

On principal normal vector fields of submanifolds in a Riemannian manifold of constant curvature

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Introduction.

In [9], the author proved the complete integrability of the distribution of principal tangent vector spaces of a hypersurface in a Riemannian manifold of constant curvature and applied it to the investigation of minimal hypersurfaces in a sphere with some special properties. The principal tangent vector space at a point of a hypersurface in a Riemannian manifold is the linear subspace of vectors which are eigenvectors corresponding to a fixed eigenvalue of the 2nd fundamental form at the point. If we consider this eigenvalue as the length of a normal vector to the hypersurface, we can generalize the above consideration to any submanifold in a Riemannian manifold of constant curvature.

In the present paper, he will prove a more generalized theorem than Theorem 2 in [9] and study the properties of the integral submanifolds of the distribution of principal tangent vector spaces corresponding to a principal normal vector field.

§ 1. Preliminary.

For any C^∞ vector bundle $E \rightarrow M$ over a C^∞ differentiable manifold M , we denote the set of C^∞ cross sections by $\Gamma(E, M)$.

Let $\bar{M} = \bar{M}^{n+p}$ be an $(n+p)$ -dimensional C^∞ Riemannian manifold and $M = M^n$ be an n -dimensional immersed C^∞ submanifold in \bar{M} by an immersion $\phi: M \rightarrow \bar{M}$. Let $P: \phi^*T(\bar{M}) \rightarrow T(\bar{M})$ be the projection defined by the orthogonal decomposition:

$$T_{\phi(x)}(\bar{M}) = \phi_*(T_x(M)) + N_x, \quad x \in M \text{ and put } P^\perp = 1 - P.$$

We denote the normal bundle of M in \bar{M} by the immersion ϕ by $N(M, \bar{M})$ whose total space is $\bigcup_{x \in M} x \times N_x \subset M \times T(\bar{M})$. Then

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$$\phi^*T(\bar{M}) = T(M) \oplus N(M, \bar{M}).$$

In the following, we denote simply $\mathcal{X}(M) = \Gamma(T(M), M)$ and $\mathcal{X}^\perp(M) = \Gamma(N(M, \bar{M}), M)$. We denote the covariant differentiation operators for \bar{M} and M by $\bar{\nabla}$ and ∇ respectively. For the vector bundle $N(M, \bar{M})$, we have the naturally induced metric connection from the one of \bar{M} and denote the corresponding covariant differential operator by ∇^\perp . According to Martz [2] for an $X \in \mathcal{X}(M)$, we have the following decomposition of $\bar{\nabla}_X$.

$$(1.1) \quad \bar{\nabla}_X = \nabla_X + T_X \quad \text{on } \mathcal{X}(M),$$

where

$$\nabla_X = P\bar{\nabla}_X \quad \text{and} \quad T_X = P^\perp\bar{\nabla}_X$$

and

$$(1.2) \quad \bar{\nabla}_X = T_X + \nabla_X^\perp \quad \text{on } \mathcal{X}^\perp(M),$$

where

$$T_X = P\bar{\nabla}_X \quad \text{and} \quad \nabla_X^\perp = P^\perp\bar{\nabla}_X.$$

T_X is called the shape operator of M in \bar{M} due to O'Neill.

Applying P and P^\perp to the equation of definition of curvature tensor of \bar{M} ,

$$\bar{R}_{X,Y} = [\bar{\nabla}_X, \bar{\nabla}_Y] - \bar{\nabla}_{[X,Y]}$$

considered only for $X, Y \in \mathcal{X}(M)$, and substituting the decompositions (1.1) and (1.2) of $\bar{\nabla}_X, \bar{\nabla}_Y$, we have the following formulas:

(i) On $\mathcal{X}(M)$

$$(1.3) \quad P\bar{R}_{X,Y} = R_{X,Y} + [T_X, T_Y] \quad (\text{the Gauss equation})$$

$$(1.4) \quad P^\perp\bar{R}_{X,Y} = T_X\nabla_Y - T_Y\nabla_X + \nabla_X^\perp T_Y - \nabla_Y^\perp T_X - T_{[X,Y]} \\ (\text{the second Codazzi-Mainardi equation}),$$

where $R_{X,Y} = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ is the curvature tensor of M and $[T_X, T_Y] = T_X T_Y - T_Y T_X$.

