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ON PROBABILITIES IN CERTAIN MULTIVARIATE DISTRIBUTIONS: THEIR DEPENDENCE ON CORRELATIONS

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1. SUMMARY AND TWO KNOWN LEMMAS

Recently, C. G. Khatri [5] presented an inequality for probabilities of convex symmetric regions in multivariate normal distributions; unfortunately, his proof was incorrect (as it was shown by Šidák [9]), and the general correctness of his inequality is an open question. In the present paper, starting with stronger assumptions, we are able to prove a stronger result: namely, if all correlations ϱ_{ij} in the underlying normal distributions have the product form $\varrho_{ij} = b_i b_j$ (where $-1 \leqq b_i, b_j \leqq 1$), the probabilities in question are, roughly speaking, non-decreasing functions of the absolute values of the correlations; an immediate consequence is essentially Khatri's inequality for this special case.

After further strengthening the assumptions, namely considering equicorrelated normal distributions, we show that the probabilities of certain regions (which now need be neither convex nor symmetric) are again non-decreasing functions of the correlations.

Finally we note that our method of proofs gives also easily some probability inequalities for certain special cases of multivariate exponential and Poisson distributions.

We shall need the following two lemmas.

Lemma 1. *Let $g_1(\mathbf{v})$ and $g_2(\mathbf{v})$ be two functions of a real random vector $\mathbf{v}$. If the respective expectations in subsequent formulas exist, then*

$$Eg_1(\mathbf{v})\, g_2(\mathbf{v}) \geqq Eg_1(\mathbf{v})\, Eg_2(\mathbf{v})$$

provided for any two points $\mathbf{v}_1$ and $\mathbf{v}_2$, either $g_1(\mathbf{v}_1) \geqq g_1(\mathbf{v}_2)$ and $g_2(\mathbf{v}_1) \geqq g_2(\mathbf{v}_2)$, or $g_1(\mathbf{v}_1) \leqq g_1(\mathbf{v}_2)$ and $g_2(\mathbf{v}_1) \leqq g_2(\mathbf{v}_2)$, while

$$Eg_1(\mathbf{v})\, g_2(\mathbf{v}) \leqq Eg_1(\mathbf{v})\, Eg_2(\mathbf{v})$$

provided for any two points $\mathbf{v}_1$ *and* $\mathbf{v}_2$, *either* $g_1(\mathbf{v}_1) \geqq g_1(\mathbf{v}_2)$ *and* $g_2(\mathbf{v}_1) \leqq g_2(\mathbf{v}_2)$, *or* $g_1(\mathbf{v}_1) \leqq g_1(\mathbf{v}_2)$ *and* $g_2(\mathbf{v}_1) \geqq g_2(\mathbf{v}_2)$.

This lemma is due to C. G. Khatri [4], p. 1859, Lemma 5. In fact, we shall use this lemma only for one-dimensional variables v rather than for vectors $\mathbf{v}$, and only the first part of it. However, if some (but not all) component regions in the theorems and proofs to follow were replaced by their complements, it would be appropriate to use the second part of the lemma, and we should obtain analogous results but with reversed signs of inequalities; the details of this easy modification are left to the reader.

Lemma 2. *If* $\mathbf{Z} = (Z_1, \ldots, Z_n)$ *is a random vector with density* $\varphi(\mathbf{z})$ *such that* $\varphi(\mathbf{z}) = \varphi(-\mathbf{z})$ *and the set* $\{\mathbf{z}; \varphi(\mathbf{z}) \geq c\}$ *is convex for every non-negative c, and if* $\mathscr{C}$ *is a convex set, symmetric about the origin, then* $P\{\mathbf{Z} + a\mathbf{b} \in \mathscr{C}\}$ *is a non-increasing function of the parameter* $a(0 \leqq a < \infty)$ *for any vector* $\mathbf{b}$.

This lemma is due to T. W. Anderson [1]. We shall use it for the case where $\mathbf{Z}$ has a multivariate normal distribution with zero means, and it can be seen that the assumptions concerning φ are satisfied in this case.

