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ON PROBABILITY LAWS OF SOLUTIONS TO DIFFERENTIAL SYSTEMS DRIVEN BY A FRACTIONAL BROWNIAN MOTION

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This article investigates several properties related to densities of solutions $(X_t)_{t\in[0,1]}$ to differential equations driven by a fractional Brownian motion with Hurst parameter $H > 1/4$. We first determine conditions for strict positivity of the density of X_t. Then we obtain some exponential bounds for this density when the diffusion coefficient satisfies an elliptic type condition. Finally, still in the elliptic case, we derive some bounds on the hitting probabilities of sets by fractional differential systems in terms of Newtonian capacities.

1. Introduction. Let $B = (B^1, \dots, B^d)$ be a d-dimensional fractional Brownian motion indexed by [0, 1], with Hurst parameter $H > 1/4$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Recall that this means that the components B^i are i.i.d. and that each B^i is a centered Gaussian process satisfying

$$\mathbb{E}\left[\left(B_t^i - B_s^i\right)^2\right] = |t-s|^{2H}. \tag{1}$$

In particular, for any $H > 1/4$, the path $t \mapsto B_t$ is almost surely $(H-\varepsilon)$-Hölder continuous for any $\varepsilon > 0$ and for $H = 1/2$ the process $B = B^H$ coincides with the usual d-dimensional Brownian motion.

We are concerned here with the following class of equations driven by B:

$$X_t^x = x + \int_0^t V_0(X_s^x)\,ds + \sum_{i=1}^d \int_0^t V_i(X_s^x)\,dB_s^i, \qquad t \in [0,1], \tag{2}$$

where x is a generic initial condition and $\{V_i; 0 \le i \le d\}$ is a collection of smooth vector fields of $\mathbb{R}^n$. Owing to the fact that the family $\{B^H; 0 < H < 1\}$ is a very natural generalization of Brownian motion, this kind of system is increasingly used in applications and has also been thoroughly analyzed in the last past years at a theoretical level.

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Among the contributions to the study of (2) which seem most relevant to our purposes let us first mention the resolution of the equation, with Young type integration methods for $H > 1/2$ (cf., [33]) and rough paths techniques for $H \in (1/4, 1/2)$ (see, e.g., [20]). Then once equation (2) is solved, a natural question to address is to get some information on the law of the random variable X_t^x when $t \in (0, 1]$. To this respect, we have to distinguish several cases:

- When $H > 1/2$ and under ellipticity assumptions on the vector fields V_i, existence and smoothness of the density are shown in [23, 30]. The Hörmander's case for $H > 1/2$ is treated in [5].
- When $H \in (1/4, 1/2)$, the integrability of the Jacobian established in [15] immediately yields smoothness of the density in the elliptic case. The hypoelliptic case is handled in the series of papers [13, 22, 24], culminating by the reference [14] which gives a Hömander's type criterion for a wide class of Gaussian processes including fBm with $H \in (1/4, 1/2)$.
- Concentration results and exponential bounds on the density are treated in particular cases: gradient bounds in the case $H > 1/2$ are obtained in [8], and an upper bound for the density in a skew-symmetric situation is addressed in [9].

Let us also mention several attempts of small time asymptotics for the density of X_t^x, like the expansions contained in [4, 7, 28].

The current article should be seen as another step toward a better understanding of the law of X^x as a process when the coefficients of equation (2) satisfy different kind of ellipticity conditions.

The following assumption will prevail until the end of the paper.

HYPOTHESIS 1.1. *The vector fields $V_0, \dots, V_d$ are $C_b^\infty(\mathbb{R}^n)$ (bounded together with all their derivatives).*

Let us now range our nondegeneracy conditions in increasing order of restrictions: the first kind of assumption is a rather mild control-type hypothesis which can be traced back to [10] and [12].

HYPOTHESIS 1.2. *Let $\mathcal{H}$ be the Hilbert space related to our fBm B (see the definition at Section 2.2) and define a map $\Phi : \mathcal{H} \to \mathcal{C}(\mathbb{R}^n)$ such that for all $h \in \mathcal{H}$, $\Phi(h)$ is defined by the ordinary differential equation*

$$\Phi(h)_t = x + \int_0^t V_0\big(\Phi(h)_s\big)\, ds + \sum_{i=1}^d \int_0^t V_i\big(\Phi(h)_s\big)\, d\mathcal{R}h_s^i,$$

which is understood in the (p-var) Young sense and where the isometry $\mathcal{R}$ is defined by relation (14). Then for any $y \in \mathbb{R}^n$, there exists an element $h \in \mathcal{H}$ such that $\Phi(h)_t = y$ and $\Phi(h)$ is a submersion.

Hypothesis 1.2 is a variant of Hörmander's condition, and it has been shown in [10], Theorem II.1, that it is equivalent to the strict positivity of the density function of X_t^x in case of nondegenerate equations driven by Brownian motions. More precisely, as pointed out in [12], page 28, Hypothesis 1.2 is, for instance, satisfied if the following condition is met: for every $x \in \mathbb{R}^n$ and every nonvanishing $\lambda \in \mathbb{R}^d$, the vectors $V_1(x), \dots, V_d(x)$ and $[V_1, Y](x), \dots, [V_d, Y](x)$ span $\mathbb{R}^n$, where we have set $Y = \sum_{i=1}^d \lambda_i V_i$.

This provides a handy geometric interpretation of this assumption and the usual diffusion case tends to indicate that Hypothesis 1.2 should be minimal in order to establish strict positivity of the density for X_t^x.

The second assumption we shall invoke is of elliptic type, and can be stated as follows.

HYPOTHESIS 1.3. *The vector fields $V_1, \dots, V_d$ of equation* (2) *form an elliptic system, that is,*

$$v^* V(x) V^*(x) v \geq \lambda |v|^2 \qquad \text{for all } v, x \in \mathbb{R}^n, \tag{3}$$

where we have set $V = (V_j^i)_{i=1,\dots,n; j=1,\dots,d}$ and where λ designates a strictly positive constant.

With this set of hypotheses in hand, we obtain the following results:

(1) We first give some general conditions in order to check that the density p_t of X_t^x is strictly positive on $\mathbb{R}^n$.

THEOREM 1.4. *Consider the solution X^x to equation* (2) *driven by a d-dimensional fBm with Hurst parameter $H > 1/4$. Assume that Hypotheses* 1.1, 1.2 *are satisfied and that the Malliavin matrix of X_t^x, $t > 0$, is invertible with inverse in L^p, for every $p \geq 1$. Let $t \in (0, 1]$ and consider the density $p_t : \mathbb{R}^n \to \mathbb{R}_+$ of the random variable X_t^x. Then $p_t(y) > 0$ for all $y \in \mathbb{R}^n$.*

Note that the nondegeneracy condition on the Malliavin matrix assumed in the above theorem is achieved under Hörmander's condition (cf., [5, 14]).

(2) Next we derive some Gaussian or sub-Gaussian type upper bounds for the density p_t of the random variable X_t^x.

THEOREM 1.5. *Let X^x be the solution to equation* (2) *driven by a d-dimensional fBm B with Hurst parameter $H > 1/4$, assume that $V_1, \dots, V_d$ satisfy the elliptic condition* (3) *and let $t \in (0, 1]$. Then the density p_t of X_t^x satisfies the following inequality*:

$$p_t(y) \leq c_1 t^{-nH} \exp\left(-\frac{|y-x|^{(2H+1)\wedge 2}}{c_2 t^{2H}}\right) \qquad \text{for all } y \in \mathbb{R}^n, \tag{4}$$

for two strictly positive constants c_1, c_2.

Observe that we have put an emphasis in computing the correct exponents in all terms of relation (4). Namely, the terms t^{-nH} and t^{2H} (resp., outside and inside the exponential terms) can be considered as optimal, since they correspond to what one obtains in the fractional Brownian case, that is, nondegenerate constant coefficients $V_1, \dots, V_d$ and $V_0 \equiv 0$. As far as the exponent of $|y - x|$ within the exponential is concerned, the quadratic Gaussian term we get in the regular case (namely $H > 1/2$) is also optimal, while the exponent $2H + 1$ of the irregular case ($H < 1/2$) is due to some poorer concentration properties of the random variable X_t^x (see Proposition 2.10 below for more details).

(3) Finally, we complete this paper by studying the relationship between capacities of sets in $\mathbb{R}^n$ and hitting probabilities for equation (2) seen as a system. Indeed, we are interested in solving a classical problem on potential theory for stochastic processes which is the following: can we relate the hitting probabilities of X^x solution to equation (2) with a Newtonian capacity? In other words, we wish to know if there exists $\alpha \in \mathbb{R}$ such that for all Borel sets $A \subset \mathbb{R}^n$

$$\mathbb{P}\big(X^x(\mathbb{R}_+) \cap A \neq \varnothing\big) > 0 \quad \Longleftrightarrow \quad \mathrm{Cap}_\alpha(A) > 0.$$

For the sake of readability, let us briefly recall the definition of Newtonian capacity: for all Borel sets $A \subset \mathbb{R}^n$, we define $\mathcal{P}(A)$ to be the set of all probability measures with compact support in A. For $\mu \in \mathcal{P}(A)$, we let $\mathcal{E}_\alpha(\mu)$ denote the α-dimensional energy of μ, that is,

$$\mathcal{E}_\alpha(\mu) := \iint \mathrm{K}_\alpha\big(|x - y|\big)\mu(dx)\mu(dy), \tag{5}$$

where K_α denotes the α-dimensional Newtonian kernel, that is,

$$\mathrm{K}_\alpha(r) := \begin{cases} r^{-\alpha}, & \text{if } \alpha > 0, \\ \log(N_0/r), & \text{if } \alpha = 0, \\ 1, & \text{if } \alpha < 0, \end{cases} \tag{6}$$

where $N_0 > 0$ is a constant. For all $\alpha \in \mathbb{R}$ and Borel sets $A \subset \mathbb{R}^n$, we then define the α-dimensional capacity of A as

$$\mathrm{Cap}_\alpha(A) := \Big[\inf_{\mu \in \mathcal{P}(A)} \mathcal{E}_\alpha(\mu)\Big]^{-1}, \tag{7}$$

where by convention we set $1/\infty := 0$. In particular, it is easily seen from definitions (5)–(7) that for any $x \in \mathbb{R}^n$ we have $\mathrm{Cap}_\alpha(\{x\}) > 0$ if and only if $\alpha < 0$.

Let us now go back to our fBm situation: recall that for a n-dimensional fractional Brownian motion $B = (B(t), t \geq 0)$ with Hurst parameter $H \in (0, 1)$, the following is well known (see, e.g., [32] and the references therein):

$$B \text{ hits points in } \mathbb{R}^n \qquad \text{a.s. if and only if } n < \frac{1}{H}. \tag{8}$$

Moreover, for all $0 < a < b$, $\eta > 0$, there exist positive constants c_3, c_4 such that for any Borel set $A \subset \mathbb{R}^n$,

$$c_3 \operatorname{Cap}_{n-(1/H)}(A) \le \mathbb{P}\big(B([a,b]) \cap A \neq \varnothing\big) \le c_4 \operatorname{Cap}_{n-(1/H)-\eta}(A).$$

As in the case of density functions, our aim is to obtain similar bounds for the solution to equation (2), where B is a fBm with $H > \frac{1}{4}$. We shall get the following.

