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On primitive Schubert indices of genus g and weight $g-1$

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Introduction

Let $\alpha_0, \alpha_1, \cdots, \alpha_{g-1}$ be integers satisfying $0 \leqq \alpha_0 \leqq \alpha_1 \leqq \cdots \leqq \alpha_{g-1} \leqq g-1$. Then $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_{g-1})$ is called a *Schubert index of genus* g. Moreover, we call $\sum_{i=0}^{g-1} \alpha_i$ the *weight of* α, which is denoted by $w(\alpha)$. We denote by $H(\alpha)$ the complement of the set $\{\alpha_i+i+1 \mid i=0, 1, \cdots, g-1\}$ in $\boldsymbol{N}$ where $\boldsymbol{N}$ is the additive semigroup of non-negative integers. If $H(\alpha)$ becomes a subsemigroup of $\boldsymbol{N}$, then it is said that α *satisfies the semigroup condition.* Let C be a complete non-singular 1-dimensional algebraic variety over the field $\boldsymbol{C}$ of complex numbers (which is called a *smooth curve*) of genus g. For any point P of C, a non-negative integer n is called a *gap at* P if

$$h^0(C, \mathcal{O}_C((n-1)P)) = h^0(C, \mathcal{O}_C(nP)),$$

i.e., there exists a holomorphic differential form on C vanishing to order $n-1$ at P. Then the number of gaps at P is equal to g. Let $m_1 < m_2 < \cdots < m_g$ be the gaps at P. If we set $\alpha_{i-1}(P)=m_i-i$ for $i=1, 2, \cdots, g$, then $\alpha(P)=(\alpha_0(P), \cdots, \alpha_{g-1}(P))$ is a Schubert index of genus g satisfying the semigroup condition.

Recall that Schubert indices of genus g are partially ordered by $\alpha \leqq \beta$ if $\alpha_i \leqq \beta_i$, $i=0, 1, \cdots, g-1$ where $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_{g-1})$ and $\beta=(\beta_0, \beta_1, \cdots, \beta_{g-1})$. We say that a Schubert index α is *primitive* if every Schubert index β with $\beta \leqq \alpha$ satisfies the semigroup condition. Let $\mathcal{M}_{g,1}$ be the moduli space of pointed smooth curves of genus g. For any Schubert index α of genus g we may define a locally closed subset of $\mathcal{M}_{g,1}$ by $\mathcal{C}_\alpha=\{(C, P) \in \mathcal{M}_{g,1} \mid \alpha(P)=\alpha\}$. Then the weight $w(\alpha)$ of α gives an upper bound for the codimension of any component of $\mathcal{C}_\alpha$. To simplify the discussion a point $x=(C, P) \in \mathcal{M}_{g,1}$ is said to be *dimensionally proper* if $\mathcal{C}_{\alpha(P)}$ has codimension $w(\alpha(P))$ in a neighborhood of x. Then Eisenbud-Harris [2] showed that for any primitive Schubert index α of genus g and weight $\leqq g-2$ there exists a dimensionally proper point with Schubert index α. In this paper we will show the following:

MAIN THEOREM. *For any primitive Schubert index* α *of genus* g *and weight* $g-1$ *there exists a dimensionally proper point with Schubert index* α.

We show in §1 that using the result of Eisenbud-Harris [2] Main Theorem is reduced to the following:

For any odd integer g *there exists a dimensionally proper point with Schubert index* $\alpha(g)$ *where we set* $\alpha(g)=(0^{(g+1)/2}, 2^{(g-1)/2})$.

Let $\varphi_g : \boldsymbol{C}[X_1, \cdots, X_{(g+1)/2}] \longrightarrow \boldsymbol{C}[t^h]_{h\in H(\alpha(g))}$ be the $\boldsymbol{C}$-algebra homomorphism defined by $\varphi_g(X_i)=t^{a_i}$ where $\{a_1<a_2<\cdots<a_{(g+1)/2}\}$ is the minimal set of generators for the semigroup $H(\alpha(g))$ determined by the Schubert index $\alpha(g)$. In §2 we determine the generators for the ideal $\operatorname{Ker}\varphi_g$. Combining it with Corollary 4.9 in Komeda [3] it is proved that $C_{\alpha(g)}\neq\emptyset$. Moreover, it follows from Coppens [1] that if $C_{\alpha(g)}\neq\emptyset$, any point of $C_{\alpha(g)}$ is dimensionally proper.