(ii) On $\mathcal{X}^\perp(M)$

$$(1.5) \quad P\bar{R}_{X,Y} = \nabla_X T_Y - \nabla_Y T_X + T_X \nabla_Y^\perp - T_Y \nabla_X^\perp - T_{[X,Y]} \\ (\text{the first Codazzi-Mainardi equation}),$$

$$(1.6) \quad P^\perp\bar{R}_{X,Y} = R_{X,Y}^\perp + [T_X T_Y] \quad (\text{the Ricci equation}),$$

where $R_{X,Y}^\perp = \nabla_X^\perp \nabla_Y^\perp - \nabla_Y^\perp \nabla_X^\perp - \nabla_{[X,Y]}^\perp$ is the curvature tensor of $N(M, \bar{M})$.

Now, take a fixed point $x \in M$ and a normal vector $v \in N_x$. A non zero tangent vector $u \in M_x = T_x(M)$ is called a *principal tangent vector* for v if

$$(1.7) \quad T_u z = \langle u, z \rangle v \quad \text{for any } z \in M_x,$$

where \langle, \rangle denotes the inner product of M . And v is called a *principal normal vector* of M in \bar{M} at the point. If v is a principal normal vector at $x \in M$, the set of all principal tangent vectors for v and the zero vector is clearly a linear subspace of M_x which we call the *principal tangent vector space* for v and denote it by $E(x, v)$. On the principal tangent vector spaces, we have the following.

LEMMA 1. *If v_1 and v_2 are principal normal vectors at $x \in M$ and $v_1 \neq v_2$, then $E(x, v_1) \perp E(x, v_2)$.*

PROOF. Let $u_1 \in E(x, v_1)$ and $u_2 \in E(x, v_2)$. By means of (1.7), we get

$$T_{u_1}u_2 = \langle u_1, u_2 \rangle v_1 \quad \text{and} \quad T_{u_2}u_1 = \langle u_2, u_1 \rangle v_2.$$

Since the operator T is self adjoint, we have $T_{u_1}u_2 = T_{u_2}u_1$, hence $\langle u_1, u_2 \rangle (v_1 - v_2) = 0$, which follows $\langle u_1, u_2 \rangle = 0$. q. e. d.

Now, let $F(\bar{M})$ and $F(M)$ be the orthonormal frame bundles over \bar{M} and M respectively, here we consider M has the induced Riemannian metric from \bar{M} through ϕ . Let B be the set of elements $b = (x, e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p})$, where $(x, e_1, \dots, e_n) \in F(M)$ and $(\phi(x), \phi_*e_1, \dots, \phi_*e_n, e_{n+1}, \dots, e_{n+p}) \in F(\bar{M})$. B is considered naturally C^∞ manifold and $B \rightarrow M$ is a principal fibre bundle over M . We denote the basic forms and connection forms on $F(\bar{M})$ of \bar{M} by

$$\bar{\omega}_A, \bar{\omega}_{AB} = -\bar{\omega}_{BA}, \quad A, B = 1, 2, \dots, n+p$$

and the induced forms on B through the natural mapping $B \rightarrow F(\bar{M})$ by the notations omitted bars. We have the structure equations for \bar{M} :

$$(1.8) \quad \begin{cases} d\bar{\omega}_A = \sum_B \bar{\omega}_B \wedge \bar{\omega}_{BA}, \\ d\bar{\omega}_{AB} = \sum \bar{\omega}_{AC} \wedge \bar{\omega}_{CB} + \bar{\Omega}_{AB} \end{cases}$$

and on B

$$(1.9) \quad \begin{cases} \omega_\alpha = 0, & \alpha = n+1, \dots, n+p \\ \omega_{i\alpha} = \sum_j A_{\alpha ij} \omega_j, & i = 1, 2, \dots, n, \end{cases}$$

where¹⁾

$$(1.10) \quad A_{\alpha ij} = A_{\alpha ji}.$$

Then using the frame $b = (x, e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p})$, the shape operator T_x will be expressed as follows:

1) In the following, the indices run as follows:

$i, j, k, \dots = 1, 2, \dots, n$;

$\alpha, \beta, \gamma, \dots = n+1, \dots, n+p$;

$A, B, C, \dots = 1, 2, \dots, n+p$

with some exceptions.

$$(1.11) \quad T_{e_i}(e_j) = \sum_{\alpha} \omega_{j\alpha}(e_i) e_{\alpha} = \sum_{\alpha} A_{\alpha ij} e_{\alpha}$$

and

$$(1.12) \quad T_{e_i}(e_{\alpha}) = \sum_j \omega_{\alpha j}(e_i) e_j = - \sum_j A_{\alpha ij} e_j$$

and the bilinearity of $T_u(v)$ in u and v .

