the same block of $\alpha$; $s_1$ and $s_2$ belong to the same block of $\beta$; $s'_2$ and $s'_3$ belong to the same block of $\alpha$ and so on; $s'_{m-1}$ and $s'_m$ belong to the same block of $\alpha$; $s_m$ and $j'$ belong to the same block of $\beta$.

One can think about the elements of $\mathfrak{C}_n$ as "microchips" or "generalized microchips" with $n$ pins on the left hand side (corresponding to the elements of $M$) and $n$ pins on the right hand side (corresponding to the elements of $M'$). For $\alpha \in \mathfrak{C}_n$ we connect two pins of the corresponding chip if and only if they belong to the same set of the partition $\alpha$. The operation described above can then be viewed as a "composition" of such chips: having $\alpha, \beta \in \mathfrak{C}_n$ we identify (connect) the right pins of $\alpha$ with the corresponding left pins of $\beta$, which uniquely defines a connection of the remaining pins (which are the left pins of $\alpha$ and the right pins of $\beta$). An example of multiplication of two chips from $\mathfrak{C}_n$ is given on Figure 1. Note that, performing the operation we can obtain some "dead circles" formed by some identified pins from $\alpha$ and $\beta$. These circles should be disregarded (however they play an important role in representation theory as they allow to deform the multiplication in the semigroup algebra). From this interpretation it is fairly obvious that the composition of elements from $\mathfrak{C}_n$ defined above is associative. On the level of associative algebra, the partition algebra was defined in [Mar1] and then studied by several authors especially in recent years, see for example [Bl, Mar2, MarEl, MarWo, Pa, Xi]. Purely as a semigroup it seems that $\mathfrak{C}_n$ appeared in [Maz2].

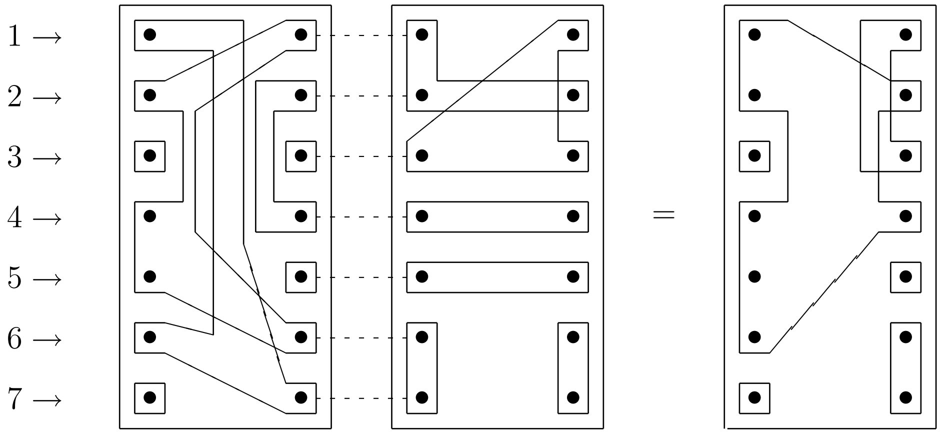


Figure 1: Multiplication of elements of $\mathfrak{C}_n$.

Let $\alpha \in \mathfrak{C}_n$ and $X$ be a block of $\alpha$. The block $X$ will be called

- a *line* provided that $|X| = 2$ and $X$ intersects with both $M$ and $M'$;
- a *generalized line* provided that $X$ intersects with both $M$ and $M'$;
- a *bracket* if $|X| = 2$ and either $X \subset M$ or $X \subset M'$;
- a *generalized bracket* if $|X| \geq 2$ and either $X \subset M$ or $X \subset M'$;
- a *point* if $|X| = 1$.

By a *Brauer-type semigroup* we will mean a "natural" subsemigroup of the semigroup $\mathfrak{C}_n$. Here are some examples:

(E1) The subsemigroup, consisting of all elements $\alpha \in \mathfrak{C}_n$ such that each block of $\alpha$ is a line. This subsemigroup is canonically identified with $S_n$ and is the group of units of $\mathfrak{C}_n$.

(E2) The subsemigroup, consisting of all elements $\alpha \in \mathfrak{C}_n$ such that each block of $\alpha$ is a either a line or a point. This subsemigroup is canonically identified with the *symmetric inverse semigroup* $\mathcal{IS}_n$.

(E3) The subsemigroup $\mathfrak{B}_n$, consisting of all elements $\alpha \in \mathfrak{C}_n$ such that each block of $\alpha$ is a either a line or a bracket. This is the classical *Brauer semigroup*, see [Ke, Maz1].

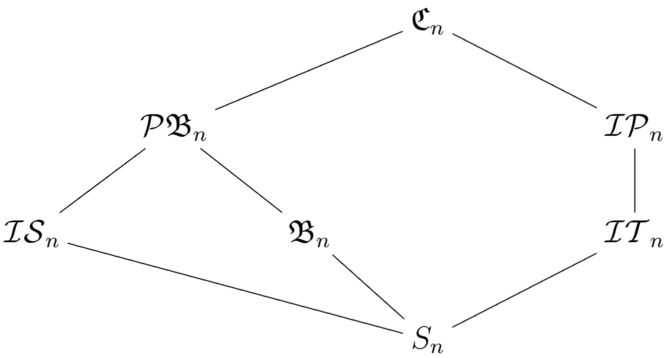


Figure 2: Inclusions for classical Brauer-type semigroups

(E4) The subsemigroup $\mathcal{P}\mathfrak{B}_n$, consisting of all elements $\alpha \in \mathfrak{C}_n$ such that each block of $\alpha$ is a either a line or a bracket or a point. This is the *partial analogue of the Brauer semigroup*, see [Maz1].

(E5) The subsemigroup $\mathcal{IP}_n$, consisting of all $\alpha \in \mathfrak{C}_n$ such that each block of $\alpha$ is a generalized line. In this form the semigroup $\mathcal{IP}_n$ appeared in [Mal2, Mal3]. It is easy to see that the semigroup $\mathcal{IP}_n$ is isomorphic to the *dual symmetric inverse monoid* $\mathcal{I}_M^*$ from [FL].

(E6) The subsemigroup $\mathcal{IT}_n$, consisting of all $\alpha \in \mathfrak{C}_n$ such that each block $X$ of $\alpha$ is a generalized line and $|X \cap M| = |X \cap M'|$. In this form the semigroup $\mathcal{IT}_n$ appeared in [Mal3]. The semigroup $\mathcal{IT}_n$ is isomorphic to the *greatest factorizable inverse submonoid* $\mathcal{F}_M^*$ of $\mathcal{I}_M^*$ from [FL].

All the semigroups described above are regular. $S_n$ is a group. The semigroups $IS_n$, $\mathcal{IP}_n$ and $\mathcal{IT}_n$ are inverse, while $\mathfrak{C}_n$, $\mathfrak{B}_n$ and $\mathcal{P}\mathfrak{B}_n$ are not. The partially ordered set consisting of these semigroups, with the partial order given by inclusions, is illustrated on Figure 2.

In what follows we will need some easy combinatorial results for Brauer-type semigroups. For $\alpha \in \mathfrak{C}_n$ we define the *rank* $\mathrm{rk}(\alpha)$ of $\alpha$ as the number of generalized lines in $\alpha$, that is the number of blocks in $\alpha$ intersecting with both $M$ and $M'$. Note that for the semigroups $S_n$, $\mathcal{IS}_n$, $\mathfrak{B}_n$, $\mathcal{P}\mathfrak{B}_n$ and $\mathfrak{C}_n$ ranks of the elements classify the $\mathcal{D}$-classes (this is obvious for $S_n$, for $\mathcal{IS}_n$ this is an easy exercise, for $\mathfrak{B}_n$ and $\mathcal{P}\mathfrak{B}_n$ this can be found in [Maz1], and for $\mathfrak{C}_n$ it can be obtained by arguments similar to those from [Maz1] for $\mathfrak{B}_n$).

For the semigroup $\mathcal{IT}_n$ we will need a different notion. Let $X$ be a set and $X = \cup_{i=1}^k X_k$ be a decomposition of $X$ into a union of pairwise disjoint subsets. For each $i$, $1 \leq i \leq n$, let $m_i$ denote the number of subsets of this decomposition, whose cardinality equals $i$. The tuple $(m_1, \dots, m_{|X|})$ will be called the *type* of the decomposition. Consider an element, $\alpha \in \mathcal{IT}_n$. By definition $\alpha$ is a decomposition of $M \cup M'$ into a disjoint union of subsets, whose intersections with $M$ and $M'$ have the same cardinality. Let $(m_1, \dots, m_{2n})$ be the type of this decompositions (note that $m_i \neq 0$ only if $i$ is even). The element $\alpha$ induces a decomposition of $M$ into disjoint subsets, whose blocks are intersections of the blocks of $\alpha$ with $M$. By the *type* of $\alpha$ we will mean the type of this decomposition of $M$, which is obviously equal to $(m_2, m_4, \dots, m_{2n})$. The types of elements from $\mathcal{IT}_n$ correspond bijectively to partitions of $n$ (a *partition*, $\lambda \vdash n$, of $n$ is a tuple, $\lambda = (\lambda_1, \dots, \lambda_k)$, of positive integers such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ and $\lambda_1 + \dots + \lambda_k = n$). The types of the elements classify the $\mathcal{D}$-classes in $\mathcal{IT}_n$, see [FL, Section 3].