2. NORMAL DISTRIBUTIONS WITH SPECIAL CORRELATION STRUCTURE

C. G. Khatri [5], Theorems 1 and 2, published the following theorem.

Let the random vector $\mathbf{X} = (X_1, \ldots, X_p)$, *having a p-variate normal distribution with zero means and any covariance matrix, be partitioned as* $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_q)$ *where* $\mathbf{X}_k = (X_{p_1+\cdots+p_{k-1}+1}, \ldots, X_{p_1+\cdots+p_k})$, $k = 1,2, \ldots, q$, *with* $p_1 + \ldots + p_q = p$, *and (as a convention)* $p_1 + \ldots + p_0 = 0$. *Let* $\mathscr{D}_k$, $k = 1,2, \ldots, q$, *be a convex and symmetric region in* $\mathbf{X}_k$ *about the origin in p-dimensional space containing the whole axes* $-\infty < x_j < \infty$ *due to all other variates. Then*

$$(1) \qquad P\{\bigcap_{k=1}^{q} \mathscr{D}_k\} \geqq P\{\mathscr{D}_1\} \, P\{\bigcap_{k=2}^{q} \mathscr{D}_k\} \geqq \prod_{k=1}^{q} P\{\mathscr{D}_k\}\,.$$

Further, if $\overline{\mathscr{D}}_k$ *is the complementary region of* $\mathscr{D}_k$, *then*

$$(2) \qquad P\{\bigcap_{k=1}^{q} \overline{\mathscr{D}}_k\} \geqq P\{\overline{\mathscr{D}}_1\} \, P\{\bigcap_{k=2}^{q} \overline{\mathscr{D}}_k\} \geqq \prod_{k=1}^{q} P\{\overline{\mathscr{D}}_k\}\,.$$

Unfortunately, the proof of these inequalities given in (5) was incorrect (as it was shown by Šidák [9]). The general correctness of the inequality (1) is an open question, while (2) is known to be generally false (cf. the counterexample in Šidák [8]).

Let us also remark that Khatri's expression "a convex and symmetric region in $\mathbf{X}_k$..." is somewhat unclear. On inspecting his proofs in [4], [5] one finds that his requirements on $\mathscr{D}_k$ are that they must have the form

$$\mathscr{D}_k = \mathscr{R}_{p_1} \times \ldots \times \mathscr{R}_{p_{k-1}} \times \mathscr{A}_k \times \mathscr{R}_{p_{k+1}} \times \ldots \times \mathscr{R}_{p_q}$$

(with obvious modifications for $\mathscr{D}_1$, $\mathscr{D}_q$) where $\mathscr{R}_{p_i}$ is the Euclidean p_i-dimensional space, and $\mathscr{A}_k$ is some subset of $\mathscr{R}_{p_k}$ which is convex and symmetric about the origin. Hence Khatri's assertions in [4], [5] should be modified in this sense, and we have e. g.

$$P\{\bigcap_{k=1}^{q} \mathscr{D}_k\} = P\{\mathbf{X}_1 \in \mathscr{A}_1, \ldots, \mathbf{X}_q \in \mathscr{A}_q\} .$$

In order to use a more precise and direct notation, we shall work in the present paper with the sets $\mathscr{A}_k$.

Theorem 1. *Let the random vector $\mathbf{X} = (X_1, \ldots, X_p)$ be partitioned as $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$ where $\mathbf{X}_1 = (X_1, \ldots, X_{p_1})$, $\mathbf{X}_2 = (X_{p_1+1}, \ldots, X_p)$, and let I_1, I_2 be the sets of indices $I_1 = \{1, \ldots, p_1\}$, $I_2 = \{p_1 + 1, \ldots, p\}$. Let $\mathbf{X}$ have a p-variate normal distribution with zero means, arbitrary variances and with the correlation matrix $R(\lambda) = \|\varrho_{ij}(\lambda)\|$ depending on a parameter $\lambda (0 \leqq \lambda \leqq 1)$ as follows: under the probability law P_λ, we have $\varrho_{ij}(\lambda) = \varrho_{ji}(\lambda) = b_i b_j$ whenever $i \neq j$ and either $i, j \in I_1$ or $i, j \in I_2$, and $\varrho_{ij}(\lambda) = \varrho_{ji}(\lambda) = \lambda\, b_i b_j$ whenever either $i \in I_1, j \in I_2$ or $i \in I_2, j \in I_1$; here $b_1, \ldots, b_p$ are some fixed numbers satisfying $-1 \leqq b_i \leqq 1, i = 1, \ldots, p$. If $\mathscr{A}_1$, and $\mathscr{A}_2$, are convex regions symmetric about the origin in the p_1-dimensional space, and $(p\text{-}p_1)$-dimensional space, respectively, then*