§2. Principal normal vector fields.

Now, we suppose that a principal normal vector field $V \in \Gamma(N(M, \bar{M}))$ is given. Then, $\dim E(x, V(x))$, $x \in M$, is clearly an upper semi-continuous positive integer valued function on M . Hence for its minimal value $m (> 0)$, the point set M_0 of x such that $m = \dim E(x, V(x))$ is an open subset of M .

LEMMA 2. *If V is a C^{∞} principal normal vector field of M in \bar{M} , then $E(M, V) = \bigcup_{x \in M_0} E(x, V(x))$ makes a C^{∞} m -dimensional distribution on M_0 .*

PROOF. Take any point $x_0 \in M_0$ and a local cross section of the bundle $B \rightarrow M$ about x_0 . Then, any vector $u \in E(x, V(x))$ is a solution of the equations

$$(2.1) \quad \sum_j \{A_{\alpha ij}(x) - v_{\alpha}(x) \delta_{ij}\} u_j = 0, \quad i = 1, 2, \dots, n; \alpha = n+1, \dots, n+p,$$

where $u = \sum_i u_i e_i$ and $V(x) = \sum_{\alpha} v_{\alpha}(x) e_{\alpha}(x)$, because we have

$$T_u z = \sum_{\alpha} A_{\alpha ij} u_i z_j e_{\alpha}, \quad z = \sum_j z_j e_j$$

by (1.11). Since the system of linear equations in u_1, \dots, u_n has rank $n-m$, the distribution $E(x, V(x))$ is C^{∞} about x_0 . q. e. d.

LEMMA 3. *Let M be an n -dimensional C^{∞} submanifold immersed in an $(n+p)$ -dimensional C^{∞} Riemannian manifold \bar{M} of constant curvature \bar{c} with a C^{∞} principal normal vector field V such that $E(M, V)$ has constant dimension $m > 1$. Then, the m -dimensional distribution $E(M, V)$ is completely integrable if and only if $\nabla_u^{\perp} V = 0$ for any $u \in E(M, V)$.*

PROOF. By Lemma 2, $E(M, V)$ is an m -dimensional C^{∞} distribution on M . We take any two tangent vector fields $X, Y \in \mathfrak{X}(M)$ such that $X(x), Y(x) \in E(x, V(x))$, $x \in M$, which we denote simply by $X, Y \in E(M, V)$. For any $Z \in \mathfrak{X}(M)$, we have

$$(2.2) \quad T_X Z = \langle X, Z \rangle V, \quad T_Y Z = \langle Y, Z \rangle V.$$

By means of the second Codazzi-Mainardi equation (1.4), from (2.2) we have

$$\begin{aligned} (T_{[X, Y]} + P^{\perp} \bar{R}_{X, Y}) Z &= T_X \nabla_Y Z - T_Y \nabla_X Z + \nabla_X^{\perp} T_Y Z - \nabla_Y^{\perp} T_X Z \\ &= \{\langle X, \nabla_Y Z \rangle - \langle Y, \nabla_X Z \rangle\} V + P^{\perp} (\bar{\nabla}_X \langle Y, Z \rangle V) - P^{\perp} (\bar{\nabla}_Y \langle X, Z \rangle V) \\ &= \langle \nabla_X Y - \nabla_Y X, Z \rangle V + \langle Y, Z \rangle \nabla_X^{\perp} V - \langle X, Z \rangle \nabla_Y^{\perp} V, \end{aligned}$$

that is

$$(2.3) \quad T_{[X, Y]}Z = \langle [X, Y], Z \rangle V + \langle Y, Z \rangle \nabla_X^\perp V - \langle X, Z \rangle \nabla_Y^\perp V - P^\perp(\bar{R}_{X, Y}Z).$$