For the semigroup $\mathcal{P}\mathfrak{B}_n$ we will need a more complicated technical tool. Although $\mathcal{D}$-classes are classified by ranks we will need to distinguish elements of a given rank, so we introduce the notion of a type. For $\alpha \in \mathcal{P}\mathfrak{B}_n$ let $r$ denote the number of lines in $\alpha$; $b_1$ the number of brackets in $\alpha$, contained in $M$; $b_2$ the number of brackets in $\alpha$, contained in $M'$; $p_1$ the number of points in $\alpha$, contained in $M$; $p_2$ the number of points in $\alpha$, contained in $M'$. Obviously $n = r + 2b_1 + p_1 = r + 2b_2 + p_2$. Define the *type* of $\alpha$ as follows:

$$\mathrm{type}(\alpha) = \begin{cases} (b_2, b_1 - b_2, 0, p_1), & b_1 \geq b_2; \\ (b_1, 0, b_2 - b_1, p_2), & b_2 > b_1. \end{cases}$$

We will need the following explicit combinatorial formulae for the number of elements of a given rank or type.

**Proposition 1.** *(a) For $k \in \{0, \dots, n\}$ the number of elements of rank $k$ in $\mathcal{IS}_n$ equals $\binom{n}{k}^2 k!$.*

*(b) For $k \in \{1, \dots, n\}$ the number of elements of rank $k$ in $\mathfrak{B}_n$ equals $0$ if $n - k$ is odd and $\frac{(n!)^2}{2^{2l}(l!)^2 k!}$ if $n - k = 2l$ is even.*

*(c) The number of elements of $\mathcal{IT}_n$ of type $(m_1, \dots, m_n)$ equals*

$$\frac{(n!)^2}{\displaystyle\prod_{i=1}^{n} (m_i!(i!)^{2m_i})}.$$

*(d) For all non-negative integers $k, m, t$ such that $2k + 2m + t \leq n$ the number of elements of the type $(k, m, 0, t)$ in $\mathcal{P}\mathfrak{B}_n$ is equal to the number of elements of the type $(k, 0, m, t)$ in $\mathcal{P}\mathfrak{B}_n$ and equals*

$$\frac{(n!)^2}{k!2^k(t+2m)!(k+m)!2^{k+m}t!(n-2k-2m-t)!}.$$

*Proof.* This is a straightforward combinatorial calculation. □

**Remark 2.** The semigroup $\mathfrak{C}_n$ can be also connected to some other semigroups of binary relations. As we have already mentioned, the subsemigroup $\mathcal{IP}_n$ of $\mathfrak{C}_n$ is isomorphic to the dual symmetric inverse monoid $\mathcal{I}_M^*$ from [FL], which is the semigroup of all difunctional binary relations under the operation of taking the smallest difunctional binary relations, containig the product of two given relations. The semigroup $\mathcal{IT}_n$ is isomorphic to the greatest factorizable inverse submonoid of $\mathcal{I}_M^*$, that is to the semigroup $E(\mathcal{I}_M^*)S_n$. One can also deform the multiplication in $\mathfrak{C}_n$ in the following way: given $\alpha, \beta \in \mathfrak{C}_n$ define $\gamma = \alpha \star \beta$ as follows: all blocks of $\gamma$ are either points or generalized lines, and for $i, j \in M$ the elements $i$ and $j'$ belong to the same block of $\gamma$ if and only if $i$ belongs to some block $X$ of $\alpha$ and $j'$ belongs to some block $Y$ of $\beta$ such that $X \cap M' = (Y \cap M)'$. It is straightforward that this deformed multiplication is associative and hence we get a new semigroup, $\tilde{\mathfrak{C}}_n$. This semigroup is an inflation of Vernitsky's inverse semigroup $(D_X, \diamond)$, see [Ve], which is a subsemigroup of $\tilde{\mathfrak{C}}_n$ in the natural way. An isomorphic object can be obtained if instead of points one requires that $\gamma$ contains at most one generalized bracket, which is a subset of $M$, and at most one generalized bracket, which is a subset of $M'$.

# 3 Presentation for $\mathfrak{B}_n$

For $i = 1, \dots, n-1$ we denote by $s_i$ the elementary transposition $(i, i+1) \in S_n$, and by $\pi_i$ the element $\{i, i+1\} \cup \{i', (i+1)'\} \cup \bigcup_{j \neq i, i+1} \{j, j'\}$ of $\mathfrak{B}_n$ (the elementary *atom* from [Maz1]). It is easy to see (and can be derived from the results of [Maz1] and [Mal1]) that $\mathfrak{B}_n$ is generated by $\{s_i\} \cup \{\pi_i\}$ as a monoid. Moreover, $\mathfrak{B}_n$ is even generated by $\{s_i\}$ and, for example, $\pi_1$. However, we think that the set $\{s_i\} \cup \{\pi_i\}$ is more natural as a system of generators for $\mathfrak{B}_n$, for example because of the connection between Brauer and Temperley-Lieb algebras (and analogy with the singular braid monoid, see [Ba, Bi]). In this section we obtain a presentation for $\mathfrak{B}_n$ with respect to this system of generators.

Let $T$ denote the monoid with the identity element $e$, generated by the elements $\sigma_i$, $\theta_i$, $i = 1, \dots, n-1$, subject to the following relations (where

$i, j \in \{1, 2, \dots, n-1\}$):

$$\sigma_i^2 = e; \quad \sigma_i\sigma_j = \sigma_j\sigma_i, \; |i-j| > 1; \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \; |i-j| = 1; \tag{3.1}$$

$$\theta_i^2 = \theta_i; \quad \theta_i\theta_j = \theta_j\theta_i, \; |i-j| > 1; \quad \theta_i\theta_j\theta_i = \theta_i, \; |i-j| = 1; \tag{3.2}$$

$$\theta_i\sigma_i = \sigma_i\theta_i = \theta_i, \quad \theta_i\sigma_j = \sigma_j\theta_i, \; |i-j| > 1; \tag{3.3}$$

$$\sigma_i\theta_j\sigma_i = \sigma_j\theta_i\sigma_j, \quad \theta_i\sigma_j\theta_i = \theta_i, \; |i-j| = 1; \tag{3.4}$$

$$\text{and} \;\; \sigma_i\sigma_{i+1}\theta_i\theta_{i+2} = \sigma_{i+2}\sigma_{i+1}\theta_i\theta_{i+2}. \tag{3.5}$$

**Theorem 3.** *The map $\sigma_i \mapsto s_i$ and $\theta_i \to \pi_i$, $i = 1, \dots, n-1$, extends to an isomorphism, $\varphi : T \to \mathfrak{B}_n$.*

The rest of the section will be devoted to the proof of Theorem 3.

It is a direct calculation to verify that the generators $s_i$ and $\pi_i$ of $\mathfrak{B}_n$ satisfy the relations, corresponding to (3.1)–(3.5). Thus the map $\sigma_i \mapsto s_i$ and $\theta_i \mapsto \pi_i$, $i = 1, \dots, n-1$, extends to an epimorphism, $\varphi : T \twoheadrightarrow \mathfrak{B}_n$. Hence, to prove Theorem 3 we have only to show that $|T| = |\mathfrak{B}_n|$. To do this we will have to study the structure of the semigroup $T$ in details.

Let $W$ denote the free monoid, generated by $\sigma_i$, $\theta_i$, $i = 1, \dots, n-1$, and $\psi : W \twoheadrightarrow T$ denote the canonical projection. Let $\sim$ be the corresponding congruence on $W$, that is $v \sim w$ provided that $\psi(v) = \psi(w)$. We start with the following description of units in $T$:

**Lemma 4.** *The elements $\sigma_i$, $i = 1, \dots, n-1$, generate the group $G$ of units in $T$, which is isomorphic to the symmetric group $S_n$.*

*Proof.* Let $v, w \in W$ be such that $v \sim w$. Assume further that $v$ contains some $\theta_i$. Since $\theta$'s allways occur on both sides in the relations (3.2)–(3.5) and do not occur in the relations (3.1), it follows that $w$ must contain some $\theta_j$. In particular, the submonoid, generated in $W$ by $\sigma_i$, $i = 1, \dots, n-1$, is a union of equivalence classes with respect to $\sim$. Using the well-known Coxeter presentation of the symmetric group we obtain that $\sigma_i$, $i = 1, \dots, n-1$, generate in $T$ a copy of the symmetric group. All elements of this group are obviously units in $T$. On the other hand, if $v, w \in W$ and $v$ contains some $\theta_i$, then $vw$ contains $\theta_i$ as well. By the above arguments, $vw$ can not be equivalent to the empty word. Hence $v$ is not invertable in $T$. The claim of the lemma follows. $\square$

In what follows we will identify the group $G$ of units in $T$ with $S_n$ via the isomorphism, which sends $\sigma_i \in G$ to $s_i$. There is a natural action of $S_n$ on $T$ by inner automorphisms of $T$ via conjugation: $x^g = g^{-1}xg$ for each $x \in T$, $g \in S_n$.