$$P(\lambda) = P_\lambda\{\mathbf{X}_1 \in \mathscr{A}_1, \mathbf{X}_2 \in \mathscr{A}_2\} \tag{3}$$

is a non-decreasing function of $\lambda (0 \leqq \lambda \leqq 1)$. If $\bar{\mathscr{A}}_1$, $\bar{\mathscr{A}}_2$ are complements of $\mathscr{A}_1$, $\mathscr{A}_2$, respectively, then also

$$\bar{P}(\lambda) = P_\lambda\{\mathbf{X}_1 \in \bar{\mathscr{A}}_1, \mathbf{X}_2 \in \bar{\mathscr{A}}_2\} \tag{4}$$

is a non-decreasing function of $\lambda (0 \leqq \lambda \leqq 1)$.

Proof. We shall prove the first part of the theorem concerning (3). Evidently, we may suppose also that the variances of all X_i's are equal to 1. Introduce the following model : for $0 \leqq \lambda < \lambda + h \leqq 1$ let

$$\begin{aligned} X_i^* &= (1 - b_i^2)^{1/2} Y_i + (1 - \lambda - h)^{1/2} b_i W_1 + \lambda^{1/2} b_i U + h^{1/2} b_i V \quad \text{for} \quad i \in I_1 , \\ &= (1 - b_i^2)^{1/2} Y_i + (1 - \lambda - h)^{1/2} b_i W_2 + \lambda^{1/2} b_i U + h^{1/2} b_i V \quad \text{for} \quad i \in I_2 , \end{aligned}$$

$$\begin{aligned} X_i^{**} &= (1 - b_i^2)^{1/2} Y_i + (1 - \lambda - h)^{1/2} b_i W_1 + \lambda^{1/2} b_i U + h^{1/2} b_i V_1 \quad \text{for} \quad i \in I_1 , \\ &= (1 - b_i^2)^{1/2} Y_i + (1 - \lambda - h)^{1/2} b_i W_2 + \lambda^{1/2} b_i U + h^{1/2} b_i V_2 \quad \text{for} \quad i \in I_2 , \end{aligned}$$

where all variables $Y_1, \ldots, Y_p, W_1, W_2, U, V, V_1, V_2$ have independent $N(0,1)$ distributions. By an easy calculation it can be then shown that the distribution of the vector $\mathbf{X}^*$ (or $\mathbf{X}^{**}$) coincides with that of $\mathbf{X}$ under $P_{\lambda+h}$ (or P_λ, respectively).

In the sequel, we shall use the notation E_v for the expectation over the random variable V(i. e., if $f(v)$ is a function of a real variable v, $\varphi(v)$ the density of V, then $E_v f(v) = \int f(v)\, \varphi(v) \mathrm{d}v$), and similarly for E_u. Further, put

$$Z_i = (1 - b_i^2)^{1/2} Y_i + (1 - \lambda - h)^{1/2}\, b_i W_k \quad \text{for} \quad i \in I_k, \quad k = 1, 2\,.$$