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§1. Reduction to the Schubert indices $(0^{(g+1)/2}, 2^{(g-1)/2})$.

First we give a few properties of primitive Schubert indices. Let $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_{g-1})$ be a Schubert index of genus g with $\alpha_0=\alpha_1=\cdots=\alpha_{i-1}=0$ and $\alpha_i>0$. Then we say that $i+1$ is the *first non-gap of* α and that $g+\alpha_{g-1}$ is the *last gap of* α. By Proposition 1.1 in [2], α is primitive if and only if twice its first non-gap is larger than its last gap.

LEMMA 1. *Let* $\alpha=(0^p, \alpha_1, \cdots, \alpha_q)$ *be a primitive Schubert index of genus* $p+q$ *with* $\alpha_1>0$, *and let* $\beta=(0^{p+1}, \beta_1, \cdots, \beta_q)$ *be a Schubert index of genus* $p+q+1$. *If* β_q *is equal to* α_q *or* α_q+1, *then* β *is primitive.*

PROOF. The first non-gap and the last gap of α are equal to $p+1$ and $p+q+\alpha_q$ respectively. Since α is primitive, we have $2(p+1)>p+q+\alpha_q$. Hence we get

$$2(p+2)-(p+1+q+\beta_q)=2(p+1)-(p+q+\alpha_q)+1+(\alpha_q-\beta_q)>0,$$

which implies that β is primitive. *Q.E.D.*

LEMMA 2. *If* $\alpha=(0^p, \alpha_1, \alpha_2, \cdots, \alpha_q)$ *is a primitive Schubert index with* $\alpha_1>0$, *then so is* $\beta=(0^p, \alpha_2, \cdots, \alpha_q)$.

PROOF. Since α is primitive, we have $2(p+1)>p+q+\alpha_q$. Hence we get $2(p+1)-(p+q-1+\alpha_q)>1$, which implies that β is primitive. *Q.E.D.*

Eisenbud and Harris (Theorem 5.4 in [2]) showed the following, which was a key lemma to prove that for any Schubert index α of genus g and weight $\leqq g-2$ there exists a dimensionally proper point with Schubert index α.

If $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_{g-2})$ *is a Schubert index of genus* $g-1$ *such that* $C_\alpha\subset\mathcal{M}_{g-1,1}$ *contains a dimensionally proper point, then so does* $C_\beta\subset\mathcal{M}_{g,1}$ *if the Schubert index* $\beta=(\beta_0, \beta_1, \cdots, \beta_{g-1})$ *satisfies one of the following:*

1) $\beta_0=0,\ \beta_i=\alpha_{i-1}\ (i=1, \cdots, g-1)$,

2) *for some* $0<j\leqq g-1,\ \beta_0=0,\ \beta_j=\alpha_{j-1}+1,\ \beta_i=\alpha_{i-1}\ (i=1, \cdots, g-1, i\neq j)$ *and* β *satisfies the semigroup condition.*

We note that the case $j=g-1$ in 2) is excluded from Theorem 5.4 in [2], but its proof shows that this case is also O.K.

For any odd integer $g\geqq 3$ we denote by $\alpha(g)$ the Schubert index $(0^{(g+1)/2}, 2^{(g-1)/2})$.

PROPOSITION 1. *If for any odd integer* $g\geqq 3$, $C_{\alpha(g)}\subset\mathcal{M}_{g,1}$ *contains a dimensionally proper point, then for any primitive Schubert index* β *of genus* h *and weight* $h-1$, *so does* C_β.

PROOF. Let $\beta=(0^{h-n}, \beta_1, \cdots, \beta_n)$ be a primitive Schubert index of genus h and weight $h-1$ with $\beta_1>0$. First we suppose that $\beta_1\geqq 2$. Then we have a sequence

$$\begin{aligned}\gamma^{(0)} &= \alpha(2n+1) = (0^{n+1}, 2^n) \longrightarrow \gamma^{(1)} = (0^{n+2}, 2^{n-1}, 3) \longrightarrow \gamma^{(2)}\\ &= (0^{n+3}, 2^{n-1}, 4) \longrightarrow \cdots \longrightarrow \gamma^{(\beta_n-2)} = (0^{n+\beta_n-1}, 2^{n-1}, \beta_n) \longrightarrow \gamma^{(\beta_n-1)}\\ &= (0^{n+\beta_n}, 2^{n-2}, 3, \beta_n) \longrightarrow \cdots \longrightarrow \gamma^{(\beta_n+\beta_{n-1}-4)}\\ &= (0^{n+\beta_n+\beta_{n-1}-3}, 2^{n-2}, \beta_{n-1}, \beta_n) \longrightarrow \cdots \longrightarrow \gamma^{(h-2n-1)}\\ &= (0^{h-n}, \beta_1, \cdots, \beta_n) = \beta,\end{aligned}$$