**Lemma 5.** *The $S_n$-stabilizer of $\theta_1$ is the subgroup $H$ of $S_n$, consisting of all permutations, which preserve the set $\{1, 2\}$. This subgroup is isomorphic to $S_2 \times S_{n-2}$.*

*Proof.* We have $\sigma_j\theta_1\sigma_j = \theta_j$, $j \neq 2$, by (3.3). Since $\sigma_j$, $j \neq 2$, generate $H$, we obtain that all elements of $H$ stabilize $\theta_1$. In particular, the $S_n$-orbit of $\theta_1$ consists of at most $|S_n|/|H| = \binom{n}{2}$ elements. At the same time, it is easy to see that the $S_n$-orbit of $\varphi(\theta_1)$ consists of exactly $\binom{n}{2}$ different elements and hence $H$ must coincide with the $S_n$-stabilizer of $\theta_1$. □

Since $S_n$ acts on $T$ via automorphisms and $\theta_1$ is an idempotent, all elements in the $S_n$-orbit of $\theta_1$ are idempotents. From Lemma 5 it follows that the elements of the $S_n$-orbit of $\theta_1$ are in the natural bijection with the cosets $H\backslash S_n$. By the definition of $H$, two elements, $x, y \in S_n$, are contained in the same coset if and only if $x(\{1, 2\}) = y(\{1, 2\})$.

**Lemma 6.** *The $S_n$-orbit of $\theta_1$ contains all $\theta_i$, $i = 1, \dots, n-1$. Moreover, for $w \in S_n$ we have $w^{-1}\theta_1 w = \theta_i$ if and only if $w(\{1, 2\}) = \{i, i+1\}$.*

*Proof.* We use induction on $i$ with the case $i = 1$ being trivial. Let $i > 1$ and assume that $\theta_{i-1}$ is contained in our orbit. Then $\theta_i = \sigma_{i-1}\sigma_i\theta_{i-1}\sigma_i\sigma_{i-1}$ and hence $\theta_i$ is contained in our orbit as well. Hence all $\theta_i$ indeed belong to the $S_n$-orbit of $\theta_1$. The second claim follows from

$$\sigma_{i-1}\sigma_i\sigma_{i-2}\sigma_{i-1}\cdots\sigma_1\sigma_2(\{1, 2\}) = \{i, i+1\}, \tag{3.6}$$

which is obtained by a direct calculation. This completes the proof. □

For $w \in S_n$ such that $w(\{1, 2\}) = \{i, j\}$, where $i < j$, we set $\epsilon_{i,j} = w^{-1}\theta_1 w$, which is well defined by Lemma 5.

**Lemma 7.** *Suppose $\{i, j\} \cap \{p, q\} = \varnothing$. Then $\epsilon_{i,j}\epsilon_{p,q} = \epsilon_{p,q}\epsilon_{i,j}$.*

*Proof.* Since all elements $\epsilon_{i,j}$ are obtained from $\theta_1$ via automorphisms, it is enough to show that $\theta_1$ commutes with all elements $\epsilon_{i,j}$ such that $\{i, j\} \cap \{1, 2\} = \varnothing$. Take any $v \in S_n$ such that $v(\{1, 2\}) = \{1, 2\}$ and $v(\{i, j\}) = \{3, 4\}$. Such $v$ obviously exists. Then $\theta_1$ commutes with $\epsilon_{i,j}$ if and only if $v^{-1}\theta_1 v = \theta_1$ commutes with $v^{-1}\epsilon_{i,j}v = \theta_3$. The statement now follows from (3.2). □

**Lemma 8.** *Suppose $\{i, j\} \cap \{p, q\} \neq \varnothing$. Then $\epsilon_{i,j}\epsilon_{p,q} = u\theta_1 v$ for certain $u, v \in S_n$.*

*Proof.* If $\{i,j\} = \{p,q\}$ the statement is obvious as $\epsilon_{i,j}$ is an idempotent. Assume $|\{i,j\} \cap \{p,q\}| = 1$. Since all elements $\epsilon_{i,j}$ are obtained from $\theta_1$ via automorphisms, it is enough to consider the case when $\{i,j\} = \{1,2\}$, $p = 2$ and $q > 2$. Consider $v \in S_n$ such that $v(1) = 1, v(2) = 2$ and $v(q) = 3$. Then, using (3.3) and (3.4) we have

$$v^{-1}\theta_1\epsilon_{p,q}v = \theta_1\theta_2 = \theta_1\sigma_1\sigma_2\theta_1\sigma_2\sigma_1 = \theta_1\sigma_2\theta_1\sigma_2\sigma_1 = \theta_1\sigma_2\sigma_1.$$

The statement follows. □

For each $k$, $1 \le k \le [\frac{n}{2}]$, set $\delta_k = \theta_1\theta_3\dots\theta_{2k-1}$. Set also $\delta_0 = e$. The elements $\delta_i$, $0 \le i \le [\frac{n}{2}]$, will be called *canonical.* The group $S_n \times S_n$ acts naturally on $T$ via $(g,h)(x) = g^{-1}xh$ for $x \in T$ and $(g,h) \in S_n \times S_n$.

**Lemma 9.** *Every $S_n \times S_n$-orbit contains a canonical element.*

*Proof.* Let $x \in T$. If $x \in S_n$ the statement is obvious. Assume that $x \notin S_n$. By Lemma 6 we can write $x = w\theta_1g_1\theta_1g_2\dots\theta_1g_k$ for some $k \ge 1$ and $w, g_1, \dots, g_k \in S_n$. Moreover, we may assume that $x$ can not be written as a product of $\theta_1$'s and elements of $S_n$, which contains less than $k$ occurrences of $\theta_1$. We have

$$\begin{aligned} x = w(g_1\dots g_k)(g_1\dots g_k)^{-1}\theta_1(g_1\dots g_k)\cdot \\ \cdot(g_2\dots g_k)^{-1}\theta_1(g_2\dots g_k)\dots(g_{k-1}g_k)^{-1}\theta_1(g_{k-1}g_k)g_k^{-1}\theta_1g_k, \end{aligned} \tag{3.7}$$

and hence we can write

$$x = u\epsilon_{i_1,j_1}\dots\epsilon_{i_k,j_k}, \tag{3.8}$$

where $u = wg_1\dots g_k$ and $\{i_t,j_t\}$=$\{(g_t\dots g_k)(1),\ (g_t\dots g_k)(2)\}$, $1 \le t \le k$. Since $x$ is chosen such that it can not be reduced to an element of $T$ which contains less that $k$ entries of $\theta_1$, from Lemma 7 and Lemma 8 it follows that $\{i_t,j_t\} \cap \{i_s,j_s\} = \varnothing$ for any two factors $\epsilon_{i_t,j_t}$, $\epsilon_{i_s,j_s}$ in (3.8). This implies that the $S_n \times S_n$-orbit of $x$ contains $\epsilon_{i_1,j_1}\dots\epsilon_{i_k,j_k}$ with $\{i_t,j_t\} \cap \{i_s,j_s\} = \varnothing$ for all $s \ne t$.

Now consider some $v \in S_n$ such that $v(i_1) = 1$, $v(j_1) = 2$, $v(i_2) = 3$ and so on, $v(j_k) = 2k$. Then the element $v^{-1}\epsilon_{i_1,j_1}\cdots\epsilon_{i_k,j_k}v$ is canonical by definition. This completes the proof. □

**Remark 10.** From the proof of Lemma 9 it follows that each $x \in T$ can be written in the form $x = w\theta_1g_1\theta_1g_2\dots\theta_1g_k$, where $k \le \lfloor\frac{n}{2}\rfloor$.