THEOREM 1.6. *Let X^x be the solution to equation* (2) *driven by a d-dimensional fBm B with Hurst parameter $H > 1/4$. Fix $0 < a < b \le 1$, $M > 0$, and $\eta > 0$ Then whenever $V_1, \dots, V_d$ satisfy the elliptic condition* (3), *there exists two strictly positive constants c_5, c_6 depending on a, b, H, M, n, η such that for all compact sets $A \subseteq [-M, M]^n$,*

$$c_5 \operatorname{Cap}_{n-(1/H)}(A) \le \mathbb{P}\big(X^x([a,b]) \cap A \neq \varnothing\big) \le c_6 \operatorname{Cap}_{n-(1/H)-\eta}(A). \tag{9}$$

REMARK 1.7. We have used the self-similarity property of fractional Brownian motions in order to obtain correct order (in t) in the various bounds for the density functions of X^x, which is essential to prove the above characterization of hitting probabilities. At the price of additional work and less precise estimates, we believe the above result can be extended to more general Gaussian noises. This will be discussed in a later work.

Moreover, as a corollary of Theorem 1.6, we easily get that if Hypothesis 1.3 is met, then if $n < \frac{1}{H}$ the process X^x hits points in $\mathbb{R}^n$ with strictly positive probability, while if $n > \frac{1}{H}$ the process X^x does not hit points in $\mathbb{R}^n$ a.s.

Let us say a few words about the methodology we have followed in order to obtain the results above. Our computations lie into the landmark of stochastic analysis for Gaussian processes, and we try to apply general Malliavin calculus tools which yield global recipes in order to get strict positivity [2] or upper bounds [29], Chapter 2, for densities of random variables defined on the Wiener space. We also invoke the references [16, 17], which establish nice relationships between stochastic analysis and potential theory for processes. This being said, our technical efforts will mainly be focused on the following points:

- An accurate Karhunen–Loeve expansion of fBm which will enable us to obtain the strict positivity of the density p_t.
- A combination of rough paths estimates and a sharp analysis of some covariance matrices related to fBm in order to obtain our exponential upper bounds.
- A thorough analysis of bivariate densities for the hitting probabilities of X^x.

All those points will obviously be detailed in the next sections.

Here is how our article is structured: Section 2 gathers some material on fBm and rough differential equations which prove to be useful in the sequel. Section 3 is devoted to establish criteria for the strict positivity of the density of X_t^x and our

Gaussian upper bounds for p_t are handled in Section 4. Finally, we get the bounds on hitting probabilities in Section 5, where in particular all the previous tools are used.

Notation. Throughout this paper, unless otherwise specified, we use $|\cdot|$ for Euclidean norms and $\|\cdot\|_{L^p}$ for the L^p norm with respect to the underlying probability measure $\mathbb{P}$.

Consider a finite-dimensional vector space V. The space of V-valued Hölder continuous functions defined on [0, 1], with Hölder continuity exponent $\gamma \in (0, 1)$, will be denoted by $\mathcal{C}^\gamma(V)$, or just $\mathcal{C}^\gamma$ when this does not yield any ambiguity. For a function $g \in \mathcal{C}^\gamma(V)$ and $0 \le s < t \le 1$, we shall consider the semi-norms

$$\|g\|_{s,t,\gamma} = \sup_{s \le u < v \le t} \frac{|g_v - g_u|_V}{|v-u|^\gamma}. \tag{10}$$

The semi-norm $\|g\|_{0,1,\gamma}$ will simply be denoted by $\|g\|_\gamma$.

Generic universal constants will be denoted by c, C independently of their exact values.

2. Preliminary material. Recall that a fractional Brownian motion B is a d-dimensional centered Gaussian process with independent components B^i such that $\mathbb{E}[(B_t^i - B_s^i)^2]$ is given by (1). Let us also point out that B admits a representation of Volterra type, namely

$$B_t^i = \int_0^t K(t,u)\, dW_u^i, \qquad i = 1, \dots, d, \tag{11}$$

for a d-dimensional Wiener process W and a kernel K (whose exact expression is given, e.g., in [29]) such that for any $t \in [0, 1]$ we have $K(t, \cdot) \in L^2([0, 1])$. We denote by R the common covariance of the B^i, defined by

$$\begin{aligned} R_{st} = \mathbb{E}[B_s^i B_t^i] &= \int_0^s K(s,u)K(t,u)\, du \\ &= \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \end{aligned} \tag{12}$$

for $s, t \in [0, 1]$. In the remainder of the paper, we assume that the process B is realized on an abstract Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \mathcal{C}_0([0, 1]; \mathbb{R}^d)$. Namely, $\Omega = \mathcal{C}_0([0, 1])$ is the Banach space of continuous functions vanishing at 0 equipped with the supremum norm, $\mathcal{F}$ is the Borel sigma-algebra and $\mathbb{P}$ is the unique probability measure on Ω such that the canonical process $B = \{B_t = (B_t^1, \dots, B_t^d), t \in [0, 1]\}$ is a centered Gaussian process with covariance R given by (12).

2.1. *Rough path above* $\mathbf{B}$. We consider here B together with its iterated integrals as a rough path, and we refer to [20, 27] for further details on this concept. Let us just mention here a few basic facts.

Let $T^N(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d) \oplus \cdots \oplus (\mathbb{R}^d)^{\otimes N}$ be the truncated step-N tensor algebra. For paths in $T^N(\mathbb{R}^d)$ starting at the fixed point $e := 1 + 0 + \cdots + 0$, one can define p-variation metrics. We call this metric $d_{p\text{-var}}$, and its corresponding norm is denoted by $\|\cdot\|_{p\text{-var}}$.

A geometric p-rough path $\mathbf{x}$ is a path in $T^{\lfloor 1/\beta \rfloor}(\mathbb{R}^d)$ which can be approximated by lifts of smooth paths in the $d_{p\text{-var}}$ metric. Given a rough path $\mathbf{x}$, the projection on the first level is an $\mathbb{R}^d$-valued path and will be denoted by $\pi_1(\mathbf{x})$. It can be seen that rough paths actually take values in the smaller set $G^N(\mathbb{R}^d) \subset T^N(\mathbb{R}^d)$, where $G^N(\mathbb{R}^d)$ denotes the free step-N nilpotent Lie group with d generators. The Carnot–Caratheodory metric turns $(G^N(\mathbb{R}^d), d)$ into a metric space.

Let us now turn to the fBm case: according to the considerations above, in order to prove that a lift of a d-dimensional fBm as a geometric rough path exists it is sufficient to build enough iterated integrals of B by a limiting procedure. Toward this aim, a lot of the information concerning B is encoded in the rectangular increments of the covariance function R [defined by (12)], which are given by

$$R_{uv}^{st} \equiv \mathbb{E}\big[\big(B_t^1 - B_s^1\big)\big(B_v^1 - B_u^1\big)\big].$$

We then call two-dimensional ρ-variation of R the quantity

$$V_\rho(R)^\rho \equiv \sup\left\{\left(\sum_{i,j}\big|R_{s_i s_{i+1}}^{t_j t_{j+1}}\big|^\rho\right)^{1/\rho}; (s_i), (t_j) \in \Pi\right\},$$

where Π stands again for the set of partitions of $[0, 1]$. The following result is now well known for fractional Brownian motion (see [20], Chapter 15).

PROPOSITION 2.1. *For a fractional Brownian motion with Hurst parameter H, we have $V_\rho(R) < \infty$ for all $\rho \geq 1/(2H)$. Consequently, for $H > 1/4$ the process B admits a lift $\mathbf{B}$ as a geometric rough path of order p for any $p > 1/H$.*

2.2. *Malliavin calculus tools*. Gaussian techniques are obviously essential in the analysis of densities for solutions to (2), and we proceed here to introduce some of them. These lines follow the classical analysis for Gaussian rough paths as explained in [20] and [29]. We refer to those books for further details.

2.2.1. *Wiener space associated to fBm*. Let $\mathcal{E}$ be the space of $\mathbb{R}^d$-valued step functions on $[0, 1]$, and $\mathcal{H}$ the closure of $\mathcal{E}$ for the scalar product

$$\big\langle (\mathbf{1}_{[0,t_1]}, \ldots, \mathbf{1}_{[0,t_d]}), (\mathbf{1}_{[0,s_1]}, \ldots, \mathbf{1}_{[0,s_d]})\big\rangle_{\mathcal{H}} = \sum_{i=1}^d R(t_i, s_i),$$

where R is defined by (12). Then if $(e_1,\dots,e_d)$ designates the canonical basis of $\mathbb{R}^d$, one constructs an isometry $K_H^*:\mathcal{H}\to L^2([0,1])$ such that $K_H^*(\mathbf{1}_{[0,t]}e_i)=\mathbf{1}_{[0,t]}\ K_H(t,\cdot)e_i$, where the kernel $K=K_H$ has been introduced in (11), and verifies that $\mathbb{E}[B_s^iB_t^i]=\int_0^{s\wedge t}K(t,r)K(s,r)\,dr$. When $H>\frac{1}{2}$, it can be shown that $L^{1/H}([0,1])\subset\mathcal{H}$, and when $H<\frac{1}{2}$ one has $C^\gamma\subset\mathcal{H}\subset L^2([0,1])$ for all $\gamma>\frac{1}{2}-H$. We shall also use the following representations of the inner product in $\mathcal{H}$: for $H>1/2$ and $\phi,\psi\in\mathcal{H}$, we have

$$\langle\phi,\psi\rangle_{\mathcal{H}}=H(2H-1)\int_0^1\int_0^1|s-t|^{2H-2}\langle\phi_s,\psi_t\rangle_{\mathbb{R}^d}\,ds\,dt. \tag{13}$$

In order to deduce that $(\Omega,\mathcal{H},\mathbb{P})$ defines an abstract Wiener space, we remark that $\mathcal{H}$ is continuously and densely embedded in Ω. To this aim, define first the space $\bar{\mathcal{H}}$ as

$$\bar{\mathcal{H}}=\left\{\ell:[0,1]\to\mathbb{R}^d;\ell_t=\int_0^t K(t,u)\phi_u\,du\text{ with }\phi\in L^2\big([0,1]\big)\right\},$$

where K is defined by 11. It is worth noticing at this point that the space $\bar{\mathcal{H}}$ yields the accurate notion of Cameron–Martin space in the fBm context (for Brownian motion one obtains $\mathcal{H}=L^2([0,1])$ and $\bar{\mathcal{H}}=W^{1,2}([0,1])$). Then one proves that the operator $\mathcal{R}:=\mathcal{R}_H:\mathcal{H}\to\bar{\mathcal{H}}$ given by

$$\mathcal{R}\psi:=\int_0^{\cdot}K(\cdot,s)\big[K_H^*\psi\big](s)\,ds \tag{14}$$

defines a dense and continuous embedding from $\mathcal{H}$ into Ω; this is due to the fact that $\mathcal{R}_H\psi$ is H-Hölder continuous (for details, see [30], page 399). Let us now quote from [20], Chapter 15, a result relating the 2-d regularity of R and the regularity of $\bar{\mathcal{H}}$.

PROPOSITION 2.2. *Let B be a fBm with Hurst parameter $H\in(1/4,1/2)$. Then one has $\bar{\mathcal{H}}\subset\mathcal{C}^{\rho\text{-var}}$ for $\rho>(H+1/2)^{-1}$. Furthermore, the following quantitative bound holds*:

$$\|h\|_{\bar{\mathcal{H}}}\ge\frac{\|h\|_{\rho\text{-var}}}{(V_\rho(R))^{1/2}}.$$

REMARK 2.3. Proposition 2.2 can be sharpened to $\bar{\mathcal{H}}\subset\mathcal{C}^{\rho\text{-var}}$ for all $\rho\ge(H+1/2)^{-1}$. Moreover, the above embedding inequality holds true for all $\rho\ge(H+1/2)^{-1}$ if the present ρ-variation is replaced by the mixed $(1,\rho)$-variation. We refer interested readers to [18] for more details.