Let $\mathbf{X}_1^*, \mathbf{X}_2^*, \mathbf{X}_1^{**}, \mathbf{X}_2^{**}, \mathbf{Z}_1, \mathbf{Z}_2, \mathbf{b}_1, \mathbf{b}_2$ have an analogous meaning of partitioned vectors as $\mathbf{X}_1, \mathbf{X}_2$. Consider now, for arbitrary but fixed u, the probability

$$\begin{aligned}(5)\qquad & P\{\mathbf{Z}_k + \lambda^{1/2}\mathbf{b}_k u + h^{1/2}\mathbf{b}_k V \in \mathscr{A}_k, \quad k = 1, 2\} = \\ & = E_v P\{\mathbf{Z}_k + \lambda^{1/2}\mathbf{b}_k u + h^{1/2}\mathbf{b}_k v \in \mathscr{A}_k, \quad k = 1, 2\} = \\ & = E_v \prod_{k=1}^{2} P\{\mathbf{Z}_k + (\lambda^{1/2} u + h^{1/2} v)\, \mathbf{b}_k \in \mathscr{A}_k\}\,.\end{aligned}$$

Let us now concentrate on the probabilities

$$(6)\qquad P\{\mathbf{Z}_k + (\lambda^{1/2} u + h^{1/2} v)\, \mathbf{b}_k \in \mathscr{A}_k\}, \quad k = 1, 2\,,$$

occurring in the last line of (5). Both vectors $\mathbf{Z}_1$ and $\mathbf{Z}_2$ have multivariate normal distributions with zero means, and therefore, by Lemma 2, both probabilities (6) as functions of v are non-decreasing for $-\infty < v \leqq -\lambda^{1/2} h^{-1/2} u$ and non-increasing for $-\lambda^{1/2} h^{-1/2} u \leqq v < \infty$. Moreover, these probabilities are symmetric functions of v about the point $v = -\lambda^{1/2} h^{-1/2} u$.

By these last assertions, it can be now immediately seen that the probabilities (6) satisfy the conditions on the functions g_1, g_2 in Lemma 1. Therefore, by Lemma 1,

$$\begin{aligned}(7)\qquad & E_v \prod_{k=1}^{2} P\{\mathbf{Z}_k + (\lambda^{1/2} u + h^{1/2} v)\, \mathbf{b}_k \in \mathscr{A}_k\} \geqq \\ & \geqq \prod_{k=1}^{2} E_v P\{\mathbf{Z}_k + (\lambda^{1/2} u + h^{1/2} v)\, \mathbf{b}_k \in \mathscr{A}_k\} = \\ & = \prod_{k=1}^{2} P\{\mathbf{Z}_k + \lambda^{1/2}\mathbf{b}_k u + h^{1/2}\mathbf{b}_k V \in \mathscr{A}_k\} = \\ & = P\{\mathbf{Z}_k + \lambda^{1/2}\mathbf{b}_k u + h^{1/2}\mathbf{b}_k V_k \in \mathscr{A}_k, \quad k = 1, 2\}\,.\end{aligned}$$

Combining now (5) and (7), and applying the expectation E_u, we get

$$(8)\qquad P(\lambda + h) = P\{\mathbf{X}_1^* \in \mathscr{A}_1, \mathbf{X}_2^* \in \mathscr{A}_2\} \geqq P\{\mathbf{X}_1^{**} \in \mathscr{A}_1, \mathbf{X}_2^{**} \in \mathscr{A}_2\} = P(\lambda)\,,$$

which proves the first part of Theorem 1.

The proof of the second part concerning (4) runs in a completely analogous way. The only substantial change is that we are now led, in place of (6), to the probabilities

$$(9)\qquad P\{\mathbf{Z}_k + (\lambda^{1/2} u + h^{1/2} v)\, \mathbf{b}_k \in \bar{\mathscr{A}}_k\}, \quad k = 1, 2\,,$$

which are non-increasing for $-\infty < v \leqq -\lambda^{1/2} h^{-1/2} u$ and non-decreasing for

$-\lambda^{1/2}h^{-1/2}u \leqq v < \infty$. However, Lemma 1 is clearly again applicable, and the rest of the proof is analogous.