**Lemma 11.** *The $S_n \times S_n$-orbit of the canonical element $\delta_k$, $0 \le k \le [\frac{n}{2}]$, contains at most*

$$\frac{(n!)^2}{2^{2k}(k!)^2(n-2k)!}$$

*elements.*

*Proof.* It is enough to show that the stabilizer of $\delta_k$ under the $S_n \times S_n$-action contains at least $(k!)^2 2^{2k}(n-2k)!$ elements. Set

$$\Sigma_i^0 = \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i},\ 1 \leq i \leq k-1;$$
$$\Sigma_i^1 = \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i}\sigma_{2i-1},\quad 1 \leq i \leq k-1.$$

Then both $\Sigma_i^0$ and $\Sigma_i^1$ swap the sets $\{2i-1, 2i\}$ and $\{2i+1, 2i+2\}$. It follows that the group $H$, generated by all $\Sigma_i^0$, consists of all permutations of the set $\{1,2\}, \{3,4\}, \dots, \{2k-1, 2k\}$ and is therefore isomorphic to the group $S_k$. It is further easy to see that the group $\tilde{H}$, generated by all $\Sigma_i^0$ and $\Sigma_i^1$, is isomorphic to the wreath product $H \wr S_2$. From (3.5) and (3.3) it follows that the left multiplication with both $\Sigma_i^0$ and $\Sigma_i^1$ stabilizes $\delta_k$. Therefore for each element of $\tilde{H}$ the left multiplication with this element stabilizes $\delta_k$ as well. Similarly one proves that the right multiplication with each element from $\tilde{H}$ stabilizes $\delta_k$. Apart from this, from (3.3) we have that the conjugation by any element from the group $H' = \langle \sigma_{2k+1}, \dots, \sigma_{n-1} \rangle \simeq S_{n-2k}$ stabilizes $\delta_k$.

Observe that the group, generated by the left copy of $\tilde{H}$, the right copy of $\tilde{H}$, and the $H'$ is a direct product of these three componets. Using the product rule we derive that the cardinality of the stabilizer of $\delta_k$ is at least

$$(|H \wr S_2|)^2 |S_{n-2k}| = (k!)^2 2^{2k}(n-2k)!,$$

and the proof is complete. $\square$

**Corollary 12.**

$$|T| \leq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n!)^2}{2^{2k}(k!)^2(n-2k)!}.$$

*Proof.* The proof follows from Lemma 11 and Remark 10 by a direct calculation. $\square$

*Proof of Theorem 3.* Comparing Corollary 12 and Proposition 1(b) we have $|T| \leq |\mathfrak{B}_n|$. Since $\varphi : T \to \mathfrak{B}_n$ is surjective we have $|T| \geq |\mathfrak{B}_n|$. Hence $|T| = |\mathfrak{B}_n|$ and $\varphi$ is an isomorphism. $\square$

# 4 Presentation for $\mathcal{IT}_n$

For $i \in \{1, 2, \dots, n-1\}$ let $\varrho_i$ denote the element $\{i, i+1, i', (i+1)'\} \cup \bigcup_{j \neq i, i+1} \{j, j'\} \in \mathcal{IT}_n$. By [Mal3, Proposition 9], the elements $\{\sigma_i\}$ and $\{\varrho_i\}$ generate $\mathcal{IT}_n$ (and even $\{\sigma_i\}$ and, say $\varrho_1$, do).

Let $T$ denote the monoid with the identity element $e$, generated by the elements $\sigma_i$, $\tau_i$, $i = 1, \dots, n-1$, subject to the following relations (where $i, j \in \{1, 2, \dots, n-1\}$):

$$\sigma_i^2 = e; \quad \sigma_i\sigma_j = \sigma_j\sigma_i, \ |i-j| > 1; \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \ |i-j| = 1; \tag{4.1}$$

$$\tau_i^2 = \tau_i; \quad \tau_i\tau_j = \tau_j\tau_i, \ i \neq j; \tag{4.2}$$

$$\tau_i\sigma_i = \sigma_i\tau_i = \tau_i; \quad \tau_i\sigma_j = \sigma_j\tau_i, \ |i-j| > 1; \tag{4.3}$$

$$\sigma_i\tau_j\sigma_i = \sigma_j\tau_i\sigma_j \ \text{ and } \ \tau_i\sigma_j\tau_i = \tau_i\tau_j, \ |i-j| = 1. \tag{4.4}$$

**Theorem 13.** *The map $\sigma_i \mapsto s_i$ and $\tau_i \to \varrho_i$, $i = 1, \dots, n-1$, extends to an isomorphism, $\varphi : T \to \mathcal{IT}_n$.*

The rest of the section will be devoted to the proof of Theorem 13.

It is a direct calculation to verify that the generators $s_i$ and $\varrho_i$ of $\mathcal{IT}_n$ satisfy the relations, corresponding to (4.1)–(4.4). Thus the map $\sigma_i \mapsto s_i$ and $\tau_i \mapsto \varrho_i$, $i = 1, \dots, n-1$, extends to an epimorphism, $\varphi : T \twoheadrightarrow \mathcal{IT}_n$. Hence, to prove Theorem 13 we have only to show that $|T| = |\mathcal{IT}_n|$. As in the previous section, to do this we will study the structure of $T$ in details. Let $W$ denote the free monoid, generated by $\sigma_i$, $\tau_i$, $i = 1, \dots, n-1$, $\psi : W \twoheadrightarrow T$ denote the canonical projection, and $\sim$ be the corresponding congruence on $W$. The first part of our arguments is very similar to that from the previous Section.

**Lemma 14.** *The elements $\sigma_i$, $i = 1, \dots, n-1$, generate the group $G$ of units in $T$, which is isomorphic to the symmetric group $S_n$ (and will be identified with $S_n$ in the sequel).*

*Proof.* Analogous to that of Lemma 4. □

There are two natural actions on $T$:

(I) The group $S_n$ acts on $T$ by inner automorphisms via conjugation.

(II) The group $S_n \times S_n$ acts on $T$ via $(g, h)(x) = g^{-1}xh$ for $x \in T$ and $(g, h) \in S_n \times S_n$.

**Lemma 15.** *The $S_n$-stabilizer of $\tau_1$ is the subgroup $H$ of $S_n$, consisting of all permutations, which preserve the set $\{1, 2\}$. This subgroup is isomorphic to $S_2 \times S_{n-2}$.*

*Proof.* Analogous to that of Lemma 5. □

Since $S_n$ acts on $T$ via automorphisms and $\tau_1$ is an idempotent, all elements in the $S_n$-orbit of $\tau_1$ are idempotents. From Lemma 15 it follows that the elements of the $S_n$-orbit of $\tau_1$ are in the natural bijection with the cosets $H\backslash S_n$. By the definition of $H$, two elements, $x, y \in S_n$, are contained in the same coset if and only if $x(\{1,2\}) = y(\{1,2\})$.

**Lemma 16.** *The $S_n$-orbit of $\tau_1$ contains all $\tau_i$, $i = 1, \dots, n-1$. Moreover, for $w \in S_n$ we have $w^{-1}\tau_1 w = \tau_i$ if and only if $w(\{1,2\}) = \{i, i+1\}$.*

*Proof.* Analogous to that of Lemma 6. □

**Lemma 17.** *All elements in the $S_n$-orbit of $\tau_1$ commute.*

*Proof.* Since all elements in the $S_n$-orbit of $\tau_1$ are obtained from $\tau_1$ via automorphisms, it is enough to show that $\tau_1$ commutes with all elements in this orbit. Let $w \in S_n$ be such that $w(\{1,2\}) = \{i,j\}$. If $\{i,j\} = \{1,2\}$ then $w^{-1}\tau_1 w = \tau_1$ by Lemma 16 and hence we may assume $\{i,j\} \neq \{1,2\}$.

Take any $v \in S_n$ such that

- $v(\{1,2\}) = \{1,2\}$ and $v(\{i,j\}) = \{3,4\}$ if $\{i,j\} \cap \{1,2\} = \varnothing$;
- $v(\{1,2\}) = \{1,2\}$ and $v(\{i,j\}) = \{2,3\}$ if $\{i,j\} \cap \{1,2\} \neq \varnothing$.

Such $v$ obviously exists. Then $\tau_1$ commutes with $w^{-1}\tau_1 w$ if and only if $v^{-1}\tau_1 v$ commutes with $v^{-1}w^{-1}\tau_1 wv$. Using our choice of $v$ and Lemma 16 we have $v^{-1}\tau_1 v = \tau_1$ and $v^{-1}w^{-1}\tau_1 wv = \tau_j$, where $j = 3$ if $\{i,j\} \cap \{1,2\} = \varnothing$, and $j = 2$ otherwise. The statement now follows from (4.2). □

For $w \in S_n$ such that $w(\{1,2\}) = \{i,j\}$, where $i < j$, we set $\varepsilon_{i,j} = w^{-1}\tau_1 w$, which is well defined by Lemma 15.