Since \bar{M} is of constant curvature \bar{c} , we have

$$(2.4) \quad \bar{R}_{X, Y}Z = \bar{c}(\langle Y, Z \rangle X - \langle X, Z \rangle Y)$$

in general. Thus, (2.3) is equivalent to

$$(2.5) \quad T_{[X, Y]}Z = \langle [X, Y], Z \rangle V + \langle Y, Z \rangle \nabla_X^\perp V - \langle X, Z \rangle \nabla_Y^\perp V.$$

Therefore $[X, Y] \subset E(M, V)$ if and only if

$$(2.6) \quad \langle Y, Z \rangle \nabla_X^\perp V = \langle X, Z \rangle \nabla_Y^\perp V$$

for any $Z \in \mathcal{X}(M)$. This condition is dependent only on $X(x), Y(x)$ at each $x \in M$. Since $m > 1$, we can take $Z(x)$ such that $\langle Y(x), Z(x) \rangle \neq 0$, $\langle X(x), Z(x) \rangle = 0$ if $X(x) \wedge Y(x) \neq 0$. Then (2.6) implies

$$\nabla_{X(x)}^\perp V = 0.$$

Therefore, if $E(M, V)$ is completely integrable, then $\nabla_u^\perp V = 0$ for any $u \in E(M, V)$. Conversely, if this condition is satisfied, then $E(M, V)$ is completely integrable from (2.5). q. e. d.

LEMMA 4. Let M, \bar{M} and V be supposed as in Lemma 3. Then, the principal normal vector field V satisfies $\nabla_u^\perp V = 0$ for any $u \in E(M, V)$.

PROOF. For a fixed point $x_0 \in M$, we can take a sufficiently small neighborhood of x_0 and a local cross section $(x, e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p})$ of the bundle $B \rightarrow M$ defined on this neighborhood such that $e_1, \dots, e_m \subset E(M, V)$. Then, putting $V = \sum_\alpha v_\alpha e_\alpha$, by (1.11) and (1.7) we have

$$(2.7) \quad T_{e_\alpha}(e_j) = \sum_\alpha A_{\alpha\alpha j} e_\alpha = \delta_{\alpha j} \sum_\alpha v_\alpha e_\alpha, \quad \alpha = 1, 2, \dots, m$$

or

$$A_\alpha = ((A_{\alpha ij})) = \begin{pmatrix} v_\alpha & 0 & 0 \\ 0 & v_\alpha & \\ & 0 & \bar{A}_\alpha \end{pmatrix}, \quad \bar{A}_\alpha = ((A_{\alpha rt}),$$

$$r, t = m+1, \dots, n; \quad \alpha = n+1, \dots, n+p.^{2)}$$

On the other hand, we get from $\omega_{i\alpha} = \sum_j A_{\alpha ij} \omega_j$ by means of exterior derivative and the structure equation (1.8) replaced with

$$(2.8) \quad \bar{\Omega}_{AB} = -\bar{c} \bar{\omega}_A \wedge \bar{\omega}_B,$$

2) In the following, a, b, c, \dots run from 1 to m and r, s, t, \dots run from $m+1$ to n .

$$\begin{aligned} d\omega_{i\alpha} &= d(\sum_j A_{\alpha ij}\omega_j) = \sum_j dA_{\alpha ij} \wedge \omega_j + \sum_j A_{\alpha ij} d\omega_j \\ &= \sum_j dA_{\alpha ij} \wedge \omega_j + \sum_{k,j} A_{\alpha ik}\omega_{kj} \wedge \omega_j \end{aligned}$$

and

$$\begin{aligned} d\omega_{i\alpha} &= \sum_j \omega_{ij} \wedge \omega_{j\alpha} + \sum_\beta \omega_{i\beta} \wedge \omega_{\beta\alpha} - \bar{c}\omega_i \wedge \omega_\alpha \\ &= \sum_{k,j} \omega_{ik} A_{\alpha kj} \wedge \omega_j + \sum_{j,\beta} \omega_{\alpha\beta} A_{\beta ij} \wedge \omega_j, \end{aligned}$$

hence

$$\sum_j \{dA_{\alpha ij} + \sum_k A_{\alpha kj}\omega_{ki} + \sum_k A_{\alpha ik}\omega_{kj} + \sum_\beta A_{\beta ij}\omega_{\beta\alpha}\} \wedge \omega_j = 0.$$