Corollary 1. *Let the random vector* $\mathbf{X} = (X_1, \ldots, X_p)$ *be partitioned as* $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_q)$ *where* $\mathbf{X}_k = (X_{p_1+\ldots+p_{k-1}+1}, \ldots, X_{p_1+\ldots+p_k})$, $k = 1, 2, \ldots, q$, *with* $p_1 + \ldots p_q = p$, *and (as a convention)* $p_1 + \ldots + p_0 = 0$. *Let* $\mathbf{X}$ *have a p-variate normal distribution with zero means, arbitrary variances and with the correlations* $\varrho_{ij} = b_i b_j$, $i \neq j$; $i, j = 1, \ldots, p$, *where* $-1 \leqq b_i \leqq 1$, $i = 1, \ldots, p$. *If* $\mathscr{A}_k$, $k = 1, \ldots, q$, *are convex regions symmetric about the origin in the* p_k*-dimensional space, respectively, then*

$$(10) \quad P\{\mathbf{X}_1 \in \mathscr{A}_1, \mathbf{X}_2 \in \mathscr{A}_2, \ldots, \mathbf{X}_q \in \mathscr{A}_q\} \geqq \geqq P\{\mathbf{X}_1 \in \mathscr{A}_1\} \, P\{\mathbf{X}_2 \in \mathscr{A}_2, \ldots, \mathbf{X}_q \in \mathscr{A}_q\} \geqq \ldots \geqq \prod_{k=1}^{q} P\{\mathbf{X}_k \in \mathscr{A}_k\} .$$

If $\bar{\mathscr{A}}_k$, $k = 1, \ldots, q$, *is the complement of* $\mathscr{A}_k$, *then*

$$(11) \quad P\{\mathbf{X}_1 \in \bar{\mathscr{A}}_1, \mathbf{X}_2 \in \bar{\mathscr{A}}_2, \ldots, \mathbf{X}_q \in \bar{\mathscr{A}}_q\} \geqq \geqq P\{\mathbf{X}_1 \in \bar{\mathscr{A}}_1\} \, P\{\mathbf{X}_2 \in \bar{\mathscr{A}}_2, \ldots, \mathbf{X}_q \in \bar{\mathscr{A}}_q\} \geqq \ldots \geqq \prod_{k=1}^{q} P\{\mathbf{X}_k \in \bar{\mathscr{A}}_k\} .$$

Proof. It suffices to put $\lambda = 1$ and $\lambda = 0$ in Theorem 1.

The first assertion concerning (3), and the second assertion concerning (4), of Theorem 1 generalize Theorem 1 in Šidák [7], and Lemma 2 in Šidák [8], respectively, while (11) in Corollary 1 generalizes Theorem 2 in Khatri [4]; all of these previous results concerned the case of one-dimensional $\mathscr{A}_k$'s.

3. EQUICORRELATED NORMAL DISTRIBUTIONS

In previous literature devoted to probability inequalities for multivariate normal distributions, almost all results concerned the case of convex symmetric sets. However, if we sufficiently strengthen the requirements on the distributions, namely, if we consider equicorrelated normal distributions, we are also able to prove a result for sets of other types, which need be neither convex nor symmetric.

Theorem 2. *Let the random vector* $\mathbf{X} = (X_1, \ldots, X_p)$ *have, under the probability law* P_ϱ, *a p-variate normal distribution with equal mean values, equal variances, and with all correlations equal to* $\varrho \geqq 0$. *If* $\mathscr{E}$ *is an arbitrary Borel set on the real line, then*

$$(12) \qquad P(\varrho) = P_\varrho\{X_1 \in \mathscr{E}, \ X_2 \in \mathscr{E}, \ldots, X_p \in \mathscr{E}\}$$

is a non-decreasing function of $\varrho (0 \leqq \varrho \leqq 1)$.

The proof is very similar to that of Theorem 1. Since $\mathscr{E}$ is arbitrary, we may clearly suppose that all mean values are 0 and all variances 1. Then we can introduce the following model: for $0 \leqq \varrho < \varrho + h \leqq 1$ let

$$X_i^* = (1 - \varrho - h)^{1/2} Y_i + \varrho^{1/2} U + h^{1/2} V \quad \text{for} \quad i = 1, \ldots, p\,,$$
$$X_i^{**} = (1 - \varrho - h)^{1/2} Y_i + \varrho^{1/2} U + h^{1/2} V_i \quad \text{for} \quad i = 1, \ldots, p\,,$$

where all variables $Y_1, \ldots, Y_p, U, V, V_1, \ldots, V_p$ have independent $N(0,1)$ distributions. Evidently, the distribution of the vector $\mathbf{X}^*$ (or $\mathbf{X}^{**}$) coincides with that of $\mathbf{X}$ under $P_{\varrho+h}$ (or P_ϱ, respectively).