where $w(\gamma^{(i+1)})=w(\gamma^{(i)})+1$ for $i=0, 1, \cdots, h-2n-2$, and all Schubert indices in the above are primitive because of Lemma 1. It follows from the above result of [2] and the assumption that $C_\beta\subset\mathcal{M}_{h,1}$ contains a dimensionally proper point.

Next we suppose that $\beta_1=\cdots=\beta_l=1$ and $\beta_{l+1}\geqq 2$ for some $1\leqq l\leqq n-1$. Then we have a sequence

$$\begin{aligned}\delta &= \delta^{(0)} = (0^{h-n}, \beta_{l+1}, \cdots, \beta_n) \longrightarrow \delta^{(1)} = (0^{h-n}, 1, \beta_{l+1}, \cdots, \beta_n) \longrightarrow \delta^{(2)}\\ &= (0^{h-n}, 1, 1, \beta_{l+1}, \cdots, \beta_n) \longrightarrow \cdots \longrightarrow \delta^{(l)} = (0^{h-n}, 1^l, \beta_{l+1}, \cdots, \beta_n) = \beta,\end{aligned}$$

where $w(\delta^{(i+1)})=w(\delta^{(i)})+1$ for $i=0, 1, \cdots, l-1$. Since β is primitive, it follows from Lemma 2 that all Schubert indices in the above are primitive. By the above $C_\delta\subset\mathcal{M}_{h-l,1}$ contains a dimensionally proper point, which implies that so does C_β. *Q.E.D.*

§2. On the Schubert indices $\alpha(g)=(0^{(g+1)/2}, 2^{(g-1)/2})$.

Let $H=H(g)=H(\alpha(g))$ be the subsemigroup of N defined by $\alpha(g)$. Then the minimal set of generators for H is

$$\{a_1 = p+1,\ a_2=p+2,\ a_3 = g+6,\ a_4 = g+7, \cdots, a_p = g+3+p\},$$

where we set $p=(g+1)/2$.

Let $\mathcal{M}_g^{(\upsilon)}$ be the fine moduli space of the smooth curves of genus g with a level υ-structure for some fixed $\upsilon \geqq 3$ and $p: \mathcal{X}_g \to \mathcal{M}_g^{(\upsilon)}$ the associated smooth family of curves. For any positive integers n and r which are relatively prime, we set

$$G_{n,r} = \{x \in \mathcal{X}_g \mid n \text{ is the first non-gap of } x \text{ on } p^{-1}(p(x)) \text{ and } n+r \text{ is the first non-gap of } x \text{ relatively prime to } n \text{ and } h^0(p^{-1}(p(x)), (n+r)x) = h+2 \text{ where } n+r = hn+\varepsilon \text{ with } 0<\varepsilon<n\}.$$

Then Coppens [1] showed that if $G_{n,r} \neq \emptyset$, then $G_{n,r}$ is equidimensional of dimension $g-4+2n+r-h$. To prove that $C_{\alpha(g)} \subset \mathcal{M}_{g,1}$ contains a dimensionally proper point it suffices to show that $C_{\alpha(g)} \neq \emptyset$, because applying the above Coppens' result to our case $n=(g+1)/2+1$ and $r=1$ we get

$$\dim C_{\alpha(g)} \leqq \dim G_{(g+1)/2+1,1} = g-4+2((g+1)/2+1)+1-1 = 2g-1$$

and we know that any irreducible component of $C_{\alpha(g)}$ has dimension larger than or equal to

$$3g-2-w(\alpha(g)) = 3g-2-(g-1) = 2g-1\,.$$

Let φ be the $\boldsymbol{C}$-algebra homomorphism from $\boldsymbol{C}[X_1, X_2, \cdots, X_p]$ to $\boldsymbol{C}[t^h]_{h \in H}$ sending X_i to t^{a_i}. Then we have the following:

PROPOSITION 2. *The ideal* $I=\operatorname{Ker}\varphi$ *is generated by*

$$X_2^3-X_1X_3,\ X_2X_j-X_1X_{j+1}\ (3 \leqq j \leqq p-1)\,,$$
$$X_2X_p-X_1^4,\ X_3X_j-X_2^2X_{j+1}\ (3 \leqq j \leqq p-1)\,,$$
$$X_3X_p-X_2^2X_1^3,\ X_iX_j-X_{i-1}X_{j+1}\ (4 \leqq i \leqq p-1,\ i \leqq j \leqq p-1)\,,$$
$$X_iX_p-X_{i-1}X_1^3\ (4 \leqq i \leqq p)\,.$$

PROOF. We consider the set

$$\{a_1<a_2,\ 2a_1<a_1+a_2<2a_2<a_3<a_4<\cdots<a_p<3a_1\}.$$

Then we have

$$a_2 = a_1+1,\ a_3 = 2a_2+1,\ a_4 = a_3+1,\ a_5 = a_4+1, \cdots, a_p = a_{p-1}+1\,,$$
$$3a_1 = a_p+1\,.$$

Hence the ideal $I=\operatorname{Ker}\varphi$ contains

$$X_2X_2^2-X_1X_3,\ X_2X_j-X_1X_{j+1}\ (3 \leqq j \leqq p-1),$$
$$X_2X_p-X_1X_1^3,\ X_3X_j-X_2^2X_{j+1}\ (3 \leqq j \leqq p-1)\,,$$
$$X_3X_p-X_2^2X_1^3,\ X_iX_j-X_{i-1}X_{j+1}\ (4 \leqq i \leqq p-1,\ i \leqq j \leqq p-1)\,,$$
$$X_iX_p-X_{i-1}X_1^3\ (4 \leqq i \leqq p)\,.$$

Let J be the ideal generated by the above elements. Let $3 \leqq i \leqq j$. If $i+j \leqq p+1$, then

$$\begin{aligned} X_i X_j &\equiv X_{i-1} X_{j+1} \equiv X_{i-2} X_{j+2} \equiv \cdots \equiv X_4 X_{j+i-4} \equiv X_3 X_{j+i-3} \\ &\equiv X_2^2 X_{j+i-2} \equiv X_2 X_1 X_{j+i-1} = X_1 X_2 X_{j+i-1} \bmod J. \end{aligned}$$

If $i+j=p+2$, then

$$X_i X_j \equiv X_{i-1} X_{j+1} \equiv \cdots \equiv X_2^2 X_{j+i-2} = X_2^2 X_p \equiv X_2 X_1^4 = X_1^4 X_2 \bmod J.$$

If $i+j=p+3$, then

$$X_i X_j \equiv X_{i-1} X_{j+1} \equiv \cdots \equiv X_3 X_{j+i-3} = X_3 X_p \equiv X_2^2 X_1^3 = X_1^3 X_2^2 \bmod J.$$

If $i+j=p+4$, then

$$\begin{aligned} X_i X_j &\equiv X_{i-1} X_{j+1} \equiv \cdots \equiv X_4 X_{j+i-4} = X_4 X_p \equiv X_3 X_1^3 \\ &\equiv X_2^3 X_1^2 = X_1^2 X_2^3 \bmod J. \end{aligned}$$

If $i+j \geqq p+5$, then

$$\begin{aligned} X_i X_j &\equiv X_{i-1} X_{j+1} \equiv \cdots \equiv X_{i+j-p} X_p \equiv X_{i+j-p-1} X_1^3 \equiv X_{i+j-p-2} X_2 X_1^2 \\ &= X_1^2 X_2 X_{i+j-p-2} \bmod J. \end{aligned}$$

Therefore, if $3 \leqq i \leqq j$, we have

$$X_i X_j \equiv X_1 X_2 M \quad \bmod J,$$

where M is a monomial.

We may take as generators for the ideal $I = \operatorname{Ker} \varphi$ the following type:

$$F = \prod_i X_i^{\nu_i} - \prod_i X_i^{\mu_i}, \quad \nu_i \mu_i = 0 \qquad \text{for all } i.$$

We set $L = \prod_i X_i^{\nu_i}$ and $R = \prod_i X_i^{\mu_i}$. Then we may assume that F is one of the following types:

(1) $L = X_1 L_0$ and $R = X_i X_j R_0$, $3 \leqq i \leqq j$, where R_0 contains neither X_1 nor X_2.