By E. Cartan's lemma, we can put

$$(2.9) \quad dA_{\alpha ij} + \sum_k A_{\alpha kj}\omega_{ki} + \sum_k A_{\alpha ik}\omega_{kj} + \sum_\beta A_{\beta ij}\omega_{\beta\alpha} = \sum_k B_{\alpha ijk}\omega_k$$

and we see that

$$(2.10) \quad B_{\alpha ijk} \text{ is symmetric with respect to } i, j, k$$

by using (1.10) in addition. Putting $i=j=a$ in (2.9) we get

$$(2.11) \quad dv_\alpha + \sum_{\beta=n+1}^{n+p} v_\beta \omega_{\beta\alpha} = \sum_{k=1}^n B_{\alpha aak}\omega_k,$$

and putting $i=a, j=b, a \neq b$, in (2.9) we get

$$\sum_{k=1}^n B_{\alpha abk}\omega_k = 0,$$

hence

$$(2.12) \quad B_{\alpha abk} = 0, \quad a \neq b.$$

Making use of (2.10) and (2.12), the right hand side of (2.11) can be written as

$$\begin{aligned} dv_\alpha + \sum_{\beta=n+1}^{n+p} v_\beta \omega_{\beta\alpha} &= B_{\alpha aaa}\omega_a + \sum_{r=m+1}^n B_{\alpha aar}\omega_r, \\ a &= 1, 2, \dots, m; \quad \alpha = n+1, \dots, n+p. \end{aligned}$$

Since $m > 1$, it must be

$$(2.13) \quad \begin{cases} B_{\alpha aaa} = 0 \\ B_{\alpha 11r} = B_{\alpha 22r} = \dots = B_{\alpha mmr} (= B_{\alpha r}). \end{cases}$$

Hence, we get

$$(2.14) \quad dv_\alpha + \sum_\beta v_\beta \omega_{\beta\alpha} = \sum_{r=m+1}^n B_{\alpha r}\omega_r.$$

This equation implies that for any $u \in E(M, V)$

$$V_u^\perp V = \sum_{r=m+1}^n B_{\alpha r} \omega_r(u) e_\alpha = 0,$$

since $e_\alpha \in E(M, V)$.

q. e. d.

Thus, we get the following

THEOREM 1. *Let M be an n -dimensional C^∞ submanifold immersed in an $(n+p)$ -dimensional C^∞ Riemannian manifold \bar{M} of constant curvature \bar{c} with a C^∞ principal normal vector field V . Let M_0 be the open subset of points x of M such that $\dim E(x, V(x)) = \text{minimum}$. Then, $E(x, V(x))$, $x \in M_0$, is a completely integrable distribution on M_0 .*

REMARK. When $p=1$, this theorem contains Theorem 2 in [9], in which we supposed that the multiplicities of principal curvatures are all constant.

§ 3. Integral submanifolds of $E(M, V)$.

In this section, we consider only the case in Theorem 1 with $M_0 = M$ and $\dim E(M, V) > 1$, i. e.

$$(\alpha) \left\{ \begin{array}{l} M \text{ is an } n\text{-dimensional } C^\infty \text{ submanifold immersed in an} \\ (n+p)\text{-dimensional } C^\infty \text{ Riemannian manifold } \bar{M} \text{ of constant} \\ \text{curvature } \bar{c} \text{ with a } C^\infty \text{ principal normal vector field } V \neq 0 \\ \text{such that } E(M, V) \text{ is an } m\text{-dimensional distribution and } m > 1 \end{array} \right.$$

$E(M, V)$ is C^∞ by Lemma 2 and completely integrable by Theorem 1. We denote the integral submanifold of $E(M, V)$ through $x \in M$ by $M^m(x) = M^m$.