**Lemma 18.** *Let $\{i,j,k\} \subset \{1,2,\dots,n\}$ and $i < j < k$. Then*

$$\varepsilon_{i,j}\varepsilon_{j,k} = \varepsilon_{i,k}\varepsilon_{j,k} = \varepsilon_{i,j}\varepsilon_{i,k}.$$

*Proof.* We prove that $\varepsilon_{i,j}\varepsilon_{j,k} = \varepsilon_{i,k}\varepsilon_{j,k}$ and the second equality is proved by analogous arguments. Let $w \in S_n$ be such that $w(i) = 1$, $w(j) = 2$, $w(k) = 3$. Conjugating by $w$ we reduce our equality to the equality $\tau_1\tau_2 = \sigma_2\tau_1\sigma_2\tau_2$ Using (4.3) and (4.4) we have

$$\sigma_2\tau_1\sigma_2\tau_2 = \sigma_1\tau_2\sigma_1\tau_2 = \sigma_1\tau_1\tau_2 = \tau_1\tau_2.$$

The claim follows. □

For $i, j \in M$ set $\varepsilon_{i,i} = e$ and $\varepsilon_{i,j} = \varepsilon_{j,i}$ if $j < i$. For a non-empty binary relation, $\rho$, on $M$ set

$$\varepsilon_\rho = \prod_{i\rho j} \varepsilon_{i,j}.$$

**Corollary 19.** *Let $\rho$ be non-empty binary relation on $M$ and $\rho^*$ be the reflexive-symmetric-transitive closure of $\rho$. Then $\varepsilon_\rho = \varepsilon_{\rho^*}$*

*Proof.* Follows easily from Lemma 17, Lemma 18 and the fact that all $\varepsilon_{i,j}$'s are idempotents. □

Let $\lambda : \{1, \dots, n\} = X_1 \cup \dots \cup X_k$ be a decomposition of $M$ into an unordered union of pairwise disjoint sets. With this decomposition we associate the equivalence relation $\rho_\lambda$ on $M$, whose equivalence classes coincide with $X_i$'s.

**Corollary 20.** *Let $\lambda$ and $\mu$ be two decompositions of $M$ as above. Assume that the types of $\lambda$ and $\mu$ coincide. Then $\varepsilon_{\rho_\lambda}$ and $\varepsilon_{\rho_\mu}$ are conjugate in $T$.*

*Proof.* Let $v \in S_n$ be an element, which maps $\lambda$ to $\mu$ (such element exists since the types of $\lambda$ and $\mu$ are the same). One easily sees that $v^{-1}\varepsilon_{\rho_\lambda} v = \varepsilon_{\rho_\mu}$. The statement follows. □

A decomposition, $\lambda : \{1, \dots, n\} = X_1 \cup \dots \cup X_k$, is called *canonical* provided that (up to a permutation of the blocks) we have $|X_1| \geq |X_2| \geq \dots \geq |X_k|$, $X_1 = \{1, 2, \dots, l_1\}$, $X_2 = \{l_1 + 1, l_1 + 2, \dots, l_1 + l_2\}$ and so on. Note that in this case $\lambda$ can also be viewed as a *partition* of $n$. The element $\varepsilon_{\rho_\lambda}$ will be called *canonical* provided that $\lambda$ is canonical.

**Lemma 21.** *Every $S_n \times S_n$-orbit contains a canonical element.*

*Proof.* Because of Corollary 20 it is enough to show that every $S_n \times S_n$-orbit contains $\varepsilon_{\rho_\lambda}$ for some decomposition $\lambda$. Let $x \in T$. If $x \in S_n$, then the statement is obvious. Let $x \in T \setminus S_n$. From Lemma 16 we have that the semigroup $T$ is generated by $S_n$ and $\tau_1$. Hence we have $x = w\tau_1 g_1 \tau_1 g_2 \cdots \tau_1 g_k$ for some $w, g_1, \dots, g_k \in S_n$. Therefore

$$\begin{aligned} x = w(g_1 \dots g_k)(g_1 \dots g_k)^{-1}\tau_1(g_1 \dots g_k)\cdot \\ \cdot (g_2 \dots g_k)^{-1}\tau_1(g_2 \dots g_k) \dots (g_{k-1}g_k)^{-1}\tau_1(g_{k-1}g_k)g_k^{-1}\tau_1 g_k, \end{aligned}$$

and hence we can write $x = u\varepsilon_{i_1,j_1} \dots \varepsilon_{i_k,j_k}$, where $u = wg_1 \dots g_k$ and

$$\{i_t, j_t\} = \{(g_t \dots g_k)(1), (g_t \dots g_k)(2)\}, \ 1 \leq t \leq k.$$

Define the equivalence relation $\rho$ as the reflexive-symmetric-transitive closure of the relation $\{(i_1, j_1), \dots, (i_k, j_k)\}$ and $\lambda$ be the corresponding decomposition of $\{1, 2, \dots, n\}$. From Corollary 19 we get that the $S_n \times S_n$-orbit of $x$ contains $\varepsilon_\rho = \varepsilon_{\rho_\lambda}$. This completes the proof. $\square$

**Lemma 22.** *Let $\lambda$ be a canonical decomposition of $\{1, 2, \dots, n\}$. For $i = 1, \dots, n$ set $\lambda^{(i)} = |\{j : |X_j| = i\}|$. Then the $S_n \times S_n$-stabilizer of $\varepsilon_\lambda$ contains at least*

$$\prod_{i=1}^{n} (\lambda^{(i)}!(i!)^{2\lambda^{(i)}})$$

*elements.*

*Proof.* Fix $i \in \{1, 2, \dots, n\}$. Let $X_a, X_{a+1} \dots, X_b$ be all blocks of $\lambda$ of cardinality $i$. Then for any non-maximal element $j$ of any of $X_a, X_{a+1} \dots, X_b$, using Lemma 17, the definition of $\varepsilon_\lambda$, and (4.3) we have $\sigma_j \varepsilon_\lambda = \varepsilon_\lambda \sigma_j = \varepsilon_\lambda$. Moreover, for any $w \in S_n$, which stabilizes all elements outside $X_a \cup X_{a+1} \cup \dots \cup X_b$ and maps each $X_s$ to some $X_t$, we have $w(\lambda) = \lambda$ and hence $w^{-1} \varepsilon_\lambda w = \varepsilon_\lambda$. This gives us exactly $\lambda^{(i)}!(i!)^{2\lambda^{(i)}}$ elements of the $S_n \times S_n$-stabilizer. The statement of the lemma now follows by applying the product rule since for different $i$ the elements above act on pairwise disjoint subsets of $\{1, \dots, n\}$. $\square$

**Corollary 23.**

$$|T| \leq \sum_{\lambda \vdash n} \frac{(n!)^2}{\prod_{i=1}^{n} (\lambda^{(i)}!(i!)^{2\lambda^{(i)}})}.$$

*Proof.* Canonical elements of $T$ are in bijection with partitions $\lambda \vdash n$ by construction. By Lemma 21, every $S_n \times S_n$-orbit contains a canonical element. We have $|S_n \times S_n| = (n!)^2$. By Lemma 22, the stabilizer of a canonical element, corresponding to $\lambda$, contains at least $\prod_{i=1}^{n} (\lambda^{(i)}!(i!)^{2\lambda^{(i)}})$ elements. The statement now follows by applying the sum rule. $\square$

*Proof of Theorem 13.* Comparing Corollary 23 and Proposition 1(c) we have $|T| \leq |\mathcal{IT}_n|$. Since $\varphi : T \to \mathcal{IT}_n$ is surjective we have $|T| \geq |\mathcal{IT}_n|$. Hence $|T| = |\mathcal{IT}_n|$ and $\varphi$ is an isomorphism. $\square$

**Remark 24.** From the above arguments it follows that the inequality obtained in Lemma 22 is in fact an equality. From the proof of Lemma 22 one easily derives that the $S_n \times S_n$-stabilizer of $\varepsilon_\lambda$ is isomorphic to the direct product of wreath products $S_{\lambda^{(i)}} \wr (S_i \times S_i)$.

**Remark 25.** Following the arguments of the proof of Theorem 13 one easily proves the following presentation for the symmetric inverse semigroup $\mathcal{IS}_n$: $\mathcal{IS}_n$ is generated, as a monoid, by $\sigma_1, \dots, \sigma_{n-1}, \vartheta_1, \dots, \vartheta_n$ subject to the following relations:

$$\sigma_i^2 = e; \quad \sigma_i\sigma_j = \sigma_j\sigma_i, \ |i-j| > 1; \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \ |i-j| = 1; \tag{4.5}$$

$$\vartheta_i^2 = \vartheta_i; \quad \vartheta_i\vartheta_j = \vartheta_j\vartheta_i \ i \neq j; \tag{4.6}$$

$$\sigma_i\vartheta_i = \vartheta_{i+1}\sigma_i; \ \sigma_i\vartheta_j = \vartheta_j\sigma_i, \ j \neq i, i+1; \quad \vartheta_i\sigma_i\vartheta_i = \vartheta_i\vartheta_{i+1}. \tag{4.7}$$

The classical presentation for $\mathcal{IS}_n$ usually involves only one additional generator (namely $\vartheta_1$) and can be found for example in [Li, Chapter 9].