For an arbitrary but fixed u consider now the probability

$$(13) \qquad P\{(1 - \varrho - h)^{1/2} Y_i + \varrho^{1/2} u + h^{1/2} V \in \mathscr{E}, \quad i = 1, \ldots, p\} =$$
$$= E_v P\{(1 - \varrho - h)^{1/2} Y_i + \varrho^{1/2} u + h^{1/2} v \in \mathscr{E}, \quad i = 1, \ldots, p\} =$$
$$= E_v \prod_{i=1}^{p} P\{(1 - \varrho - h)^{1/2} Y_i + \varrho^{1/2} u + h^{1/2} v \in \mathscr{E}\}\,.$$

All probabilities in the product in the last line of (13) are equal, so that they, as well as their products, certainly satisfy the conditions on the functions g_i in Lemma 1. Therefore, by induction applying Lemma 1, we get

$$(14) \qquad E_v \prod_{i=1}^{p} P\{(1 - \varrho - h)^{1/2} Y_i + \varrho^{1/2} u + h^{1/2} v \in \mathscr{E}\} \geqq$$
$$\geqq \prod_{i=1}^{p} E_v P\{(1 - \varrho - h)^{1/2} Y_i + \varrho^{1/2} u + h^{1/2} v \in \mathscr{E}\} =$$
$$= P\{(1 - \varrho - h)^{1/2} Y_i + \varrho^{1/2} u + h^{1/2} V_i \in \mathscr{E}, \quad i = 1, \ldots, p\}\,.$$

Putting (13) and (14) together, and applying E_u, we get finally

$$(15) \qquad P(\varrho + h) = P\{X_i^* \in \mathscr{E}, \quad i = 1, \ldots, p\} \geqq$$
$$\geqq P\{X_i^{**} \in \mathscr{E}, \quad i = 1, \ldots, p\} = P(\varrho)\,.$$

Corollary 2. *Under the assumptions of Theorem 2 we have*

$$(16) \qquad P_\varrho\{X_1 \in \mathscr{E}, X_2 \in \mathscr{E}, \ldots, X_p \in \mathscr{E}\} \geqq [P_0\{X_1 \in \mathscr{E}\}]^p\,.$$

The inequality (16) and some other sharper inequalities for equicorrelated normal distributions were given in Šidák [10], Section 3, Case 1.

4. SPECIAL EXPONENTIAL AND POISSON DISTRIBUTIONS

Let us remark in this last section that our method of proofs, based on Lemma 1, can be also applied to give the following simple inequalities for special cases of multivariate exponential and Poisson distributions.

Theorem 3. *Let the random vector* $\mathbf{Z} = (Z_1, \ldots, Z_k)$ *be given by* $Z_1 = \min(Y_1, V), \ldots$ $\ldots, Z_k = \min(Y_k, V)$, *where* $Y_1, \ldots, Y_k$, V *are independent exponential variables. Then, for any numbers* $c_1, \ldots, c_k$ *we have*

$$P\{Z_1 \leqq c_1, \ldots, Z_k \leqq c_k\} \geqq \prod_{i=1}^{k} P\{Z_i \leqq c_i\}, \tag{17}$$

$$P\{Z_1 \geqq c_1, \ldots, Z_k \geqq c_k\} \geqq \prod_{i=1}^{k} P\{Z_i \geqq c_i\}. \tag{18}$$

Proof. To prove (17), we may write

$$\begin{aligned} P\{Z_1 \leqq c_1, \ldots, Z_k \leqq c_k\} &= \\ = E_v P\{\min(Y_1, v) \leqq c_1, \ldots, \min(Y_k, v) \leqq c_k\} &= \\ = E_v \prod_{i=1}^{k} P\{\min(Y_i, v) \leqq c_i\}. \end{aligned} \tag{19}$$