THEOREM 2. *Any integral submanifold M^m of the distribution $E(M, V)$ under the condition (α) is of constant curvature and totally umbilic in M^n and \bar{M}^{n+p} .*

PROOF. About any point $x_0 \in M$, we use $b = (x, e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p}) \in B$ such that $e_1, \dots, e_m \subset E(M, V)$ and $V = \lambda e_{n+1}$. By the way analogous to (2.7) and (2.14) we have

$$(3.1) \quad A_{n+1aj} = \lambda \delta_{aj} \quad \text{or} \quad \omega_{an+1} = \lambda \omega_a,$$

$$(3.2) \quad A_{\beta aj} = 0 \quad \text{or} \quad \omega_{a\beta} = 0, \quad \beta = n+2, \dots, n+p$$

and

$$(3.3) \quad d\lambda = \sum_{r=m+1}^n B_{n+1r} \omega_r,$$

$$(3.4) \quad \lambda \omega_{n+1\beta} = \sum_{r=m+1}^n B_{\beta r} \omega_r, \quad \beta = n+2, \dots, n+p.$$

From (3.1) and (3.2), we get

$$\begin{aligned}
d\omega_{an+1} &= \sum_{b=1}^m \omega_{ab} \wedge \omega_{bn+1} + \sum_{r=m+1}^n \omega_{ar} \wedge \omega_{rn+1} \\
&\quad + \sum_{\beta=n+2}^{n+p} \omega_{a\beta} \wedge \omega_{\beta n+1} - \bar{c} \omega_a \wedge \omega_{n+1} \\
&= \lambda \sum_b \omega_{ab} \wedge \omega_b + \sum_r \omega_{ar} \wedge \omega_{rn+1}, \\
d(\lambda \omega_a) &= d\lambda \wedge \omega_a + \lambda \sum_{j=1}^n \omega_j \wedge \omega_{ja},
\end{aligned}$$

hence

$$(3.5) \quad d\lambda \wedge \omega_a + \lambda \sum_{r=m+1}^n \omega_r \wedge \omega_{ra} - \sum_{r=m+1}^n \omega_{ar} \wedge \omega_{rn+1} = 0.$$

Analogously, from $d\omega_{a\beta} = 0$ we get

$$(3.6) \quad \lambda \omega_a \wedge \omega_{n+1\beta} + \sum_{r=m+1}^n \omega_{ar} \wedge \omega_{r\beta} = 0, \quad \beta = n+2, \dots, n+p.$$

From (3.3) and (3.5), we get

$$\sum_r (B_{n+1r} \omega_a - \lambda \omega_{ar} + \sum_t \omega_{at} A_{n+1tr}) \wedge \omega_r = 0.$$

From (3.4) and (3.6), we get

$$\sum_r (B_{\beta r} \omega_a + \sum_t \omega_{at} A_{\beta tr}) \wedge \omega_r = 0.$$

By means of E. Cartan's lemma, we get from these

$$\sum_t \omega_{at} (A_{n+1tr} - \lambda \delta_{tr}) + B_{n+1r} \omega_a = \sum_t C_{art} \omega_t,$$

$$\sum_t \omega_{at} A_{\beta tr} + B_{\beta r} \omega_a = \sum_t C_{\beta art} \omega_t, \quad \beta = n+2, \dots, n+p,$$

where $C_{art} = C_{atr}$ and $C_{\beta art} = C_{\beta atr}$. From the assumption (α), the solution of the following equations in u_{m+1}, \dots, u_n :

$$\sum_t u_t (A_{n+1tr} - \lambda \delta_{tr}) = 0,$$

$$\sum_t u_t A_{\beta tr} = 0$$

is $u_{m+1} = u_{m+2} = \dots = u_n = 0$. Therefore, ω_{ar} can be written as

$$(3.7) \quad \omega_{ar} = \rho_r \omega_a + \sum_{t=m+1}^n \Gamma_{art} \omega_t$$

where ρ_t, Γ_{art} are C^∞ functions on the submanifold of B whose points satisfy the conditions in the beginning of the proof.