(2) $L = X_2 L_0$ and $R = X_i X_j R_0$, $3 \leqq i \leqq j$, where R_0 contains neither X_1 nor X_2.

(3) $L = X_1 L_0$ and $R = X_2 R_0$, where L_0 (resp. R_0) does not contain X_2 (resp. X_1).

(4) Both L and R contain neither X_1 nor X_2.

If $R = X_i X_j R_0$, $3 \leqq i \leqq j$, then by the above we have $R \equiv X_1 X_2 M R_0 \bmod J$, where M is a monomial. Hence in the case (1) (resp. (2)), we may decrease the weighted degree of $F = L - R$, where the weighted degree on $\boldsymbol{C}[X_1, X_2, \cdots, X_p]$ is defined by the following: For any i, the weighted degree of X_i is a_i and for any non-zero element c of $\boldsymbol{C}$, the weighted degree of c is zero. In the case (4) we also have

$$L \equiv X_1X_2M \bmod J \quad \text{and} \quad R \equiv X_1X_2N \bmod J\,,$$

where M and N are monomials. Hence we may decrease the weighted degree of F. Lastly we consider the case (3). If $R_0=X_2^2M$, then we have

$$R = X_2^3M \equiv X_1X_3M \bmod J.$$

If $R_0=X_2M$ and M does not contain X_2, then we must have $M=X_iM_0$ for some $i\geqq 3$, which implies that

$$R = X_2^2M = X_2^2X_iM_0 \equiv \begin{cases} X_2X_1X_{i+1}M_0 \bmod J & \text{if } i\leqq p-1 \\ X_2X_1^4M_0 \bmod J & \text{if } i=p\,. \end{cases}$$

If R_0 does not contain X_2, then for some $i\geqq 3$ we have

$$R = X_2X_iM \equiv \begin{cases} X_1X_{i+1}M \bmod J & \text{if } i\leqq p-1 \\ X_1^4M \bmod J & \text{if } i=p\,, \end{cases}$$

where M is a monomial. Hence we may decrease the weighted degree of F. Therefore we get $\operatorname{Ker}\varphi=J$. *Q.E.D.*

DEFINITION. A subsemigroup S of $\boldsymbol{Z}^m$ is said to be *saturated* if the condition $nr\in S$, where n is a positive integer and r is an element of $\boldsymbol{Z}^m$, implies $r\in S$.

Let S be a subsemigroup of $\boldsymbol{Z}^{p+1}$ generated by $b_i=e_i$ for $1\leqq i\leqq p+1$, $b_{p+2}=e_1+e_2-e_3$ and $b_{p+i}=e_3+e_{i+1}-e_2$ for $3\leqq i\leqq p$, where for any $1\leqq i\leqq p+1$, e_i denotes the vector whose i-th component is equal to 1 and whose j-th component is equal to 0 if $j\neq i$.

PROPOSITION 3. *S is saturated.*

PROOF. It is sufficient to show that

$$\sum_{i=1}^{g+1} \boldsymbol{R}_+b_i\cap \boldsymbol{Z}^{p+1} = \sum_{i=1}^{g+1} \boldsymbol{N}b_i = S$$

where $\boldsymbol{R}_+$ denotes the set of non-negative real numbers. Let us take $y=\sum_{i=1}^{g+1}s_ib_i\in\boldsymbol{Z}^{p+1}$ with $s_i\in\boldsymbol{R}_+$. Then we may assume that for any i, $0\leqq s_i<1$. If we set $y=(y_1, y_2, \cdots, y_{p+1})$, then

$$y_1 = s_1+s_{p+2},\quad y_2 = s_2+s_{p+2}-(s_{p+3}+\cdots+s_{2p})\,,$$

$$y_3 = s_3-s_{p+2}+(s_{p+3}+\cdots+s_{2p}) \quad \text{and} \quad y_i = s_i+s_{p+i-1} \quad \text{for any } 4\leqq i\leqq p+1\,.$$