# 5 Presentation for $\mathcal{P}\mathfrak{B}_n$

For $i \in \{1, \dots, n\}$ let $\varsigma_i$ denote the element $\{i\} \cup \{i'\} \cup \bigcup_{j \neq i}\{j, j'\}$. Using [Maz1], it is easy to see that $\mathcal{P}\mathfrak{B}_n$ is generated by $\{\sigma_i\} \cup \{\pi_i\} \cup \{\varsigma_i\}$ (and even by $\{\sigma_i\}$, $\pi_1$ and $\varsigma_1$).

Let $T$ denote the monoid with the identity element $e$, generated by the elements $\sigma_i$, $\theta_i$, $i = 1, \dots, n-1$, and $\vartheta_i$, $i = 1, \dots, n$, subject to the relations (3.1)–(3.5), the relations from Remark 25, and the following relations (for all appropriate $i$ and $j$):

$$\theta_i\vartheta_j = \theta_i\vartheta_j, \ j \neq i, i+1; \tag{5.1}$$

$$\theta_i\vartheta_i = \theta_i\vartheta_{i+1} = \theta_i\vartheta_i\vartheta_{i+1}, \quad \vartheta_i\theta_i = \vartheta_{i+1}\theta_i = \vartheta_i\vartheta_{i+1}\theta_i; \tag{5.2}$$

$$\theta_i\vartheta_i\theta_i = \theta_i, \quad \vartheta_i\theta_i\vartheta_i = \vartheta_i\vartheta_{i+1}; \tag{5.3}$$

$$\sigma_{i+2}\sigma_{i+1}\theta_i\vartheta_{i+2}\vartheta_{i+3} = \sigma_i\sigma_{i+1}\vartheta_i\theta_i\theta_{i+2}\vartheta_{i+2} \tag{5.4}$$

**Theorem 26.** *The map $\sigma_i \mapsto s_i$, $\theta_i \to \pi_i$, $i = 1, \dots, n-1$, and $\vartheta_i \mapsto \varsigma_i$, $i = 1, \dots, n$, extends to an isomorphism, $\varphi : T \to \mathcal{P}\mathfrak{B}_n$.*

As in the previous section, one easily checks that this map extends to an epimorphism and hence to complete the proof one has to compare the cardinalities of $T$ and $\mathcal{P}\mathfrak{B}_n$.

Similarly to what was done in Section 4, using the presentation of $\mathcal{IS}_n$ given in Remark 25, one proves that elements $\sigma_i$, $i = 1, \dots, n-1$, generate the symmetric group $S_n$, and that the elements $\sigma_i$, $i = 1, \dots, n-1$; $\vartheta_i$, $i = 1, \dots, n$, generate the semigroup, which is isomorphic to $\mathcal{IS}_n$ (and which will be identified with it). As in Section 4 we consider the natural action of $S_n$ on $T$ by inner automorphisms of $T$ via conjugation: $x^g = g^{-1}xg$ for each $x \in T$, $g \in S_n$. Set $\xi_i = \theta_i\vartheta_i$, $\eta_i = \vartheta_i\theta_i$, $1 \leq i \leq n-1$.

**Lemma 27.** *The $S_n$-stabilizer of each of $\theta_1$, $\xi_1$, $\eta_1$ is the subgroup $H$ of $S_n$, consisting of all permutations, which preserve the set $\{1,2\}$. This subgroup is isomorphic to $S_2 \times S_{n-2}$.*

*Proof.* For $\theta_1$ this follows from Lemma 5. For each $j \geq 2$ we have that $\sigma_j$ commutes with both $\xi_1$ and $\eta_1$ by (3.3) and (4.7) respectively, and hence $\sigma_j\xi_1\sigma_j = \xi_1$ and $\sigma_j\eta_1\sigma_j = \eta_1$. Let $j = 1$. Then

$$\sigma_1\xi_1\sigma_1 = \sigma_1\theta_1\vartheta_1\sigma_1 = \sigma_1\theta_1\sigma_1\vartheta_2 = \theta_1\vartheta_2 = \theta_1\vartheta_1 = \xi_1;$$
$$\sigma_1\eta_1\sigma_1 = \sigma_1\vartheta_1\theta_1\sigma_1 = \vartheta_2\sigma_1\theta_1\sigma_1 = \vartheta_2\theta_1 = \vartheta_1\theta_1 = \eta_1$$

by (4.7) and (3.3). Hence $\sigma_1$ also stabilizes $\xi_1$ and $\eta_1$. Since $\sigma_j$, $j \neq 2$, generate $H$, we obtain that all elements of $H$ stabilize $\xi_1$ and $\eta_1$. In particular, the $S_n$-orbits of $\xi_1$ and of $\eta_1$ consist of at most $|S_n|/|H| = \binom{n}{2}$ elements each. At the same time, the $S_n$-orbits of $\varphi(\xi_1)$ and $\varphi(\eta_1)$ consist of exactly $\binom{n}{2}$ different elements and hence $H$ must coincide with the $S_n$-stabilizer of both $\xi_1$ and $\eta_1$. $\square$

Since $S_n$ acts on $T$ via automorphisms and $\theta_1$, $\xi_1$, $\eta_1$ are idempotents, all elements in the $S_n$-orbits of $\theta_1$, $\xi_1$, $\eta_1$ are idempotents as well. From Lemma 27 it follows that the elements of the $S_n$-orbits of $\theta_1$, $\xi_1$, $\eta_1$ are in the natural bijections with the cosets $H\backslash S_n$. By the definition of $H$, two elements, $x, y \in S_n$, are contained in the same coset if and only if $x(\{1,2\}) = y(\{1,2\})$.

**Lemma 28.** *The $S_n$-orbits of $\theta_1$, $\xi_1$, $\eta_1$ contain all elements $\theta_i$, $\xi_i$ and $\eta_i$, $i = 1, \dots, n-1$, respectively. Moreover, for $w \in S_n$ we have $w^{-1}\theta_1 w = \theta_i$ if and only if $w(\{1,2\}) = \{i, i+1\}$ and analogously for $\xi_1$ and $\eta_1$.*

*Proof.* The proof for the $S_n$-orbit of $\theta_1$ is analogous to that of Lemma 6. We prove the statement for the $S_n$-orbit of $\xi_1$. For the $S_n$-orbit of $\eta_1$ the arguments are analogous. We use induction on $i$ with the case $i = 1$ being trivial. Let $i > 1$ and assume that $\xi_{i-1}$ is contained in our orbit. Then, using (4.7), we compute

$$\xi_i = \theta_i\vartheta_i = \sigma_{i-1}\sigma_i\theta_{i-1}\sigma_i\sigma_{i-1}\vartheta_i = \sigma_{i-1}\sigma_i\theta_{i-1}\sigma_i\vartheta_{i-1}\sigma_{i-1} =$$
$$\sigma_{i-1}\sigma_i\theta_{i-1}\vartheta_{i-1}\sigma_i\sigma_{i-1} = \sigma_{i-1}\sigma_i\xi_{i-1}\sigma_i\sigma_{i-1},$$

and hence $\xi_i$ is contained in our orbit as well. The second claim follows from (3.6). This completes the proof. $\square$

For $w \in S_n$ such that $w(\{1,2\}) = \{i,j\}$, where $i < j$, we set $\epsilon_{i,j} = w^{-1}\theta_1 w$, $\mu_{i,j} = w^{-1}\xi_1 w$, $\nu_{i,j} = w^{-1}\eta_1 w$. All these elements are well defined by Lemma 27.

**Lemma 29.** *(a)* $\vartheta_i\epsilon_{i,j} = \vartheta_j\epsilon_{i,j} = \vartheta_i\vartheta_j\epsilon_{i,j} = \nu_{i,j}$*;* $\vartheta_k\epsilon_{i,j} = \epsilon_{i,j}\vartheta_k$*,* $k \notin \{i,j\}$*.*

*(b)* $\vartheta_i\mu_{i,j} = \vartheta_j\mu_{i,j} = \vartheta_i\vartheta_j\mu_{i,j} = \vartheta_i\vartheta_j$*;* $\vartheta_k\mu_{i,j} = \mu_{i,j}\vartheta_k$*,* $k \notin \{i,j\}$*.*

*Proof.* First we prove (a). Because of Lemma 28 it is enough to check that $\vartheta_1\epsilon_{1,2} = \vartheta_2\epsilon_{1,2} = \vartheta_1\vartheta_2\epsilon_{1,2} = \nu_{1,2}$ and that $\vartheta_3\epsilon_{1,2} = \epsilon_{1,2}\vartheta_3$. The latter equalities follow from (5.2) and (5.1).