From (3.7), (3.1), (3.2) and the structure equations, we get the following equalities

$$\begin{aligned}
d\omega_{ar} &\equiv d\rho_r \wedge \omega_a + \rho_r d\omega_a + \sum_t \Gamma_{art} d\omega_t \\
&\equiv d\rho_r \wedge \omega_a + \rho_r \sum_b \omega_b \wedge \omega_{ba} + \sum_{t,b} \Gamma_{art} \omega_b \wedge \omega_{bt} \\
&\equiv d\rho_r \wedge \omega_a + \rho_r \sum_b \omega_b \wedge \omega_{ba} \\
&\quad (\text{mod } \omega_{m+1}, \dots, \omega_n), \\
d\omega_{ar} &= \sum_b \omega_{ab} \wedge \omega_{br} + \sum_t \omega_{at} \wedge \omega_{tr} + \omega_{an+1} \wedge \omega_{n+1r} - \bar{c} \omega_a \wedge \omega_r \\
&\equiv \sum_b \rho_r \omega_{ab} \wedge \omega_b + \sum_t \rho_t \omega_a \wedge \omega_{tr} \\
&\quad (\text{mod } \omega_{m+1}, \dots, \omega_n),
\end{aligned}$$

hence $(d\rho_r + \sum_t \rho_t \omega_{tr}) \wedge \omega_a \equiv 0$, that is

$$(3.8) \quad d\rho_r + \sum_t \rho_t \omega_{tr} \equiv 0 \quad (\text{mod } \omega_{m+1}, \dots, \omega_n).$$

This follows that

$$(3.9) \quad d(\sum_r \rho_r \rho_r) \equiv 0 \quad (\text{mod } \omega_{m+1}, \dots, \omega_n).$$

Now, we consider any integral submanifold M^m of the distribution $E(M, V)$. On M^m , we have

$$\omega_{m+1} = \dots = \omega_n = 0$$

and

$$(3.10) \quad \omega_{ar} = \rho_r \omega_a.$$

Therefore, the curvature forms of M^m are given by

$$\begin{aligned}
(3.11) \quad d\omega_{ab} - \sum_{c=1}^m \omega_{ac} \wedge \omega_{cb} &= d\omega_{ab} - \sum_{c=1}^{n+p} \omega_{ac} \wedge \omega_{cb} \\
&\quad + \sum_{r=m+1}^n \omega_{ar} \wedge \omega_{rb} + \omega_{an+1} \wedge \omega_{n+1b} \\
&= -(\bar{c} + \sum_r \rho_r^2 + \lambda^2) \omega_a \wedge \omega_b.
\end{aligned}$$

By means of Lemma 3, λ is constant on M^m and $\sum_r \rho_r^2$ is also constant on it by (3.9). Therefore M^m is of constant curvature $\bar{c} + \sum_r \rho_r^2 + \lambda^2$. The relations

$$\omega_{ar} = \rho_r \omega_a, \quad \omega_{an+1} = \lambda \omega_a, \quad \omega_{a\beta} = 0$$

show that M^m is totally umbilic in \bar{M}^{n+p} and the first relations show that M^m is also totally umbilic in M^n . q. e. d.

COROLLARY. *Let M^n be a C^∞ submanifold of a sphere $S^{n+p} \subset R^{n+p+1}$ with a C^∞ principal normal vector field $V \neq 0$ such that $E(M, V)$ is an m -dimensional*

distribution ($m > 1$). Then $E(M, V)$ is completely integrable and any one of its integral submanifolds is contained in an m -dimensional sphere which is the intersection of S^{n+p} and an $(m+1)$ -dimensional linear subspace in R^{n+p+1} .

By Theorem 2, the tangent vector field $U = \sum_{r=m+1}^n \rho_r e_r$ of M^n is a principal normal vector field on each integral submanifold M^m of $E(M, V)$ whose principal tangent vector space is the tangent space to M^m . We call U the *induced principal normal vector field* of M^m in M^n from V . (3.8) gives us immediately the following

THEOREM 3. *The induced principal normal vector field U of any integral submanifold M^m of the distribution $E(M, V)$ under (α) is parallel in the normal vector bundle $N(M^m, M^n)$ with the induced connection from M^n .*

THEOREM 4. *For any integral submanifold M^m of $E(M, V)$ under (α) , there exists a totally geodesic submanifold $\bar{M}^{m+2}(\bar{M}^{m+1})$ of \bar{M}^{n+p} in which M^m is immersed, if U does not vanish (or vanishes near M^m in M^n).*