Let us close this section by pointing out an implication of Volterra's representation of fBm (11) in terms of filtrations. Indeed, it is readily checked that $\mathcal{F}_t\equiv\sigma(\{B_s;0\le s\le t\})$ can also be expressed as $\mathcal{F}_t=\sigma(\{W_s;0\le s\le t\})$. This filtration will be important in the sequel.

2.2.2. *Scale invariant inequalities.* The following inequalities, in particular the lower bounds, shall be used several times throughout the text. They show that one can replace the $\mathcal{H}$-norm that may be difficult to estimate by simpler quantities while keeping the correct scaling in time.

PROPOSITION 2.4. *Let $\mathcal{H}$ be the Hilbert space introduced at Section 2.2.1, depending on the Hurst parameter $H \in (0,1)$. Then*:

- *Asume $H > 1/2$. Let $\gamma > H - 1/2$. There exist constants $c_1, c_2 > 0$ such that for every continuous $f \in \mathcal{H}$, and $t \in (0,1]$,*

$$c_1 t^{2H} \frac{\min_{[0,1]} |f|^4}{\|f\|_\infty^2 + \|f\|_\gamma^2} \le \|f \mathbf{1}_{[0,t]}\|_{\mathcal{H}}^2 \le c_2 t^{2H} \|f\|_\infty^2.$$

- *Assume $H \le 1/2$ and let $\gamma > 1/2 - H$. There exist constants $c_1, c_2 > 0$ such that for every $f \in C^\gamma$, and $t \in (0,1]$,*

$$c_1 t^{2H} \min_{[0,1]} |f|^2 \le \|f \mathbf{1}_{[0,t]}\|_{\mathcal{H}}^2 \le c_2 t^{2H} \big(\|f\|_\gamma^2 + \|f\|_\infty^2\big).$$

PROOF. We first assume $H > 1/2$. The inequality $\|f \mathbf{1}_{[0,t]}\|_{\mathcal{H}}^2 \le c_2 t^{2H} \|f\|_\infty^2$ is a straightforward consequence of (13). The inequality

$$c_1 \frac{\min_{[0,1]} |f|^4}{\|f\|_\infty^2 + \|f\|_\gamma^2} \le \|f\|_{\mathcal{H}}^2$$

is proved in [5], Lemma 4.4. For $t \in (0,1]$, this inequality can be rescaled as follows:

$$\begin{aligned}
\|f \mathbf{1}_{[0,t]}\|_{\mathcal{H}}^2 &= H(2H-1) \int_0^t \int_0^t |u-v|^{2H-2} \langle f(u), f(v) \rangle \, du \, dv \\
&= H(2H-1) t^{2H} \int_0^1 \int_0^1 |u-v|^{2H-2} \langle f(tu), f(tv) \rangle \, du \, dv \\
&\ge c_1 t^{2H} \frac{\min_{[0,1]} |f_t|^4}{\|f_t\|_\infty^2 + \|f_t\|_\gamma^2} \ge c_1 t^{2H} \frac{\min_{[0,1]} |f|^4}{\|f\|_\infty^2 + \|f\|_\gamma^2},
\end{aligned}$$

where $f_t(u) = f(tu)$. This proves our claim for $H > 1/2$.

We now assume $H \le 1/2$. The fact that $\|f\|_{\mathcal{H}}^2 \ge c_1 \|f\|_2^2 \ge c_1 \min_{[0,1]} |f|^2$ is well known and the inequality easily rescales as above. The last inequality to prove is the upper bound. It is pointed in [30] that we have, for any $h_1, h_2 \in \mathcal{H}$,

$$\langle h_1, h_2 \rangle_{\mathcal{H}} = \int_0^1 h_1 \, d\mathcal{R} h_2,$$

where the right-hand side is understood in the Young sense and $\mathcal{R}$ is the isometry going from $\mathcal{H}$ to $\bar{\mathcal{H}}$. Hence, if $p^{-1} + q^{-1} > 1$ and $p > H^{-1}$, $q > (1/2 + H)^{-1}$ we have

$$\big|\langle h_1, h_2 \rangle_{\mathcal{H}}\big| \le C\big(\|h_1\|_{p-\text{var}} + \|h_1\|_\infty\big) \|\mathcal{R} h_2\|_{q\text{-var}}.$$

We now use Proposition 2.2 to get the bound

$$\|\mathcal{R}h_2\|_{q\text{-var}} \le C\|\mathcal{R}h_2\|_{\bar{\mathcal{H}}} = C\|h_2\|_{\mathcal{H}}.$$

This proves that

$$\|f\|_{\mathcal{H}}^2 \le c_2\big(\|f\|_\gamma^2 + \|f\|_\infty^2\big).$$

Again, this inequality easily rescales on the time interval $[0, t]$. □

2.2.3. *Malliavin calculus for B.* We refer to [29] for the definition of the Malliavin derivatives $\mathbf{D}F$ of a functional F on Ω. The Malliavin matrix of a smooth functional F is defined as

$$\gamma_F = \big(\langle \mathbf{D}F^i, \mathbf{D}F^j\rangle_{\mathcal{H}}\big)_{1\le i,j\le n}. \tag{15}$$

The reader is also sent to [29], Definition 2.1.1, for the definition of nondegenerate random vector.

It is a classical result that the law of a nondegenerate random vector $F = (F^1, \dots, F^n)$ admits a smooth density with respect to the Lebesgue measure on $\mathbb{R}^n$. Furthermore, the following integration by parts formula (borrowed from [29], Propositions 2.1.4 and 2.1.5) allows to get more quantitative estimates.

PROPOSITION 2.5. *Let $F = (F^1, \dots, F^n)$ be a nondegenerate random vector. Then the density $p_F(y)$ of F belongs to the Schwartz space, and for any $\sigma \subset \{1, \dots, n\}$,*

$$p_F(y) = (-1)^{n-|\sigma|}\mathbb{E}\big[\mathbf{1}_{\{F^i>y^i, i\in\sigma, F^i<y^i, i\neq\sigma\}} H_{(1,\dots,n)}(F, 1)\big] \qquad \text{for all } y \in \mathbb{R}^n,$$

where the random variables $H_{(1,\dots,n)}(F, 1)$ satisfy:

$$\big\|H_{(1,\dots,n)}(F, 1)\big\|_2 \le \big\|\gamma_F^{-1}\big\|_{n,2^{n+2}}^n \|\mathbf{D}F\|_{n,2^{n+2}}^n. \tag{16}$$

2.2.4. *Karhunen–Loeve expansions.* Karhunen–Loeve expansions are approximations of the Gaussian process B in $\bar{\mathcal{H}}$. We shall design here one of those expansions, which will be useful for further computations. It relies on the Volterra type representation (11) for B. More precisely, we shall construct an approximating sequence B^n, such that the distributions of B and $B - B^n$ are equivalent on $[0, 1]$.

To this aim, consider the Cameron–Martin space $\bar{\mathcal{H}}^W$ of the usual Brownian motion, namely $\bar{\mathcal{H}}^W = W^{1,2}([0, 1])$, and let $(h_k)_{k\ge1}$ be any orthonormal basis of $\bar{\mathcal{H}}^W$. If $\{Z_k; k \ge 1\}$ is an i.i.d. sequence of standard Gaussian random variables, it is well known (see, e.g., [31]) that the process

$$W_t = \sum_{k=1}^{+\infty} h_k(t) Z_k$$

is a Brownian motion on $[0, 1]$. Our Karhunen–Loeve approximation of W will be given by $W_t^n = \sum_{k=1}^n h_k(t) Z_k$, and we have the following result.

PROPOSITION 2.6. *Let* $0 < \tau < 1$. *There exists an orthonormal basis* $\{\ell_k; k \geq 1\}$ *of* $\bar{\mathcal{H}}^W$ *such that, setting* $W_t^n = \sum_{k=1}^n \ell_k(t) Z_k$, *the distribution of the processes* W *and* $W - W^n$ *are equivalent on* $[0, \tau]$.

REMARK 2.7. Proposition 2.6 only states an equivalence of measure for the processes W and $W - W^n$ defined on $[0, 1)$. This is due to the fact that, for our typical examples of orthonormal basis, we shall get $(W - W^n)_1 = 0$ almost surely.

PROOF OF PROPOSITION 2.6. We divide this proof in two steps.

Step 1. We first prove that if the matrix $(\int_\tau^1 \ell_i'(s)\ell_j'(s)\,ds)_{1\leq i,j\leq n}$ is invertible, then the distribution of the processes W and $W - W^n$ are equivalent on $[0, \tau]$.

For this, let us first observe that $W - W^n$ has the same distribution as the Brownian motion W conditioned by the event $[\int_0^1 \ell_k'(s)\,dW_s = 0,\ \text{for } 1 \leq k \leq n]$. Indeed, for any bounded and measurable functional F on the Wiener space, we have

$$\begin{aligned}
&\mathbb{E}\left[F(W_t, 0 \leq t \leq 1)\Big| \int_0^1 \ell_k'(s)\,dW_s = 0, 1 \leq k \leq n\right] \\
&\quad = \mathbb{E}\left[F\left(\sum_{k=1}^{+\infty} \ell_k(t) Z_k, 0 \leq t \leq 1\right)\Big| \int_0^1 \ell_k'(s)\,dW_s = 0, 1 \leq k \leq n\right] \\
&\quad = \mathbb{E}\left[F\left(\sum_{k=n+1}^{+\infty} \ell_k(t) Z_k, 0 \leq t \leq 1\right)\Big| \int_0^1 \ell_k'(s)\,dW_s = 0, 1 \leq k \leq n\right] \\
&\quad = \mathbb{E}\left[F\left(\sum_{k=n+1}^{+\infty} \ell_k(t) Z_k, 0 \leq t \leq 1\right)\right],
\end{aligned}$$

where we have invoked the independence of the families $\{\int_0^1 \ell_k'(s)\,dW_s; 1 \leq k \leq n\}$ and $\{\int_0^1 \ell_k'(s)\,dW_s; k > n\}$. It is thus readily checked that

$$\begin{aligned}
&\mathbb{E}\left[F(W_t, 0 \leq t \leq 1)\Big| \int_0^1 \ell_k'(s)\,dW_s = 0, 1 \leq k \leq n\right] \\
&\qquad = \mathbb{E}\big[F\big(W_t - W_t^n, 0 \leq t \leq 1\big)\big].
\end{aligned} \tag{17}$$

Let now $0 < \tau < 1$ and assume that the matrix $(\int_\tau^1 \ell_i'(s)\ell_j'(s)\,ds)_{1\leq i,j\leq n}$ is invertible. This invertibility implies that the conditional density of $(\int_0^1 \ell_k'(s) \times dW_s)_{1\leq k\leq n}$ given $\sigma(W_s, s \leq \tau)$ with respect to the distribution of $(\int_0^1 \ell_k'(s) \times dW_s)_{1\leq k\leq n}$ exists. Let us denote by $\eta_\tau(y)$, $y \in \mathbb{R}^n$ this density. If F is a bounded and measurable functional on the Wiener space we then have

$$\begin{aligned}
&\mathbb{E}\left[F(W_t, 0 \leq t \leq \tau)\Big| \int_0^1 \ell_k'(s)\,dW_s = 0, 1 \leq k \leq n\right] \\
&\qquad = \mathbb{E}\big[\eta_\tau(0) F(W_t, 0 \leq t \leq \tau)\big].
\end{aligned} \tag{18}$$

Gathering relations (17) and (18), we thus get that the distribution of the processes $W - W^n$ and W are equivalent on $[0, \tau]$. Our proposition is thus proved once we show that there exists an orthonormal basis $\{\ell_k; k \geq 1\}$ of $\bar{\mathcal{H}}^W$ such that for any $\tau \in [0, 1)$, the matrix $(\int_\tau^1 \ell_i'(s)\ell_j'(s)\,ds)_{1\leq i,j\leq n}$ is invertible.