Now we prove (b). Again, because of Lemma 28 it is enough to check that $\vartheta_1\mu_{1,2} = \vartheta_2\mu_{1,2} = \vartheta_1\vartheta_2\mu_{1,2} = \vartheta_1\vartheta_2$ and that $\vartheta_3\mu_{1,2} = \mu_{1,2}\vartheta_3$. Using (5.3), (5.2) and (5.1) we have

$$\vartheta_1\mu_{1,2} = \vartheta_1\theta_1\vartheta_1 = \vartheta_1\vartheta_2; \quad \vartheta_1\mu_{2,3} = \vartheta_1\theta_2\vartheta_2 = \theta_2\vartheta_1\vartheta_2 = \theta_2\vartheta_2\vartheta_1 = \mu_{2,3}\vartheta_1,$$

as required. □

**Lemma 30.** *Suppose* $\{i,j\} \cap \{p,q\} = \varnothing$*. Then* $\epsilon_{i,j}\epsilon_{p,q} = \epsilon_{p,q}\epsilon_{i,j}$*,* $\mu_{i,j}\mu_{p,q} = \mu_{p,q}\mu_{i,j}$ *and* $\epsilon_{i,j}\mu_{p,q} = \mu_{p,q}\epsilon_{i,j}$*.*

*Proof.* Following the arguments from the proof of Lemma 7 it is enough to show that $\mu_{1,2}\mu_{3,4} = \mu_{3,4}\mu_{1,2}$ and $\mu_{1,2}\epsilon_{3,4} = \epsilon_{3,4}\mu_{1,2}$, that is that $\xi_1\xi_3 = \xi_3\xi_1$ and $\xi_1\theta_3 = \theta_3\xi_1$. Using (5.1), (4.6) and (3.2) we have

$$\xi_1\xi_3 = \theta_1\vartheta_1\theta_3\vartheta_3 = \theta_1\theta_3\vartheta_1\vartheta_3 = \theta_3\theta_1\vartheta_3\vartheta_1 = \theta_3\vartheta_3\theta_1\vartheta_1 = \xi_3\xi_1,$$

and using (5.1) and (3.2) we also obtain $\xi_1\theta_3 = \theta_1\vartheta_1\theta_3 = \theta_1\theta_3\vartheta_1 = \theta_3\xi_1$, as required. □

**Lemma 31.** *Suppose* $\{i,j\} \cap \{p,q\} \neq \varnothing$*. Then each of the elements* $\epsilon_{i,j}\epsilon_{p,q}$*,* $\mu_{i,j}\mu_{p,q}$*,* $\epsilon_{i,j}\mu_{p,q}$*,* $\mu_{i,j}\epsilon_{p,q}$ *equals to the element of the form* $u\theta_1 v$ *for some* $u, v \in \mathcal{IS}_n$*.*

*Proof.* Using the argument from the proof of Lemma 8 it is enough to prove the statement only for the elements $\mu_{1,2}\mu_{2,3}$, $\mu_{1,2}\epsilon_{2,3}$, $\epsilon_{1,2}\mu_{2,3}$. We have

$$\mu_{1,2}\mu_{2,3} = \xi_1\xi_2 = \theta_1\vartheta_1\theta_2\vartheta_2 = \theta_1\vartheta_2\theta_2\vartheta_2 = \theta_1\vartheta_2\vartheta_3 = \xi_1\vartheta_3$$

by (5.2) and (5.3); and

$$\mu_{1,2}\epsilon_{2,3} = \theta_1\vartheta_1\theta_2 = \theta_1\vartheta_1\sigma_1\sigma_2\theta_1\sigma_1\sigma_2 = \theta_1\sigma_1\vartheta_2\sigma_2\theta_1\sigma_1\sigma_2 = \\ \theta_1\sigma_1\sigma_2\vartheta_3\theta_1\sigma_1\sigma_2 = \theta_1\sigma_1\sigma_2\theta_1\vartheta_3\sigma_1\sigma_2 = \theta_1\sigma_2\theta_1\vartheta_3\sigma_1\sigma_2 = \theta_1\vartheta_3\sigma_1\sigma_2$$

by (3.1), (3.4), (3.3), (4.7). Finally,

$$\epsilon_{1,2}\mu_{2,3} = \theta_1\theta_2\vartheta_2 = \theta_1\sigma_1\sigma_2\theta_1\sigma_2\sigma_1\vartheta_2 = \theta_1\sigma_2\vartheta_1\sigma_1.$$

using (3.1), (3.3) and (3.4). The statement follows. □

For each subset $\{i_1,\dots,i_k\}$ of $\{1,2,\dots,n\}$ set $\vartheta(\{i_1,\dots,i_k\}) = \vartheta_{i_1}\dots\vartheta_{i_k}$. Obviously, $\vartheta(\{i_1,\dots,i_k\})$ is an idempotent and each idempotent of $\mathcal{IS}_n$ has such a form. In the sequel we will use the obvious fact that each element of $\mathcal{IS}_n$ can be written in the form $uv$, where $u$ is an idempotent, and $v \in S_n$.

As in the previous sections we consider the $S_n \times S_n$-action on $T$ given by $(g,h)(x) = g^{-1}xh$ for $x \in T$ and $(g,h) \in S_n \times S_n$.

**Lemma 32.** *Every $S_n \times S_n$-orbit contains either $e$ or an element of the form $\vartheta(A)\gamma_{i_1,j_1}\dots\gamma_{i_s,j_s}$, where $A \subset \{1,2,\dots,n\}$, the sets $\{i_l,j_l\}$ are pairwise disjoint, and each $\gamma_{i_l,j_l}$ equals either $\epsilon_{i_l,j_l}$ or $\mu_{i_l,j_l}$.*

*Proof.* The idea of the proof is analogous to that of Lemma 9. Let $x \in T$. If $x \in S_n$ the statement is obvious. Assume that $x \notin S_n$. Since $T$ is generated by $\mathcal{IS}_n$ and $\theta_1$ we can write

$$x = wu\theta_1u_1g_1\theta_1u_2g_2\cdots\theta_1u_kg_k \tag{5.5}$$

for some $k \geq 1$, $w, g_1,\dots,g_k \in S_n$ and $u, u_1,\dots,u_k \in E(\mathcal{IS}_n)$. Moreover, we may assume that $x$ can not be written as a product of $\theta_1$'s and elements of $\mathcal{IS}_n$, which contains less than $k$ occurrences of $\theta_1$. We claim that $x$ can be written as

$$x = wu'\gamma_1^1g_1'\gamma_1^2g_2'\cdots\gamma_1^kg_k', \tag{5.6}$$

where, $w, g_1',\dots,g_k' \in S_n$, $u' \in E(\mathcal{IS}_n)$, and each $\gamma_1^i$ is equal to either $\theta_1$ or $\xi_1$. Let us prove this by induction on $k$. Let $k=1$ and $x = wu\theta_1u_1g_1$. We know that $u_1 = \vartheta(B)$ for some $B \subset \{1,\dots,n\}$. Let $A = B \setminus \{1,2\}$. Using (5.1) and (5.2) we obtain that

$$x = \begin{cases} wuu_1\theta_1g_1, & \text{if } B \cap \{1,2\} = \varnothing; \\ wu\vartheta(A)\xi_1g_1, & \text{if } B \cap \{1,2\} \neq \varnothing, \end{cases}$$

as required. Let now $k \geq 2$. Applying the basis of the induction to $\theta_1u_kg_k$ we obtain

$$x = wu\theta_1u_1g_1\theta_1u_2g_2\cdots\theta_1u_{k-1}g_{k-1}\theta_1u_kg_k = \\ wu\theta_1u_1g_1\theta_1u_2g_2\cdots\theta_1u_{k-1}g_{k-1}u_k'\gamma_1^kg_k,$$

where $u_k'$ is an idempotent of $\mathcal{IS}_n$ and $\gamma_1^k$ is either $\xi_1$ or $\theta_1$. Now, since $u_{k-1}g_{k-1}u_k' \in \mathcal{IS}_n$, we can write $u_{k-1}g_{k-1}u_k' = u_{k-1}'g_{k-1}'$ for some $g_{k-1}' \in S_n$ and $u_{k-1}' \in E(\mathcal{IS}_n)$. Now (5.6) follows by applying the inductive assumption to $wu\theta_1u_1g_1\theta_1u_2g_2\cdots u_{k-2}g_{k-2}\theta_1u_{k-1}'g_{k-1}'$.