PROOF. Using the notation in the proof of Theorem 2, furthermore we may put

$$(3.12) \quad U = \rho e_{m+1} \quad (\rho \neq 0)$$

locally, if $U \neq 0$. By Theorem 3, ρ is constant on M^m . Then (3.7) and (3.8) imply

$$(3.13) \quad \begin{cases} \omega_{am+1} = \rho \omega_a + \sum_{r=m+1}^n \Gamma_{am+1r} \omega_r, \\ \omega_{at} = \sum_{r=m+1}^n \Gamma_{atr} \omega_r, \quad a = 1, 2, \dots, m, \\ \omega_{m+1t} = \sum_{r=m+1}^n \Pi_{tr} \omega_r, \quad t = m+2, \dots, n. \end{cases}$$

Hence, we have

$$(3.14) \quad \omega_{am+1} = \rho \omega_a, \quad \omega_{at} = 0, \quad \omega_{m+1t} = 0 \quad \text{on } M^m, \quad t = m+2, \dots, n.$$

On the other hand, from (3.1), (3.2) and (3.4)

$$(3.15) \quad \begin{cases} \omega_{tn+1} = 0, & t = m+2, \dots, n, \\ \omega_{a\beta} = 0, & a = 1, 2, \dots, m, \\ \omega_{n+1\beta} = 0, & \beta = n+2, \dots, n+p. \end{cases}$$

Finally, we consider the following exterior derivative on M^m

$$0 = d\omega_{a\beta} = \omega_{am+1} \wedge \omega_{m+1\beta} = \rho \omega_a \wedge \omega_{m+1\beta},$$

by means of (3.14), (3.15), (1.8) and (2.8). Hence, we get

$$(3.16) \quad \omega_{m+1\beta} = 0, \quad \beta = n+2, \dots, n+p, \quad \text{on } M^m.$$

On the other hand, if we consider the following Pfaff equation system

$$(3.17) \quad \begin{cases} \bar{\omega}_t = 0, & \bar{\omega}_{at} = 0, & \bar{\omega}_{m+1t} = 0, & \bar{\omega}_{n+1t} = 0, \\ \bar{\omega}_\beta = 0, & \bar{\omega}_{a\beta} = 0, & \bar{\omega}_{m+1\beta} = 0, & \bar{\omega}_{n+1\beta} = 0. \end{cases}$$

$$a = 1, 2, \dots, m; \quad t = m+2, \dots, n; \quad \beta = n+2, \dots, n+p \quad \text{on } F(\bar{M}),$$

this is clearly completely integrable and its maximal integral submanifold in $F(\bar{M})$ gives an $(m+2)$ -dimensional totally geodesic submanifold \bar{M}^{m+2} in \bar{M}^{n+p} . (3.14), (3.15) and (3.16) show that M^m satisfies (3.17). Therefore we may consider that \bar{M}^{m+2} contains M^m .

Next, we consider the case $U \equiv 0$ about a point of M^m in M^n . From (3.7), (3.2) and (3.4), we have

$$(3.18) \quad \begin{aligned} \omega_{ar} = 0, \quad \omega_{a\beta} = 0, \quad \omega_{n+1r} = 0, \quad \omega_{n+1\beta} = 0 \quad \text{on } M^m, \\ a = 1, 2, \dots, m; \quad r = m+1, \dots, n; \quad \beta = n+2, \dots, n+p. \end{aligned}$$

On $F(\bar{M})$, the Pfaffian equation system

$$(3.19) \quad \begin{cases} \bar{\omega}_{r-} = 0, & \bar{\omega}_{ar} = 0, & \bar{\omega}_{n+1r} = 0, \\ \bar{\omega}_\beta = 0, & \bar{\omega}_{a\beta} = 0, & \bar{\omega}_{n+1\beta} = 0 \end{cases}$$

$$a = 1, 2, \dots, m; \quad r = m+1, \dots, n; \quad \beta = n+2, \dots, n+p,$$

is completely integrable and its maximal integral submanifold in $F(\bar{M})$ gives an $(m+1)$ -dimensional totally geodesic submanifold \bar{M}^{m+1} in \bar{M}^{n+p} . (3.18) shows that we may consider that \bar{M}^{m+1} contains M^m .

REMARK. \bar{M}^{m+2} (or \bar{M}^{m+1}) is clearly the locus of geodesics tangent to the vector space spanned by V and U at each point of M^m .

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