Hence we have $y_3=p-2, p-3, \cdots, 1, 0$. If $y_3=p-2$, then we get

$$s_3s_{p+3}\cdots s_{2p}\neq 0 \quad \text{and} \quad y_2 = s_2-y_3+s_3 = -p+2+s_3+s_2\,,$$

which imply that $y_2=-(p-3)$ and $y_i=1$ for any $4\leqq i\leqq p+1$. Hence we may assume that

$$y=(0,\ -(p-3),\ p-2,\ 1,\ \cdots,\ 1),$$

which implies that

$$y=b_{p+3}+b_{p+4}+\cdots+b_{2p}+b_2\in\sum_{i=1}^{g+1}\boldsymbol{N}b_i\,.$$

If $y_3=k\leqq p-3$, then $y_2=s_2-y_3+s_3=-k+s_2+s_3$, which implies that $y_2=-k$ or $-k+1$, and at least k elements of the set $\{s_{p+3},\cdots,s_{2p}\}$ are non-zero. Hence we may assume that

$$y=(0,\ -k,\ k,\ 1^k,\ 0^{p-2-k}),$$

which implies that

$$y=b_{p+3}+b_{p+4}+\cdots+b_{p+k+2}\in\sum_{i=1}^{g+1}\boldsymbol{N}b_i\,. \qquad Q.E.D.$$

We set $g_1=X_1^3$, $g_2=X_1$, $g_3=X_2$, $g_4=X_2^2$, $g_i=X_{i-2}$ for any $5\leqq i\leqq p+2$ and $g_{p+2+j}=X_{j+2}$ for any $1\leqq j\leqq p-2$. Let $\pi:\boldsymbol{C}[Y]=\boldsymbol{C}[Y_1,\cdots,Y_{2p}]\to\boldsymbol{C}[T^s]_{s\in S}=\boldsymbol{C}[t_1^{s_1}\cdots t_{p+1}^{s_{p+1}}]_{(s_1,\cdots,s_{p+1})\in S}$ (resp. $\eta:\boldsymbol{C}[Y]\to\boldsymbol{C}[X]=\boldsymbol{C}[X_1,\cdots,X_p]$) be the $\boldsymbol{C}$-algebra homomorphism defined by $\pi(Y_i)=T^{b_i}$ (resp. $\eta(Y_i)=g_i$).

PROPOSITION 4. *The ideal* $I=\mathrm{Ker}\,\varphi$ *is generated by the elements of* $\eta(\mathrm{Ker}\,\pi)$.

PROOF. Let $\zeta:\boldsymbol{C}[t_1,\cdots,t_{p+1}]\to\boldsymbol{C}[t^h]_{h\in H}=\boldsymbol{C}[H]$ be the $\boldsymbol{C}$-algebra homomorphism defined by

$$\zeta(t_1)=t^{3a_1},\quad \zeta(t_2)=t^{a_1},\quad \zeta(t_3)=t^{a_2},\quad \zeta(t_4)=t^{2a_2}\quad\text{and}$$
$$\zeta(t_i)=t^{a_{i-2}}\quad\text{for any } 5\leqq i\leqq p+1\,.$$

Let ζ' be the extension of ζ to $\boldsymbol{C}[T^s]_{s\in S}$. Then

$$\zeta'(T^{b_{p+2}})=\zeta(t_1)\zeta(t_2)\zeta(t_3)^{-1}=t^{3a_1+a_1-a_2}=t^{a_p}\in\boldsymbol{C}[H]\,,$$
$$\zeta'(T^{b_{p+3}})=\zeta(t_3)\zeta(t_4)\zeta(t_2)^{-1}=t^{a_2+2a_2-a_1}=t^{a_3}\in\boldsymbol{C}[H]\,,\quad\text{and}$$
$$\zeta'(T^{b_{p+i}})=\zeta(t_3)\zeta(t_{i+1})\zeta(t_2)^{-1}=t^{a_2+a_{i-1}-a_1}=t^{a_i}\in\boldsymbol{C}[H]$$

for any $4\leqq i\leqq p$. Hence we obtain $\zeta'(\boldsymbol{C}[T^s]_{s\in S})\subseteqq\boldsymbol{C}[H]$. We have a commutative diagram:

$$\begin{array}{ccc}\boldsymbol{C}[Y] & \xrightarrow{\ \eta\ } & \boldsymbol{C}[X]\\ \pi\downarrow & & \downarrow\varphi\\ \boldsymbol{C}[T^s]_{s\in S} & \xrightarrow{\ \zeta'\ } & \boldsymbol{C}[H]\end{array}$$

which implies that $\eta(\mathrm{Ker}\,\pi)\subseteqq\mathrm{Ker}\,\varphi=I$. Hence it suffices to show that the generators for I as in Proposition 2 is contained in the ideal $(\eta(\mathrm{Ker}\,\pi))$ generated by the elements of $\eta(\mathrm{Ker}\,\pi)$. Now we have the following:

$\pi(Y_3Y_4-Y_2Y_{p+3})=T^{b_3+b_4}-T^{b_2+b_{p+3}}=0$ and

$\eta(Y_3Y_4-Y_2Y_{p+3})=g_3g_4-g_2g_{p+3}=X_2^3-X_1X_3$.