Step 2. Let us now construct an orthonormal basis of $\bar{\mathcal{H}}^W$ with the desired invertibility property: let $(f_k)_{k\geq 1}$ be any basis of $L^2[0, 1]$ and denote by ℓ_k' the Gram–Schmidt orthonormalisation of $(f_k)_{k\geq 1}$. By using triangular matrices, we see that the invertibility of the matrix $(\int_\tau^1 \ell_i'(s)\ell_j'(s)\,ds)_{1\leq i,j\leq n}$ is then equivalent to the invertibility of $(\int_\tau^1 f_i(s)f_j(s)\,ds)_{1\leq i,j\leq n}$. For instance, by choosing $f_k(t) = (1-t)^{k-1}$, $k \geq 1$, some elementary calculations involving Hilbert matrices yield our claim. □

The previous result on Brownian motion has a direct implication in terms of our fractional Brownian motion B:

COROLLARY 2.8. *Let* $0 < \tau < 1$. *There exists an orthonormal basis* $\{h_k; k \geq 1\}$ *of* $\bar{\mathcal{H}}$ *such that, setting* $B_t^n = \sum_{k=1}^n h_k(t)Z_k$, *the distribution of the processes* B *and* $B - B^n$ *are equivalent on* $[0, \tau]$.

PROOF. Take the orthonormal basis $\{\ell_k; k \geq 1\}$ of $\bar{\mathcal{H}}^W$ constructed at Proposition 2.6 and set $h_k(t) = \int_0^t K(t, u)\ell_k'(u)\,du$. □

REMARK 2.9. One can trivially generalize Proposition 2.6, for example, on the time interval $[0, 2)$. This allows to construct an approximating sequence B^n such that the distributions of B and $B - B^n$ are equivalent on $[0, 1]$ instead of $[0, 1)$. Let us also observe that for later purpose, we shall not really need the equivalence of the two distributions but that absolute continuity would be enough.

2.3. *Differential equations driven by fractional Brownian motion.* Recall that we consider the following kind of equation:

$$X_t^x = x + \int_0^t V_0(X_s^x)\,ds + \sum_{i=1}^d \int_0^t V_i(X_s^x)\,dB_s^i, \tag{19}$$

where the vector fields $V_0, \ldots, V_d$ are $\mathcal{C}_b^\infty$-vector fields on $\mathbb{R}^n$ and B is our driving fBm as defined in (11).

2.3.1. *Existence, uniqueness and estimates.* It is a well-known fact (see, e.g., [20, 21]) that, whenever the vector fields V satisfy Hypothesis 1.1, equation (19) driven by a d-dimensional fBm B with Hurst parameter $H > 1/4$ admits a unique solution. Our sub-Gaussian bounds for the density will be a consequence of the following exponential bound.

PROPOSITION 2.10 (See [19]). *Consider equation* (19) *driven by a d-dimensional fBm B with Hurst parameter $H > 1/4$, and assume that the vector fields V satisfy Hypothesis* 1.1. *Then the following inequality holds true*:

$$\mathbb{P}\Big(\sup_{s\in[0,t]}|X_s^x - x| \geq \xi\Big) \lesssim \exp\Big(-\frac{c_H \xi^{(2H+1)\wedge 2}}{t^{2H}}\Big). \tag{20}$$

2.3.2. *Differentiability.* Thanks to [13, 30] and [25], the solution X_t^x is smooth in the Malliavin sense. We shall express this Malliavin derivative in terms of the Jacobian $\mathbf{J}$ of the equation, which is defined by the relation $\mathbf{J}_t^{ij} = \partial_{x_j} X_t^{x,i}$. Setting DV_j for the Jacobian of V_j seen as a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, let us recall that $\mathbf{J}$ is the unique solution to the linear equation

$$\mathbf{J}_t = \mathrm{Id}_n + \int_0^t DV_0(X_s^x)\mathbf{J}_s\,ds + \sum_{j=1}^d \int_0^t DV_j(X_s^x)\mathbf{J}_s\,dB_s^j, \tag{21}$$

and that the following results hold true (see [13], [30] and [25] for further details):

PROPOSITION 2.11. *Let X^x be the solution to equation* (19) *and suppose the V_i's satisfy Hypothesis* 1.1. *Then for every $i = 1,\dots,n$, $t > 0$, and $x \in \mathbb{R}^n$, we have $X_t^{x,i} \in \mathbb{D}^\infty(\mathcal{H})$ and*

$$\mathbf{D}_s^j X_t^x = \mathbf{J}_{s,t} V_j(X_s^x), \qquad j = 1,\dots,d, 0 \leq s \leq t,$$

where $\mathbf{D}_s^j X_t^{x,i}$ is the jth component of $\mathbf{D}_s X_t^{x,i}$, $\mathbf{J}_t = \partial_x X_t^x$ and $\mathbf{J}_{s,t} = \mathbf{J}_t \mathbf{J}_s^{-1}$.

Let us now quote the recent result [15], which gives a useful estimate for moments of the Jacobian of rough differential equations driven by Gaussian processes.

PROPOSITION 2.12. *Consider a fractional Brownian motion B with Hurst parameter $H \in (1/4, 1/2]$ and $p > 1/H$. Then for any $\eta \geq 1$, there exists a finite constant c_η such that the Jacobian $\mathbf{J}$ defined at Proposition* 2.11 *satisfies*

$$\mathbb{E}\big[\|\mathbf{J}\|_{p\text{-var};[0,1]}^\eta\big] = c_\eta. \tag{22}$$

3. Strict positivity of the density. In this section, we follow the approach developed by Ben Arous and Léandre [10] and prove the strict positivity of the density of solutions to equation (19) as stated in Theorem 1.4. We first present, at Section 3.1, the general criterion characterizing the set of points where the density is strictly positive for a nondegenerate finite-dimensional random variable F. Then we show how to apply this criterion in our fractional SDE context at Section 3.2.

3.1. *Strict positivity of the density for nondegenerate random variables*. We borrow the considerations here from [2], for which we refer for further details. Consider $(\Omega, \mathcal{F}, \mathbb{P})$ the canonical probability space associated with our fBm B.

Let us now introduce, for a given element $\underline{\ell} = (\ell_1, \dots, \ell_n) \in \mathcal{H}^n$ and a vector $z \in \mathbb{R}^n$, the shifted Gaussian process

$$\bigl(T_z^{\underline{\ell}} B\bigr)(h) = B(h) + \sum_{j=1}^{n} z_j \langle h, \ell_j \rangle_{\mathcal{H}}, \qquad h \in \mathcal{H}.$$

Cameron–Martin's theorem of change of measures shows that for any integrable random variable G we have $\mathbb{E}[G] = \mathbb{E}[G(T_z^{\underline{\ell}} B) J_z]$, where

$$J_z = \exp\Biggl(-\sum_{j=1}^{n} z_j B(\ell_j) - \frac{1}{2}\Biggl\|\sum_{j=1}^{n} z_j \ell_j\Biggr\|_{\mathcal{H}}^2\Biggr).$$

With the same $\underline{\ell} = (\ell_1, \dots, \ell_n)$ as above, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_k)$ lying in $\{1, 2, \dots, n\}^k$, let $\underline{\ell}_\alpha = (\ell_{\alpha_1}, \dots, \ell_{\alpha_k})$ and define

$$R_{\underline{\ell}_\alpha, p} F = \int_{\{|z| \le 1\}} \bigl\langle \bigl(\mathbf{D}^k F\bigr)\bigl(T_z^{\underline{\ell}} B\bigr), \ell_{\alpha_1} \otimes \cdots \otimes \ell_{\alpha_k} \bigr\rangle_{\mathcal{H}^{\otimes k}}^p \, dz,$$

for some $p > n$ and multi-index α with $|\alpha| = k \ge 0$.

With these notation in mind, our general criterion for positivity of densities can be read as follows.

THEOREM 3.1. *Let $F = (F^1, \dots, F^n)$ be a nondegenerate random variable and $\Phi : \mathcal{H} \to \mathbb{R}^n$ a $\mathcal{C}^\infty$ functional. Suppose that the following conditions hold*:

a. *For any $h \in \mathcal{H}$ there exists a sequence of measurable transformations $T_N^h : \Omega \to \Omega$ such that $\mathbb{P} \circ (T_N^h)^{-1}$ is absolutely continuous with respect to $\mathbb{P}$.*

b. *Let $\{\mathbf{D}\Phi^j(h);\ j = 1, \dots, n\}$ be the coordinates of $\mathbf{D}\Phi(h)$ in $\mathbb{R}^n$, and set*

$$\underline{\ell} = \bigl(\mathbf{D}\Phi^1(h), \dots, \mathbf{D}\Phi^n(h)\bigr).$$

Suppose that for every $\varepsilon > 0$:

(1) $\lim_{N\to\infty} \mathbb{P}\{|F \circ T_N^h - \Phi(h)| > \varepsilon\} = 0$;

(2) $\lim_{N\to\infty} \mathbb{P}\{\|(\mathbf{D}F) \circ T_N^h - (\mathbf{D}\Phi)(h)\|_{\mathcal{H}} > \varepsilon\} = 0$; *and*

(3) $\lim_{M\to\infty} \sup_N \mathbb{P}\{(R_{\underline{\ell}_\alpha, p} F) \circ T_N^h > M\} = 0$ *for some $p > n$ and all multi-index α with $|\alpha| = 0, 1, 2, 3$.*

c. *Finally, for a fixed $y \in \mathbb{R}^n$ assume that there exists an $h \in \mathcal{H}$ such that $\Phi(h) = y$ and for the deterministic Malliavin matrix $\gamma_\Phi(h)$ of Φ at h, one has* $\det \gamma_\Phi(h) > 0$.

Then the density of F at y satisfies $p(y) > 0$.

PROOF. The theorem is borrowed from [2], with a slight modification of the definition of $R_{\underline{\ell}_\alpha,p}F$. The legitimacy of making such modification is seen directly from the proof of Proposition 4.2.2 in [2]. □

3.2. *Strict positivity of the density for solutions to fractional SDE's.* This section is devoted to the proof of Theorem 1.4. The idea is to apply the general Theorem 3.1 to $F = X_t^x$ for each fixed $t > 0$, where X^x is the solution to equation (19) and where we still work under Hypotheses 1.1 and 1.2. We moreover assume that X_t^x is nondegenerate in the Malliavin sense, that is, its Malliavin is invertible with inverse in $L^p(\Omega)$ for $p \geq 1$. In this context, some natural definitions of the maps T_N^h and of the functional Φ are as follows:

(i) For any $h \in \mathcal{H}$, we simply define T_N^h by the identity

$$T_N^h(B) = B - B^N + \mathcal{R}h,$$

where B^N has been defined at Proposition 2.6 and Corollary 2.8 and with $\mathcal{R}h^i$ defined by (14).

(ii) The map Φ is defined as the evaluation of a function at $t \in (0, 1]$. Namely, $\Phi(h)$ is solution to the ordinary differential equation

$$\Phi(h)_t = x + \int_0^t V_0(\Phi(h)_s)\,ds + \sum_{i=1}^d \int_0^t V_i(\Phi(h)_s)\,d\mathcal{R}h_s^i, \tag{23}$$

understood in the (p-var) Young sense.

In what follows, we need to check the above Φ and T_N^h satisfy conditions in Theorem 3.1.