Similarly to (3.7) we can rewrite (5.6) as follows:

$$x = wu'(g_1'\cdots g_k')(g_1'\cdots g_k')^{-1}\gamma_1^1(g_1'\cdots g_k')\cdot \\ \cdot(g_2'\cdots g_k')^{-1}\gamma_1^2(g_2'\cdots g_k')\cdots(g_{k-1}'g_k')^{-1}\gamma_1^{k-1}(g_{k-1}'g_k')g_k'^{-1}\gamma_1^kg_k',$$

and therefore we can write

$$x = vu'\gamma_{i_1,j_1}\cdots\gamma_{i_k,j_k}, \tag{5.7}$$

where $v = wg'_1\cdots g'_k$, $\{i_t,j_t\}$=$\{(g'_t\cdots g'_k)(1),(g'_t\cdots g'_k)(2)\}$, $1\leq t\leq k$, and each $\gamma_{i_l,j_l}$ is equal to either $\epsilon_{i_l,j_l}$ or $\mu_{i_l,j_l}$. Since $x$ is initially chosen such that it can not be reduced to an element of $T$, which contains less that $k$ entries of $\theta_1$, from Lemma 31 it follows that $\{i_t,j_t\}\cap\{i_l,j_l\}=\varnothing$ for any two factors $\gamma_{i_t,j_t}$, $\gamma_{i_l,j_l}$ in (5.7). This implies that the $S_n\times S_n$-orbit of $x$ contains $u'\gamma_{i_1,j_1}\cdots\gamma_{i_s,j_s}$ such that $\{i_t,j_t\}\cap\{i_l,j_l\}=\varnothing$ for all $l\neq t$. The statement follows. □

**Corollary 33.** *Any $S_n\times S_n$- orbit contains either $e$ or an element of the form $\vartheta(A)\gamma_{i_1,j_1}\cdots\gamma_{i_s,j_s}$, such that*

*(i) the sets $\{i_l,j_l\}$ are pairwise disjoint;*

*(ii) each $\gamma_{i_l,j_l}$ equals to either $\epsilon_{i_l,j_l}$ or $\mu_{i_l,j_l}$ or $\nu_{i_l,j_l}$;*

*(iii) $A\cap\{i_1,j_1,\dots i_s,j_s\}=\varnothing$.*

*Proof.* This follows from Lemma 32 and Lemma 29. □

Now we introduce the notion of a canonical element. Let $k,l,m,t$ be some non-negative integers satisfying $2k+2l+2m+t\leq n$. Set $\delta(0,0,0,0)=e$ and if at least one of $k,l,m,t$ is not zero, set

$$\delta(k,l,m,t)=\theta_1\theta_3\cdots\theta_{2k-1}\xi_{2k+1}\xi_{2k+3}\cdots\xi_{2k+2l-1}\cdot\nu_{2k+2l+1}\nu_{2k+2l+3}\cdots\cdot \\ \cdot\nu_{2k+2l+2m-1}\vartheta_{2k+2l+2m+1}\vartheta_{2k+2l+2m+2}\cdots\vartheta_{2k+2l+2m+t}. \tag{5.8}$$

The element $\delta(k,l,m,t)$ such that $l=0$ or $m=0$ will be called a *canonical element* of type $(k,l,m,n)$.

**Corollary 34.** *Every $S_n\times S_n$-orbit contains a canonical element.*

*Proof.* Because of Corollary 33 we have to prove that, the $S_n\times S_n$-orbit of the element $\vartheta(A)\gamma_{i_1,j_1}\cdots\gamma_{i_s,j_s}$, satisfying the conditions of Corollary 33, contains a canonical element. Using conjugation, we can always reduce $\vartheta(A)\gamma_{i_1,j_1}\cdots\gamma_{i_s,j_s}$ to some $\delta(k,l,m,t)$. However, it might happen that both $m$ and $l$ will be non-zero. Without loss of generality we may assume $m\geq l\geq 1$. Using (5.4) and conjugation we get that the $S_n\times S_n$-orbit of the element $\mu_{i,j}\nu_{p,q}$ contains $\theta_{i,j}\vartheta_p\vartheta_q$ provided that $\{i,j\}\cap\{p,q\}=\varnothing$. Hence the $S_n\times S_n$-orbit of our $\delta(k,l,m,t)$ contains $\delta(k+1,l-1,m-1,t+2)$. Proceeding by induction we get that the $S_n\times S_n$-orbit of our $\delta(k,l,m,t)$ contains $\delta(k+l,0,m-l,t+2l)$, which is canonical. This completes the proof. □

**Lemma 35.** *The $S_n \times S_n$-orbits of the canonical element $\delta(k,l,0,t)$ and $\delta(k,0,l,t)$ contain at most*

$$\frac{(n!)^2}{(k+l)!2^{k+l}t!k!2^k(2l+t)!(n-2k-2l-t)!}$$

*elements.*

*Proof.* We will prove the statement for the element $\delta(k,l,0,t)$. For $\delta(k,0,l,t)$ the proof is analogous. We use the arguments similar to those from the proof of Lemma 11. It is enough to show that the stabilizer of $\delta(k,l,0,t)$ under the $S_n \times S_n$-action contains at least $(k+l)!2^{k+l}t!k!2^k(2l+t)!(n-2k-2l-t)!$ elements. Set

$$\Sigma_i^0 = \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i},\ 1 \le i \le k+l-1;$$
$$\Sigma_i^1 = \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i}\sigma_{2i-1},\quad 1 \le i \le k+l-1.$$

Then both $\Sigma_i^0$ and $\Sigma_i^1$ swap the sets $\{2i-1, 2i\}$ and $\{2i+1, 2i+2\}$. It follows that the group $H$, generated by all $\Sigma_i^0$, consists of all permutations of the set $\{1,2\}, \{3,4\}, \dots, \{2k+2l-1, 2k+2l\}$ and is therefore isomorphic to the group $S_{k+l}$. It is further easy to see that the group $\tilde{H}$, generated by all $\Sigma_i^0$ and $\Sigma_i^1$, is isomorphic to the wreath product $H \wr S_2$. From (3.5) and (3.3) it follows that the left multiplications with $\Sigma_i^0$ and $\Sigma_i^1$ stabilizes $\delta(k,l,0,t)$. Therefore the left multiplication with each element of $\tilde{H}$ stabilizes $\delta(k,l,0,t)$ as well. Now, from (4.7) and (5.2) it follows that

$$\sigma_i\eta_i = \sigma_i\vartheta_i\vartheta_{i+1}\theta_i = \vartheta_{i+1}\sigma_i\vartheta_{i+1}\theta_i = \vartheta_i\sigma_i\vartheta_i\theta_i = \vartheta_i\vartheta_{i+1}\theta_i = \eta_i.$$

for all $i = 1, \dots, n-1$. Moreover,

$$\begin{aligned}\sigma_{i+1}\eta_i\eta_{i+2} = \sigma_{i+1}\vartheta_{i+1}\theta_i\vartheta_{i+2}\theta_{i+2} = \sigma_{i+1}\vartheta_{i+1}\vartheta_{i+2}\theta_i\theta_{i+2} = \\ \vartheta_{i+1}\vartheta_{i+2}\theta_i\theta_{i+2} = \vartheta_{i+1}\theta_i\vartheta_{i+2}\theta_{i+2} = \eta_i\eta_{i+2}\end{aligned}$$

for all $i = 1, \dots, n-3$ by (5.1) and (4.7) and

$$\sigma_{i+1}\eta_i\vartheta_{i+2} = \sigma_{i+1}\vartheta_{i+1}\theta_i\vartheta_{i+2} = \sigma_{i+1}\vartheta_{i+1}\vartheta_{i+2}\theta_i = \vartheta_{i+1}\vartheta_{i+2}\theta_i = \eta_i\vartheta_{i+2}$$

for all $i = 1, \dots, n-2$ again by (5.1) and (4.7). Using this and the fact that $\eta_i$ commutes with each of $\theta_j$, $\eta_j$, $\xi_j$ whenever $|i-j| > 1$ we see that each of the elements $\sigma_i$, $2k+2l-1 \le i \le 2k+2l+t$, stabilizes $\delta(k,l,0,t)$ under the left multiplication. All these elements generate the group $H_0 \simeq S_t$, which stabilizes $\delta(k,l,0,t)$ and has trivial intersection with $\tilde{H}$. Let $H_1 = H_0 \times \tilde{H}$.

Analogously one shows that there is a group, $H_2$, isomorphic to the wreath product $(S_k \wr S_2) \times S_{2l+t}$, such that each element of this group stabilizes $\delta(k, l, 0, t)$ with respect to the right multiplication. Apart from this, from (3.3) we have that conjugation by any element from the group $H_3 = \langle \sigma_{2k+2l+t+1}, \dots, \sigma_{n-1} \rangle \simeq S_{n-2k-2l-t}$ stabilizes $\delta(k, l, 0, t)$. Observe that the group, generated by $H_1$, $H_2$ and $H_3$, is a direct product of $H_1$, $H_2$ and $H_3$. Hence, using the product rule we derive that the cardinality of the stabilizer of $\delta(k, l, 0, t)$ is at least

$$(k + l)!2^{k+l}t!k!2^k(2l + t)!(n - 2k - 2l - t)!,$$

and the proof is complete. $\square$

*Proof of Theorem 26.* Comparing Lemma 35 and Proposition 1(d) we have $|T| \leq |\mathfrak{B}_n|$. Since $\varphi : T \to \mathfrak{B}_n$ is surjective we have $|T| \geq |\mathfrak{B}_n|$. Hence $|T| = |\mathfrak{B}_n|$ and $\varphi$ is an isomorphism. $\square$

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