Since $v_1 < v_2$ implies $\min(Y_i, v_1) \leqq \min(Y_i, v_2)$, which in turn implies

$$P\{\min(Y_i, v_1) \leqq c_i\} \geqq P\{\min(Y_i, v_2) \leqq c_i\}, \quad i = 1, \ldots, k, \tag{20}$$

the probabilities in the last line of (19), as well as their products, satisfy the conditions on g_i in Lemma 1. Therefore, by induction applying Lemma 1, the probability (19) is larger or equal to

$$\prod_{i=1}^{k} E_v P\{\min(Y_i, v) \leqq c_i\} = \prod_{i=1}^{k} P\{Z_i \leqq c_i\}.$$

The proof of (18) is similar, but all inequalities in (20) are reversed.

Note that the variables $Z_1, \ldots, Z_k$ introduced in Theorem 3 have a special multivariate exponential distribution, covering for $k = 2$ the case of the general bivariate exponential distribution (cf. Marshall-Olkin [6], Theorem 3.2). Our inequalities (17) and (18) generalize the corresponding inequalities for $k = 2$ mentioned on the top of p. 38 in [6]. Some related sharper inequalities for the case where the events $\{Z_i \leqq c_i\}$ or $\{Z_i \geqq c_i\}$ are replaced by $\{Z_i \in \mathscr{A}\}$ (with the same general $\mathscr{A}$ for all i) were given by Šidák [10], Section 3, Case 5.

Theorem 4. *Let the random vector* $\mathbf{Z} = (Z_1, \ldots, Z_k)$ *be given by* $Z_1 = Y_1 + V, \ldots$ $\ldots, Z_k = Y_k + V$, *where* $Y_1, \ldots, Y_k$, V *are independent Poisson variables. Then, for any numbers* $c_1, \ldots, c_k$, *the inequalities* (17) *and* (18) *are again true.*

The proof is similar to that of Theorem 3 and is left to the reader.

Similarly as before we may note that the variables $Z_1, \ldots, Z_k$ in Theorem 4 have a special multivariate Poisson distribution, covering for $k = 2$ the case of the general bivariate Poisson distribution (cf. Haight [2], Section 3.12, or Holgate [3]). Again, if the events $\{Z_i \leqq c_i\}$ or $\{Z_i \geqq c_i\}$ are replaced by $\{Z_i \varepsilon \mathscr{A}\}$, some related sharper inequalities for this case can be found in Šidák [10], Section 3, Case 4.

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Souhrn

O PRAVDĚPODOBNOSTECH V JISTÝCH MNOHOROZMĚRNÝCH ROZLOŽENÍCH: JEJICH ZÁVISLOST NA KORELACÍCH

Zbyněk Šidák

Nedávno C. G. Khatri [5] publikoval jistou nerovnost pro pravděpodobnosti konvexních symetrických oblastí v mnohorozměrném normálním rozložení; bohužel však jeho důkaz je chybný (jak je ukázáno v [9]) a obecná platnost jeho nerovnosti zůstává otevřenou otázkou. V našem článku dokazujeme při splnění silnějších předpokladů následující silnější tvrzení: jestliže všechny korelace normálního rozložení mají součinový tvar $\varrho_{ij} = b_i b_j$ (kde $-1 \leqq b_i, b_j \leqq 1$), pak pravděpodobnost oblasti $\mathscr{A}_1 \times \mathscr{A}_2$ (kde $\mathscr{A}_1, \mathscr{A}_2$ jsou konvexní symetrické oblasti) je, zhruba řečeno, neklesající funkcí absolutních hodnot korelací; důsledkem je pak v podstatě Khatriho nerovnost pro tento speciální případ.

Zesílíme-li dále předpoklady, uvažujeme-li totiž ekvikorelovaná normální rozložení, pak pravděpodobnost oblasti $\mathscr{E} \times \mathscr{E} \times \ldots \times \mathscr{E}$ (kde $\mathscr{E}$ je libovolná, nemusí být konvexní ani symetrická) je opět neklesající funkcí korelací.

Závěrem je naší důkazové metody užito pro získání jistých nerovností pro speciální případy mnohorozměrného exponenciálního a Poissonova rozložení.

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