Recall that, according to Proposition 2.1, B admits a lift to $G^{\lfloor p \rfloor}(\mathbb{R}^d)$ as a geometric rough path for any fixed $p > 1/H$. If B^N is the Karhunen–Loeve type approximation of B discussed above, denote by $\tilde{\mathbf{B}}^N$ the lift of $\tilde{B}^N = B - B^N$ to $G^{\lfloor p \rfloor}(\mathbb{R}^d)$. We have the following.

PROPOSITION 3.2. *There exists a constant $\eta > 0$ depending on p, ρ and the process B such that*

$$\sup_N \mathbb{E}[\exp(\eta \|\tilde{\mathbf{B}}^N\|_{p\text{-var};[0,1]}^2)] < \infty.$$

Moreover, for all $q \geq 1$,

$$\|\tilde{\mathbf{B}}^N\|_{p\text{-var};[0,1]} \to 0 \qquad \text{in } L^q(\mathbb{P}) \text{ as } N \to \infty.$$

PROOF. The Gaussian tail of $\|\tilde{\mathbf{B}}^N\|_{p\text{-var};[0,1]}$ follows from Lemma 15.46 as well as Proposition 15.22 in [20]. The rest of the statement is the content of Theorem 15.47 in [20]. □

For an element $h \in \mathcal{C}^{q\text{-var}}([0,1],\mathbb{R}^d)$ such that $1/q + 1/p > 1$, we also need to introduce the translation of $\mathbf{x}$ by h [denoted $T_h(\tilde{\mathbf{B}}^N)$]. Namely, $T_h(\tilde{\mathbf{B}}^N)$ is the lift of $\pi_1(\mathbf{x}) + h$ to $G^{\lfloor p\rfloor}(\mathbb{R}^d)$. The following theorem is then an easy consequence of Theorem 9.33 and Corollary 9.35 in [20].

THEOREM 3.3. *With the notation introduced above, consider $T_h(\tilde{\mathbf{B}}^N)$. There exists a constant $\eta > 0$ depending on p, H, $\|h\|_{\mathcal{H}}$ and the process B such that*

$$\sup_N \mathbb{E}\big[\exp\big(\eta \| T_h(\tilde{\mathbf{B}}^N)\|^2_{p\text{-var};[0,1]}\big)\big] < \infty.$$

Moreover, for all $q \geq 1$,

$$d_{p\text{-var};[0,1]}\big(T_h(\tilde{\mathbf{B}}^N), \mathbf{h}\big) \to 0 \qquad \text{in } L^q(\mathbb{P}) \text{ as } N \to \infty.$$

In the statement above, $\mathbf{h}$ is the lift of h to $G^{\lfloor p\rfloor}(\mathbb{R}^d)$.

We are now ready to prove the main theorem of this section.

PROOF OF THEOREM 1.4. Recall that Φ is defined by (23), and that the solution X_t^x to equation (19) can be seen as $X_t^x = \Phi(\mathcal{R}^{-1}B)_t$. With the definition of T_N^h and that of the translation map T_h in Theorem 3.3, we have

$$X_t^x \circ T_N^h = \Phi\big(T_h(\tilde{\mathbf{B}}^N)\big) \quad \text{and} \quad \mathbf{D}^k X_t^x \circ T_N^h = \mathbf{D}^k \Phi\big(T_h(\tilde{\mathbf{B}}^N)\big) \qquad \text{for all } k \in \mathbb{N}.$$

In the above, we consider $T_h(\tilde{\mathbf{B}}^N)$ as a geometric rough path that drives the equation for Φ. Now it follows from Theorem 3.3 and the continuity of Φ and $\mathbf{D}\Phi$ in the rough path topology that

$$X_t^x \circ T_N^h \to \Phi(h) \quad \text{and} \quad \mathbf{D}X_t^x \circ T_N^h \to \mathbf{D}\Phi(h)$$

in probability. This shows that conditions b-(1) and b-(2) of Theorem 3.1 are satisfied.

For condition b-(3) of Theorem 3.1, recall that $\underline{\ell} = (\mathbf{D}\Phi^1(h), \dots, \mathbf{D}\Phi^n(h))$ and that we have set $\underline{\ell}_\alpha = (\ell_{\alpha_1}, \dots, \ell_{\alpha_k})$ for any multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \{1, 2, \dots, n\}^k$. By standard analysis, it suffices to show that for each multi-index α with $|\alpha| = 0, 1, 2, 3$,

$$\begin{aligned}(R_{\underline{\ell}_\alpha, p} X_t^x) \circ T_N^h &= \int_{\{|z|\leq 1\}} \big\langle (\mathbf{D}^k X_t^x)(T_z^{\underline{\ell}} B) \circ T_N^h, \ell_{\alpha_1} \otimes \cdots \otimes \ell_{\alpha_k}\big\rangle^p_{\mathcal{H}^{\otimes k}}\, dz \\ &= \int_{\{|z|\leq 1\}} \big\langle \mathbf{D}^k \Phi\big(T_z^{\underline{\ell}} T_h(\tilde{\mathbf{B}}^N)\big), \ell_{\alpha_1} \otimes \cdots \otimes \ell_{\alpha_k}\big\rangle^p_{\mathcal{H}^{\otimes k}}\, dz\end{aligned}$$

converges to some deterministic quantity in probability. Let

$$\hat{h} = h + \sum_{j=1}^n z^j (\mathbf{D}\Phi^j)(h).$$

The above is then reduced to show that

$$\langle \mathbf{D}^k \Phi(T_{\hat{h}} \tilde{\mathbf{B}}^N), \ell_{\alpha_1} \otimes \cdots \otimes \ell_{\alpha_k} \rangle_{\mathcal{H}^{\otimes k}} \to \langle \mathbf{D}^k \Phi(\hat{h}), \ell_{\alpha_1} \otimes \cdots \otimes \ell_{\alpha_k} \rangle_{\mathcal{H}^{\otimes k}}$$

in probability and uniformly in z for $|z| \le 1$, which follows from Theorem 3.3, continuity of $\mathbf{D}^k_{\ell_{\alpha_1} \dots \ell_{\alpha_k}} \Phi(\cdot)$ in the rough path topology and the fact that z takes values in a compact set. The proof is complete. □

4. Upper bounds for the density. The aim of this section is to study upper bounds for the density of the solution to equation (19), where B is a fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$. Specifically, we shall prove Theorem 1.5 under our elliptic Hypothesis 1.3.

Our starting point here is the integration by parts type formula given at Proposition 2.5. According to this relation applied to $F = X_t^x$, we obtain the following general upper bound for the density p_t of X_t^x:

$$p_t(y) \le c\mathbb{P}(|X_t^x - x| \ge |y - x|)^{1/2} \|\gamma_t^{-1}\|^n_{n,2^{n+2}} \|\mathbf{D} X_t^x\|^n_{n,2^{n+2}} \quad \text{for all } y \in \mathbb{R}^n, \tag{24}$$

where γ_t denotes the Malliavin matrix of X_t^x. We shall bound separately the 3 terms in relation (24): first, a direct application of inequality (20) yields

$$\mathbb{P}(|X_t^x - x| \ge |y - x|) \le \exp\left(-\frac{|y-x|^{(2H+1)\wedge 2}}{ct^{2H}}\right). \tag{25}$$

Next, we prove that there exist constants c_3 and c_4 such that for all $m \in \mathbb{N}$ and $p > 1$,

$$\|\mathbf{D} X_t^x\|_{m,p} \le c_3 t^H, \tag{26}$$

$$\|\gamma_t^{-1}\|_{m,p} \le c_4 t^{-2H}. \tag{27}$$

Plugging relations (25)-(27) into (24), this will conclude the proof of Theorem 1.5.

We start with the estimate (26). Let us first briefly review a strategy by Inahama [25], which allows to obtain a suitable characterization of the Hilbert–Schmidt norm $\|\mathbf{D}^m X_t^x\|_{\mathcal{H}^{\otimes m}}$. For simplicity, we assume $V_0 = 0$, and only work out the cases $m = 1, 2$. The general case is treated similarly.

Recall $\mathbf{J}$ is the Jacobian process. Let $\hat{B} = (\hat{B}_1, \dots, \hat{B}_d)$ be an independent copy of B and consider the $2d$-dimensional fractional Brownian motion $(B, \hat{B})$. The expectations with respect to B and $\hat{B}$ are, respectively, denoted by $\mathbb{E}$ and $\hat{\mathbb{E}}$. Set

$$\Xi_1(t) = \mathbf{J}_t \int_0^t \mathbf{J}_s^{-1} V(X_s^x)\, d\hat{B}_s,$$

and

$$\Xi_2(t) = \mathbf{J}_t \int_0^t \mathbf{J}_s^{-1} \{D^2 V(X_s^x)\langle \Xi_1(s), \Xi_1(s), dB_s\rangle + 2DV(X_t^x)\langle \Xi_1(s), d\hat{B}_s\rangle\}.$$

Here for $x \in \mathbb{R}^n$ and each $i = 1, \dots, d$, $D^2 V_i(x)$ is considered as a bilinear form on $\mathbb{R}^n \times \mathbb{R}^n$ taking values in $\mathbb{R}^n$. Thus, $D^2 V(X_s^x)\langle \Xi_1(s), \Xi_1(s), dB_s\rangle$ is understood as the real-valued quantity $D^2 V_i(X_s^x)(\Xi_1(s), \Xi_1(s))\, dB_s^i$. We refer to [25] for more details regarding this notation. This construction can be generalized to higher orders derivatives, but we stick to the case $m = 1, 2$ as mentioned above. Now we fix a sample path w of B. It is clear that $\Xi_m(w, \cdot)_t$ belongs to the inhomogeneous Wiener chaos of order m with respect to $(B, \hat{B})$ for $m = 1, 2$. Moreover, if we denote by $\hat{\mathbf{D}}$ the derivative with respect to $\hat{B}$, we have

$$\hat{\mathbf{D}}_h \Xi_1(w, \hat{B})_t = \mathbf{D}_h X_t^x(w) \quad \text{and} \quad \hat{\mathbf{D}}_{h,k} \Xi_2(w, \hat{B})_t = 2\mathbf{D}_{h,k}^2 X_t^x(w).$$

By the fact that all $\mathbb{D}^{k,p}$-norms ($k = 0, 1, \dots$) are equivalent on fixed inhomogeneous Wiener chaos, we obtain

$$\|\mathbf{D}X_t(w)\|_{\mathcal{H}} = \hat{\mathbb{E}}(\|\hat{\mathbf{D}}\Xi_1(w, \cdot)_t\|_{\mathcal{H}}^2)^{1/2} \lesssim \|\Xi_1(w, \cdot)_t\|_{1,2} \lesssim \|\Xi_1(w, \cdot)_t\|_2,$$

$$\begin{aligned}\|\mathbf{D}^2 X_t(w)\|_{\mathcal{H}\otimes\mathcal{H}} &= 2\hat{\mathbb{E}}(\|\hat{\mathbf{D}}^2 \Xi_2(w, \cdot)_t\|_{\mathcal{H}\otimes\mathcal{H}}^2)^{1/2} \lesssim 2\|\Xi_2(w, \cdot)_t\|_{2,2} \\ &\lesssim \|\Xi_2(w, \cdot)_t\|_2.\end{aligned}$$

This way, one conveniently transforms estimate of the Hilbert–Schmidt norm of $\mathbf{D}^m X_t^x$ into that of the L^2 norm of $\Xi_m(\omega, \cdot)$. In particular, for $m = 1, 2$ we have

$$\begin{aligned}\|\mathbf{D}X_t^x\|_{\mathcal{H}} &\lesssim \hat{\mathbb{E}}(|\Xi_1(t)|^2)^{1/2}, \\ \|\mathbf{D}^2 X_t^x\|_{\mathcal{H}\otimes\mathcal{H}} &\lesssim \hat{\mathbb{E}}(|\Xi_2(t)|^2)^{1/2}.\end{aligned}$$

LEMMA 4.1. *Let $H > \frac{1}{4}$. Denote by X_t^x the solution to equation* (19). *One has*

$$\|\mathbf{D}X_t^x\|_{m,p} \le c_{m,p} t^H,$$

for some constant $c_{m,p} > 0$.