$\pi(Y_3Y_{j+2}-Y_2Y_{p+j+1})=T^{b_3+b_{j+2}}-T^{b_2+b_{p+j+1}}=0$ and

$\eta(Y_3Y_{j+2}-Y_2Y_{p+j+1})=g_3g_{j+2}-g_2g_{p+j+1}=X_2X_j-X_1X_{j+1}$ for $3\leqq j\leqq p-1$.

$\pi(Y_3Y_{p+2}-Y_1Y_2)=0$ and $\eta(Y_3Y_{p+2}-Y_1Y_2)=g_3g_{p+2}-g_1g_2=X_2X_p-X_1^4$.

$\pi(Y_{p+3}Y_{j+2}-Y_4Y_{p+j+1})=0$ and

$\eta(Y_{p+3}Y_{j+2}-Y_4Y_{p+j+1})=g_{p+3}g_{j+2}-g_4g_{p+j+1}=X_3X_j-X_2^2X_{j+1}$ for $3\leqq j\leqq p-1$.

$\pi(Y_{p+3}Y_{p+2}-Y_1Y_4)=0$, because $b_{p+3}+b_{p+2}=e_1+e_4=b_1+b_4$, and

$\eta(Y_{p+3}Y_{p+2}-Y_1Y_4)=g_{p+3}g_{p+2}-g_1g_4=X_3X_p-X_1^3X_2^2$.

$\pi(Y_{p+i}Y_{j+2}-Y_{i+1}Y_{p+j+1})=0$ and $\eta(Y_{p+i}Y_{j+2}-Y_{i+1}Y_{p+j+1})$

$=g_{p+i}g_{j+2}-g_{i+1}g_{p+j+1}=X_iX_j-X_{i-1}X_{j+1}$ for $4\leqq i\leqq p-1$ and $i\leqq j\leqq p-1$.

$\pi(Y_{p+i}Y_{p+2}-Y_{i+1}Y_1)=0$, because $b_{p+i}+b_{p+2}=e_{i+1}+e_1=b_{i+1}+b_1$, and

$\eta(Y_{p+i}Y_{p+2}-Y_{i+1}Y_1)=g_{p+i}g_{p+2}-g_{i+1}g_1=X_iX_p-X_{i-1}X_1^3$ for $4\leqq i\leqq p$.

Q.E.D.

By Propositions 3 and 4, we may apply Corollary 4.9 in [**3**] to our case. Hence we get the following:

THEOREM. *For any odd integer $g\geqq 3$, we have $C_{\alpha(g)}\neq\emptyset$.*

COROLLARY. *For any odd integer $g\geqq 3$, $C_{\alpha(g)}\subset\mathcal{M}_{g,1}$ contains a dimensionally proper point. In this case, any point of $C_{\alpha(g)}$ is dimensionally proper.*

PROOF. The proof is given at the top of this section. *Q.E.D.*

Combining Corollary with Proposition 1, we get the main result in this paper.

MAIN THEOREM. *Let g be a positive integer. Then for any primitive Schubert index α of genus g and weight $g-1$, there exists a dimensionally proper point with Schubert index α.*

Let l_g (resp. m_g, resp. n_g) be the number of Schubert indices of genus g satisfying the semigroup condition (resp. Schubert indices of genus g and weight $g-1$ satisfying the semigroup condition, resp. primitive Schubert indices of genus g and weight $g-1$). Then we have the following table:

g	l_g	m_g	n_g
1	1	1	1
2	2	1	1
3	4	1	1
4	7	1	1
5	12	3	2
6	23	3	2
7	39	6	4
8	67	9	5
9	118	12	8
10	204	18	11
11	343	27	17
12	592	36	23
13	1001	51	34
14	1693	69	46
15	2857	95	65
16	4806	126	88

For example the primitive Schubert indices of genus 9 and weight 8 are the following:

$$(0^5, 2^4),\ (0^6, 2, 3, 3),\ (0^6, 2, 2, 4),\ (0^6, 1, 3, 4),\ (0^7, 4, 4),$$
$$(0^7, 3, 5),\ (0^7, 2, 6),\ (0^8, 8).$$

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