PROOF. Keep the notation as above. For $m = 1, 2$ we only need to estimate Ξ_1 and Ξ_2 by using rough paths theory. Let

$$M = (B, \hat{B}, X^x, \mathbf{J}, \mathbf{J}^{-1}). \tag{28}$$

This is a p-rough path, $p > 1/H$. The integral $\int \mathbf{J}_s^{-1} V(X_s^x)\, d\hat{B}_s$ is a rough integral of the type $\int f(M)\, d\mathbf{M}$, where f has a polynomial growth. We deduce that there exists a strictly positive r such that

$$|\Xi_1(t) - \Xi_1(s)| \le C(1 + \|\mathbf{M}\|_{p\text{-var},[0,1]})^r \|\mathbf{M}\|_{p\text{-var},[s,t]}. \tag{29}$$

We now estimate $\|\mathbf{M}\|_{p\text{-var},[s,t]}$. Denote by $D(t)$ a subdivision of the interval $[0, t]$. Define

$$\mathcal{M}_{\alpha,t,p} = \sup_{D(t)=(t_i);\, \|\mathbf{B}\|_{p\text{-var},[t_i,t_{i+1}]}^p \le \alpha} \sum_{i: t_i \in D(t)} \|\mathbf{B}\|_{p\text{-var},[t_i,t_{i+1}]}^p.$$

Then the Jacobian $\mathbf{J}$ satisfies the following growth-bound:

$$\|\mathbf{J}\|_{p\text{-var};[0,t]} + \|\mathbf{J}^{-1}\|_{p\text{-var};[0,t]} \le C\|\mathbf{B}\|_{p\text{-var},[0,t]} \exp(C\mathcal{M}_{\alpha,t,p}).$$

In addition (cf., Proposition 4.11 in [15]), $\mathbb{E}[\exp(C\mathcal{M}_{\alpha,t,p})]$ is finite for any $C > 0$. Invoking our tail estimate (20) and plugging those bounds into (29), we end up with a bound for $\|\mathbf{J}\|_{p\text{-var};[0,t]}$ of the form

$$|\Xi_1(t)| \le C(1 + \|\mathbf{M}\|_{p\text{-var},[0,1]})^r (\|\mathbf{B}\|_{p\text{-var},[0,t]} + \|\hat{\mathbf{B}}\|_{p\text{-var},[0,t]}) \exp(C\mathcal{M}_{\alpha,t,p}).$$

By scaling, we have $\|\mathbf{B}\|_{p\text{-var},[0,t]} + \|\hat{\mathbf{B}}\|_{p\text{-var},[0,t]} \overset{\text{law}}{=} t^H(\|\mathbf{B}\|_{p\text{-var},[0,1]} + \|\hat{\mathbf{B}}\|_{p\text{-var},[0,1]})$. The proof is thus completed for the case $m = 1$. In the same way, we estimate Ξ_2 as a rough integral of the type $\int \phi(M_1)\, d\mathbf{M}_1$ where ϕ has polynomial growth and M_1 is the rough path

$$M_1 = (B, \hat{B}, X^x, \mathbf{J}, \mathbf{J}^{-1}, \Xi_1).$$

Arguing as before and using previous estimates we obtain then a bound of the same type:

$$|\Xi_2(t)| \le C(1 + \|\mathbf{M}\|_{p\text{-var},[0,1]})^r (\|\mathbf{B}\|_{p\text{-var},[0,t]} + \|\hat{\mathbf{B}}\|_{p\text{-var},[0,t]}) \exp(C\mathcal{M}_{\alpha,t,p}).$$

Higher order Malliavin derivatives are treated similarly. □

4.1. *The regular case*. In this section, we treat the case where B is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. In this situation, the stochastic integral in (19) can be seen as a Young integral instead of the general rough paths type integral. Moreover, the proof of our upper bound can be summarized as follows.

PROOF OF THEOREM 1.5 IN THE REGULAR CASE. Recall that under the elliptic Hypothesis 1.3 and assuming $H > 1/2$ we wish to show that

$$p_t(y) \le c_2 t^{-nH} \exp\left(-\frac{|y-x|^2}{c_1 t^{2H}}\right) \qquad \text{for all } y \in \mathbb{R}^n. \tag{30}$$

The proof of (26) is treated in a uniform way for both the regular and irregular cases in Lemma 4.1. Hence, let us concentrate here on the proof of (27) for $0 < t \le 1$. Let

$$C_t = \int_0^t \int_0^t \mathbf{J}_u^{-1} V(X_u^x) V(X_u^x)^* (\mathbf{J}_u^{-1})^* |u-v|^{2H-2}\, du\, dv.$$

Our bound (27) is now reduced to prove that

$$y^* C_t^{-1} y \le M t^{-2H} |y|^2 \qquad \text{for } y \in \mathbb{R}^n, \tag{31}$$

for a given random variable M admitting moments of any order. To this aim, notice first that

$$y^* C_t y = \int_0^t \int_0^t \langle f_u, f_v \rangle_{\mathbb{R}^n} |u-v|^{2H-2} \, du \, dv \qquad \text{with } f_u \equiv V(X_u^x)^* (\mathbf{J}_u^{-1})^* y.$$

Furthermore, thanks to the interpolation inequality of Proposition 2.4 applied with $\gamma > H - \frac{1}{2}$, we have

$$\int_0^1 \int_0^1 \langle f_u, f_v \rangle |u-v|^{2H-2} \, du \, dv \geq c t^{2H} \frac{\min_{[0,1]} |f|^4}{\|f\|_\infty^2 + \|f\|_\gamma^2}, \tag{32}$$

where $\|f\|_\gamma$ is the γ-Hölder norm of f on the interval $[0, 1]$ as defined at (10). Furthermore, since the uniform ellipticity condition $|V(x)y|^2 \geq \lambda |y|^2$ holds true, it is readily checked that

$$\begin{aligned} |f_v|^2 \geq \lambda |\mathbf{J}_v^{-1} y|^2 \geq \lambda \|\mathbf{J}_v\|^{-2} |y|^2 \quad \text{and} \\ \|f\|_\infty + \|f\|_\gamma \leq c(1 + \|X\|_\gamma)(1 + \|\mathbf{J}^{-1}\|_\gamma)|y|. \end{aligned} \tag{33}$$

Plugging these relations into (32), we deduce that, for every $y \in \mathbb{R}^n$,

$$y^* C_t^{-1} y \leq c t^{-2H} (1 + \|X\|_\gamma)^2 (1 + \|\mathbf{J}^{-1}\|_\gamma)^2 (1 + \|\mathbf{J}\|_\gamma)^4 |y|^2,$$

from which (31), and thus (27), are easily deduced.

For the bound of Malliavin derivatives of γ_t^{-1}, note that we have

$$\mathbf{D}(\gamma_t^{-1})^{ij} = - \sum_{k,l=1}^d (\gamma_t^{-1})^{ik} (\gamma_t^{-1})^{lj} \mathbf{D} \gamma_t^{kl}. \tag{34}$$

Therefore,

$$\|\mathbf{D}(\gamma_t^{-1})^{ij}\|_{\mathcal{H}} \leq |(\gamma_t^{-1})^{ik} (\gamma_t^{-1})^{lj}| (\|\mathbf{D} X_t\|_{\mathcal{H}} + \|\mathbf{D}^2 X_t\|_{\mathcal{H}^{\otimes 2}})^2.$$

Together with the estimates for $\|\mathbf{D} X_t\|_{m,p}$ and $\|\gamma_t^{-1}\|_p$ that have been established above, we have

$$\|\gamma_t^{-1}\|_{1,p} \leq c t^{-2H}.$$

Similarly, by using equation (34) repeatedly, we conclude that for each $m \in \mathbb{N}$ and $p > 1$ there exists a constant $c_{m,p}$ such that

$$\|\gamma_t^{-1}\|_{m,p} \leq c_{m,p} t^{-2H}. \qquad \square$$

4.2. *The irregular case.* The aim of this section is to extend the results of the last section to the case where B is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$. For this, tools of rough paths theory are required to obtain the sub-Gaussian bound (4).

From the discussion above it is clear that, in order to conclude the correct asymptotic behavior (as $t \downarrow 0$) in the upper bound for the density function, we need to establish (26) and (27) for the irregular case. We have already proved (26) in both the regular and irregular cases.

The counterpart of (27) in the rough case is the content of the following lemma.

LEMMA 4.2. *Let $\frac{1}{4} < H < \frac{1}{2}$. Denote by X_t^x the solution to equation* (19), *and γ_t the Malliavin matrix of X_t^x. Under Hypothesis* 1.3, *there exists a constant $c_{m,p} > 0$ such that for all $t \in (0, 1]$ one has*

$$\|\gamma_t^{-1}\|_{m,p} \leq c_{m,p} t^{-2H}.$$

PROOF. We first prove the lemma for $m = 0$. As before the bound, we want to prove is reduced to prove that

$$y^* C_t^{-1} y \leq M t^{-2H} |y|^2 \qquad \text{for } y \in \mathbb{R}^n, \tag{35}$$

for a given random variable M admitting moments of any order, where, again, C is the reduced Malliavin matrix defined by

$$y^* C_t y = \|f\|_{\mathcal{H}}^2 \qquad \text{with } f_u \equiv \mathbf{1}_{[0,t]}(u) V(X_u^x)^* (\mathbf{J}_u^{-1})^* y.$$

From the inequality of Proposition 2.4 and the uniform ellipticity assumption, we have thus,

$$y^* C_t^{-1} y \leq c t^{-2H} (1 + \|\mathbf{J}\|_\gamma)^2 |y|^2.$$

This yields the claimed result when $m = 0$.

For $m \geq 1$, note that by Lemma 4.1 and what we have just proved, there exist constants $c_{m,p}$ and c_p such that $\|\mathbf{D} X_t^x\|_{m,p} \leq c_{m,p} t^H$ and $\|\gamma_t^{-1}\|_p \leq c_p t^{-2H}$. Putting this together with relation (34) and along the same lines as in the smooth case, we can conclude that there exists a constant $c_{m,p}$ such that $\|\mathbf{D}\gamma_t^{-1}\|_{m,p} \leq c_{m,p} t^{-2H}$, for all $m \in \mathbb{N}$ and $p > 1$. □

We can now prove our sub-Gaussian upper bound for the density $p_t(\cdot)$ of X_t^x in the rough case:

PROOF OF THEOREM 1.5 IN THE IRREGULAR CASE. Owing to inequality (20), we have

$$\mathbb{P}(|X_t^x - x| \geq |y - x|) \lesssim \exp\left(-\frac{2|y - x|^{2H+1}}{c t^{2H}}\right).$$

Now the proof follows from (24), and Lemmas 4.1 and 4.2 just as in the smooth case. □

REMARK 4.3. In order to prove Theorem 1.5, we could also have used the new expression for the density of a nondegenerate random vector obtained recently by Bally and Caramellino in [1]. This expression involves the Poisson kernel, and only requires the random vector to be twice differentiable in the Malliavin sense, in comparison with Proposition 16 where higher derivatives are involved. However, we have not included the details of this strategy here, since it yields some slightly nonoptimal coefficients in relation (4).

5. Hitting probabilities and capacities. We now turn to the evaluation of hitting probabilities for our differential system (2), that is the proof of relation (9) in Theorem 1.6. It should be noticed that the upper and lower bounds in those relations require a different methodology, and this is why they shall be studied in two separate sections.

5.1. *Lower bounds on hitting probabilities.* As established in [16], Theorem 2.1, the lower bound in (9) can be derived from a general result for the hitting probabilities of a continuous stochastic process in terms of its finite-dimensional density functions. We shall prove this general relation in our fBm context for the sake of clarity.

Specifically, suppose that $(u_t, t \geq 0)$ is a continuous stochastic process in $\mathbb{R}^n$, such that the random vector (u_t, u_s) has a joint probability density function $p_{s,t}(\cdot, \cdot)$, for all $s, t > 0$ such that $s \neq t$. As in the previous sections, we will also denote by $p_t(\cdot)$ the density of u_t, for all $t > 0$. We work under the following set of hypotheses:

(A1) For all $0 < a < b$ and $M > 0$, there exists a positive constant $C = C(a, b, M, n)$ such that for all $z \in [-M, M]^n$,

$$\int_a^b p_t(z)\, dt \geq C.$$

(A2) There exist $\beta > 0$, $H \in (0, 1)$ and $p > \beta$ such that for all $0 < a < b$, $M > 0$, one can find a constant $c = c(a, b, \beta, H, M, n, p) > 0$ such that for all $s, t \in [a, b]$ with $s \neq t$, and for every $z_1, z_2 \in [-M, M]^n$,

$$p_{s,t}(z_1, z_2) \leq \frac{c}{|t-s|^{H\beta}} \left(\frac{|t-s|^H}{|x-y|} \wedge 1 \right)^p.$$

With these assumptions in hand, our general result on hitting probabilities is the following.

THEOREM 5.1. *Suppose* (A1) *and* (A2) *are met, and fix* $0 < a < b$ *and* $M > 0$. *Then there exists a strictly positive constant* $c = c(a, b, \beta, H, M, n)$ *such that for all compact sets* $A \subseteq [-M, M]^n$,

$$\mathbb{P}\big(u([a, b]) \cap A \neq \varnothing\big) \geq c \operatorname{Cap}_\alpha(A), \tag{36}$$

where $\alpha = \beta - \frac{1}{H}$.

PROOF. We start by proving a technical lemma that gives the relationship between the upper bound in (A2) and the Newtonian kernel K_α defined by (6).

LEMMA 5.2. *Let $N > 0$, $\beta > 0$, $p > \beta$, $0 \le a < b$, and $H \in (0,1)$ be fixed. Then there exists a positive constant $C = C(a,b,\beta,N,H,p)$ such that for all $r \in [0,N]$*

$$\int_a^b \int_a^b \frac{1}{|t-s|^{H\beta}} \left(\frac{|t-s|^H}{r} \wedge 1 \right)^p ds\, dt \le C\mathrm{K}_\alpha(r), \tag{37}$$

where $\alpha = \beta - \frac{1}{H}$.

PROOF. Fix $r \in [0,N]$ and use the change of variables $u = t - s$, to see that

$$\int_a^b \int_a^b \frac{1}{|t-s|^{H\beta}} \left(\frac{|t-s|^H}{r} \wedge 1 \right)^p ds\, dt \le 2(b-a) \int_0^{b-a} u^{-H\beta} \left(\frac{u^H}{r} \wedge 1 \right)^p du.$$

Next, the change of variables $v = \frac{u^H}{r}$ implies that the right-hand side equals

$$Cr^{-\alpha} F(m) \qquad \text{where } F(m) := \int_0^m v^{-\beta-1+(1/H)} (v \wedge 1)^p\, dv,$$

with the notation $m := \frac{(b-a)^H}{r}$. Observe that $m \ge m_1 := \frac{(b-a)^H}{N} > 0$. Hence, we can split $F(m)$ into $F(m) = F(m_1) + [F(m) - F(m_1)]$. Now clearly we have $F(m_1) \le c$, and if $\beta \ne \frac{1}{H}$, then

$$F(m) - F(m_1) \le \frac{m^{1/H-\beta} - m_1^{1/H-\beta}}{1/H - \beta}.$$

Hence, if $\beta > \frac{1}{H}$, we get $F(m) - F(m_1) \le c$; if $\beta < \frac{1}{H}$, then $F(m) - F(m_1) \le Cr^{\beta-(1/H)}$; and if $\beta = \frac{1}{H}$, some similar elementary computations show that

$$F(m) - F(m_1) \le C \log(m) = c + c' \log\left(\frac{1}{r}\right).$$

Therefore, putting together these considerations we conclude the proof of relation (37), provided that the constant N_0 in (6) is sufficiently large. □

Let us now go back to the proof of Theorem 5.1: fix a compact set $A \subseteq [-M,M]^n$ and observe that whenever $\mathrm{Cap}_\alpha(A) = 0$, inequality (36) is trivially satisfied. In the remainder of the proof, we thus assume $\mathrm{Cap}_\alpha(A) > 0$. In particular, this implies that $A \ne \varnothing$. We now consider three different cases:

Case 1. $\beta < \frac{1}{H}$. Then $\alpha < 0$, and thus $\mathrm{Cap}_\alpha(A) = 1$. Therefore, it suffices to prove that there exists a positive constant $c = c(a,b,M,H,\beta,n)$ such that

$$\mathbb{P}\big(u([a,b]) \cap A \ne \varnothing\big) \ge c. \tag{38}$$

Toward this aim, for all $\epsilon \in (0,1)$ and $z \in \mathbb{R}^n$, consider the random variable

$$J_\epsilon(z) = \frac{1}{(2\epsilon)^n} \int_a^b \mathbf{1}_{\tilde{B}(z,\epsilon)}(u_t)\, dt,$$

where $\tilde{B}(z,\epsilon) = \{y \in \mathbb{R}^n : |z-y| < \epsilon\}$ and $|z| = \max_{1\le i\le n} |z_i|$. Assume now that $z \in A$. Our first aim is to prove that $\mathbb{P}(J_\epsilon(z) > 0) \ge C$, for a strictly positive constant C independent of ϵ. Indeed, hypothesis (A1) implies that there exists a positive constant $C(a,b,M,H,n)$ such that for all $\epsilon \in (0,1)$,

$$\mathbb{E}\big[J_\epsilon(z)\big] = \frac{1}{(2\epsilon)^n} \int_a^b \int_{\tilde{B}(z,\epsilon)} p_t(v)\, dv\, dt \ge C.$$

On the other hand, hypothesis (A2) and Lemma 5.2 imply that there exists a positive constant $C(a,b,M,H,\beta,n)$ such that for all $\epsilon \in (0,1)$,

$$\begin{aligned}
\mathbb{E}\big[J_\epsilon^2(z)\big] &= \frac{1}{(2\epsilon)^{2n}} \int_a^b \int_a^b \int_{\tilde{B}(z,\epsilon)} \int_{\tilde{B}(z,\epsilon)} p_{s,t}(z_1,z_2)\, dz_1\, dz_2\, ds\, dt \\
&\le \frac{c}{(2\epsilon)^{2n}} \int_{\tilde{B}(z,\epsilon)} \int_{\tilde{B}(z,\epsilon)} \mathrm{K}_\alpha(z_2 - z_1)\, dz_1\, dz_2 \le c,
\end{aligned}$$

where the last inequality is due to the fact that $\mathrm{K}_\alpha \equiv 1$ whenever $\alpha < 0$. Therefore, from the Paley–Zygmund inequality [cf., [16], inequality (2.26)], we conclude that

$$\mathbb{P}\big(J_\epsilon(z) > 0\big) \ge \frac{\mathbb{E}[J_\epsilon(z)]^2}{\mathbb{E}[J_\epsilon^2(z)]} \ge C, \tag{39}$$

where C is independent of ϵ. Moreover, the left-hand side of (39) is bounded above by $\mathbb{P}(u([a,b]) \cap A_\epsilon \neq \varnothing)$, where A_ϵ denotes the closed ϵ-enlargement of A. Then we let $\epsilon \downarrow 0$ and use the continuity of the trajectories of u to conclude that (38) holds true.

Case 2. $\beta > \frac{1}{H}$. For all $\epsilon \in (0,1)$ and $\mu \in \mathcal{P}(A)$, consider the random variable

$$J_\epsilon(\mu) = \frac{1}{(2\epsilon)^n} \int_{\mathbb{R}^n} \int_a^b \mathbf{1}_{\tilde{B}(z,\epsilon)}(u_t)\, dt\, \mu(dz).$$

Then (A1) implies the existence of a positive constant $C(a,b,M,H,n)$ such that

$$\mathbb{E}\big[J_\epsilon(\mu)\big] \ge C.$$

In order to estimate the second moment of $J_\epsilon(\mu)$, we consider the function

$$g_\epsilon(z) = (2\epsilon)^{-n} \mathbf{1}_{\tilde{B}(0,\epsilon)}(z),$$

so that we can write

$$J_\epsilon(\mu) = \int_a^b [g_\epsilon * \mu](u_t)\, dt.$$

It is readily checked that

$$\mathbb{E}\big[J_\epsilon^2(\mu)\big] = \int_{\mathbb{R}^n\times\mathbb{R}^n} [g_\epsilon * \mu](z_1)[g_\epsilon * \mu](z_2)\bigg(\int_{[a,b]^2} p_{s,t}(z_1,z_2)\,ds\,dt\bigg)\,dz_1\,dz_2,$$

and thus, owing to hypothesis (A2) and Lemma 5.2 we obtain that there exists a positive constant $c = c(a,b,M,H,\beta,n)$ such that

$$\mathbb{E}\big[J_\epsilon^2(\mu)\big] \le c\mathcal{E}_\alpha(g_\epsilon * \mu),$$

where we recall that the energy functional $\mathcal{E}_\alpha$ has been defined by relation (5). We now choose $\mu \in \mathcal{P}(A)$ such that $\mathcal{E}_\alpha(\mu) \le \frac{2}{\mathrm{Cap}_\alpha(A)}$. We also recall that, thanks to the general result [16], Theorem B.1, we have $\mathcal{E}_\alpha(g_\epsilon * \mu) \le \mathcal{E}_\alpha(\mu)$ for all $\epsilon \in (0,1)$. We thus obtain that

$$\mathbb{E}\big[J_\epsilon^2(\mu)\big] \le \frac{2c}{\mathrm{Cap}_\alpha(A)}.$$

Therefore, from the Cauchy–Schwarz inequality, we conclude that

$$\mathbb{P}\big[J_\epsilon(\mu) > 0\big] \ge \frac{\mathbb{E}[J_\epsilon(\mu)]^2}{\mathbb{E}[J_\epsilon^2(\mu)]} \ge \frac{c}{\mathrm{Cap}_\alpha(A)}, \tag{40}$$

where the positive constant c is independent of μ. As for the first case, the left-hand side of (40) is upper bounded by $\mathbb{P}(u([a,b]) \cap A_\epsilon \ne \emptyset)$, where A_ϵ denotes the closed ϵ-enlargement of A. Then we let $\epsilon \downarrow 0$ and use the continuity of the trajectories of u to assert that (36) holds true in our case 2.

Case 3. $\beta = \frac{1}{H}$. This case follows exactly along the same lines as Case 2, except for the fact that we appeal to [16], Theorem B.2, instead of [16], Theorem B.1. □

From the definition of capacity and as a consequence of Theorem 5.1, we have the following result on hitting points for the process u.

COROLLARY 5.3. *Under the hypotheses of Theorem* 5.10, *if* $\beta < \frac{1}{H}$, *the process u hits points in $\mathbb{R}^n$ with strictly positive probability*, *that is*,

$$\mathbb{P}(\exists t > 0 : u_t = x) > 0 \qquad \textit{for all } x \in \mathbb{R}^n.$$

PROOF. Observe that we have $\alpha < 0$ whenever $\beta < \frac{1}{H}$. Thus, in this case, $\mathrm{Cap}_\alpha(\{x\}) = 1$ for any $x \in \mathbb{R}^n$. On the other hand, we write $(0,\infty) = \bigcup_{m\in\mathbb{N}}[\frac{1}{m}, m]$. Then by Theorem 5.1, for all $m \ge 1$, there is $c > 0$ depending on m such that

$$\mathbb{P}\bigg(\exists t \in \bigg[\frac{1}{m}, m\bigg] : u_t = x\bigg) \ge c\,\mathrm{Cap}_\alpha(\{x\}) = c > 0.$$

Since this holds for all m, the desired result holds. □

5.2. *Bivariate density bound.* We will now apply the general result of Theorem 5.1 to the n-dimensional process solution to equation (19). In order to achieve this goal, the main remaining technical difficulty consists in proving the upper bound for the bivariate density stated at condition (A2). In this case, our strategy hinges on conditional integration by parts in the Malliavin calculus sense, which turns out to be much easier to express in terms of the underlying Wiener process W induced by the Volterra representation (11). This idea is also present in [11], and it forces us to introduce some additional notation.

We shall manipulate Malliavin derivatives with respect to both B and W. In order to distinguish them, the Malliavin derivatives with respect to W will be denoted by D and the Sobolev spaces by $D^{k,p}$. The relationship between the two kinds of derivatives are recalled in the following.

PROPOSITION 5.4. *Let $D^{1,2}$ be the Malliavin–Sobolev space corresponding to the Wiener process W. Then $\mathbb{D}^{1,2} = (K^*)^{-1} D^{1,2}$ and for any $F \in D^{1,2}$ we have $DF = K^* \mathbf{D} F$ whenever both members of the relation are well defined.*

In particular, we can compute the Malliavin derivative of $(X_t^x)_{t\geq 0}$ with respect to W as follows.

PROPOSITION 5.5. *Let X^x be the solution to equation (19) and suppose the V_i's satisfy Hypothesis 1.1. Then for every $i = 1, \dots, n$, $t > 0$, and $x \in \mathbb{R}^n$, we have $X_t^{x,i} \in D^\infty$ and*

$$D_s^j X_t^x = \mathbf{J}_t Q_{st}^j, \qquad j = 1, \dots, d, 0 \leq s \leq t,$$

where $D_s^j X_t^{x,i}$ is the jth component of $D_s X_t^{x,i}$, $\mathbf{J}_t = \partial_x X_t^x$ is defined at Proposition 2.11, and

$$(41) \qquad Q_{st}^j = \begin{cases} \displaystyle\int_s^t \partial_u K(u,s) \mathbf{J}_u^{-1} V_j(X_u)\, du, & H > 1/2, \\ K(t,s)\mathbf{J}_s^{-1} V_j(X_s) + \int_s^t (\mathbf{J}_r^{-1} V_j(X_r) - \mathbf{J}_s^{-1} V_j(X_s)) \partial_r K(r,s)\, dr, & \\ & H \leq 1/2. \end{cases}$$

Recall that we have chosen to express our conditional integration by parts formula in terms of the underlying Wiener process W, because projections on subspaces are easier to describe in an L^2 type setting. We now state this conditional integration by parts formula: for a random variable F and $t \in [0,1]$, let $\|F\|_{m,p,t}$ and $\Gamma_{F,t}$ be the quantities defined (for $m \geq 0$, $p > 0$) by

$$(42) \qquad \begin{aligned} \|F\|_{m,p,t} &= \left(\mathbb{E}_t[F^p] + \sum_{j=1}^m \mathbb{E}_t[\|D^j F\|^p_{(L_t^2)^{\otimes j}}] \right)^{1/p} \quad \text{and} \\ \Gamma_{F,t} &= (\langle DF^i, DF^j \rangle_{L_t^2})_{1 \leq i,j \leq n}, \end{aligned}$$

where we have set $L^2_t \equiv L^2([t,1])$ and $\mathbb{E}_t = \mathbb{E}(\cdot|\mathcal{F}_t)$. With this notation in hand, the following formula is borrowed from [29], Proposition 2.1.4.

PROPOSITION 5.6. *Fix $k \geq 1$. Let $F, Z_s, G \in (D^\infty)^n$ be three random vectors where $Z_s \in \mathcal{F}_s$-measurable and $(\det_{\Gamma_{F+Z_s}})^{-1}$ has finite moments of all orders. Let $g \in \mathcal{C}_p^\infty(\mathbb{R}^d)$. Then, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \{1, \dots, n\}^k$, there exists a r.v. $H^s_\alpha(F,G) \in \bigcap_{p\geq 1}\bigcap_{m\geq 0} D^{m,p}$ such that*

$$\mathbb{E}_s\big[(\partial_\alpha g)(F+Z_s)G\big] = \mathbb{E}_s\big[g(F+Z_s)H^s_\alpha(F,G)\big], \tag{43}$$

where $H^s_\alpha(F,G)$ is recursively defined by

$$H^s_{(i)}(F,G) = \sum_{j=1}^n \delta_s\big(G\big(\Gamma^{-1}_{F,s}\big)_{ij} DF^j\big),$$
$$H^s_\alpha(F,G) = H^s_{(\alpha_k)}\big(F, H^s_{(\alpha_1,\dots,\alpha_{k-1})}(F,G)\big).$$

Here, δ_s denotes the Skorohod integral with respect to the Wiener process W on the interval $[s,1]$. Furthermore, the following norm estimates hold true:

$$\big\|H^s_\alpha(F,G)\big\|_{p,s} \leq c_{p,q}\big\|\Gamma^{-1}_{F,s}DF\big\|^k_{k,2^{k-1}r,s}\|G\|^k_{k,q,s}, \tag{44}$$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

In order to get our bivariate density bound, we shall also need to work on weighted norms on the interval $[s,t]$. For instance, when $H > 1/2$, we have the following uniform scale invariant inequalities.

LEMMA 5.7. *Assume $H > 1/2$. Let $0 < \varepsilon < 1$ and $\gamma > H - \frac{1}{2}$. There exist two constants $C_1, C_2 > 0$ such that for any continuous $f : [0,1] \to \mathbb{R}^n$, and $\varepsilon \leq s < t \leq 1$, we have*

$$C_1(t-s)^{2H}\frac{\min_{[0,1]}|f|^4}{\|f\|^2_\infty + \|f\|^2_\gamma} \leq \int_s^t \left|\int_u^t \partial_v K(v,u) f(v)\,dv\right|^2 du \tag{45}$$

PROOF. For the sake of notation, we prove our lemma for real valued functions only, leaving the obvious extension to $f : [0,1] \to \mathbb{R}^n$ to the patient reader. We now proceed in several steps.

Step 1. We first prove that

$$\alpha \int_s^t \left(\int_u^t (v-u)^{H-3/2} f(v)\,dv\right)^2 du \leq \int_s^t \left(\int_u^t \partial_v K(v,u) f(v)\,dv\right)^2 du. \tag{46}$$

Using the change of variable $u = s + sx$ and $v = s + sy$, and the scaling property of the kernel K, we just need to prove that for $t \le T$,

$$\alpha \int_0^t \left(\int_u^t (v-u)^{H-3/2} f(v)\, dv \right)^2 du \\ \le \int_0^t \left(\int_u^t \partial_v K(v+1, u+1) f(v)\, dv \right)^2 du.$$

Up to a constant, the norm $\int_0^t (\int_u^t (v-u)^{H-3/2} f(v)\, dv)^2\, du$ is the norm of the reproducing Hilbert space of the Gaussian process

$$Y_t = d_H \int_0^t (t-s)^{H-1/2}\, dW_s, \tag{47}$$

where $d_H(H - 1/2) = c_H$ and the norm $\int_0^t (\int_u^t \partial_v K(v+1, u+1) f(v)\, dv)^2\, du$ is the norm of the reproducing Hilbert space of the Gaussian process

$$Z_t = \int_0^t K(t+1, s+1)\, dW_s.$$

So, to prove (46), according to Lemma 2 in [6], we just need to prove that $(Y_t)_{0 \le t \le T}$ and $(Z_t)_{0 \le t \le T}$ are equivalent in distribution. From Theorem 7 in [6], we have to prove that there exists a square integrable kernel L such that

$$K(t+1, s+1) = d_H (t-s)^{H-1/2} + d_H \int_s^t (t-r)^{H-1/2} L(r, s)\, dr.$$

Since $H > 1/2$, we can differentiate both members of the above equation with respect to t and we obtain

$$(s+1)^{1/2-H} (t-s)^{H-3/2} (t+1)^{H-1/2} - (t-s)^{H-3/2} \\ = \int_s^t (t-r)^{H-3/2} L(r, s)\, dr.$$

Hence, it suffices to take

$$L(t, s) = \frac{1}{\Gamma(H - (1/2))} D_{s+}^{H-1/2} \left[(\cdot - s)^{H-3/2} \left(\left(\frac{\cdot + 1}{s+1} \right)^{H-1/2} - 1 \right) \right](t),$$

which is easily seen to be square integrable.

Step 2. Thanks to the previous step, our result boils down to show (45) when $\partial_v K(v, u)$ is replaced by $(v-u)^{H-3/2}$. Toward this aim, notice first that by using the same arguments as in [5], Lemma 4.4, we obtain the interpolation inequality

$$\int_0^1 \left(\int_u^1 (v-u)^{H-3/2} f(v)\, dv \right)^2 du \ge C \frac{\min_{[0,1]} f^4}{\|f\|_\infty^2 + \|f\|_\gamma^2},$$

which is easy to rescale:

$$\begin{aligned}
&\int_s^t \left(\int_u^t (v-u)^{H-3/2} f(v)\,dv \right)^2 du \\
&\quad = (t-s)^{2H} \int_0^1 \left(\int_u^1 (v-u)^{H-3/2} f\bigl(s+(t-s)v\bigr)\,dv \right)^2 du \\
&\quad \geq C(t-s)^{2H} \frac{\min_{[0,1]} f_{st}^4}{\|f_{st}\|_\infty^2 + \|f_{st}\|_\gamma^2} \\
&\quad \geq C(t-s)^{2H} \frac{\min_{[0,1]} f^4}{\|f\|_\infty^2 + \|f\|_\gamma^2},
\end{aligned}$$

where we have set $f_{st}(u) = f(s+(t-s)u)$. □

In the case $H \leq 1/2$, the scale invariant inequalities we have are the following.

LEMMA 5.8. *Assume $H \leq 1/2$. Let $0 < \varepsilon < 1$. There exists constants c_1, $c_2 > 0$ such that for any $f \in C^\gamma([0,1]; \mathbb{R}^n)$, with $\gamma > 1/2 - H$ and $\varepsilon \leq s < t \leq 1$, we have*

$$c_1(t-s)^{2H} \min_{[0,1]} |f|^2 \leq \int_s^t \left| K(t,u)f(u) + \int_u^t \bigl(f(r) - f(u)\bigr)\partial_r K(r,u)\,dr \right|^2 du.$$

PROOF. Some elements of the proof are pretty similar to the proof of Lemma 5.7, so we only sketch the main arguments. We also focus here on the case of real valued functions for sake of readability.

Step 1. Set $\hat{L}(t,s) = (t-s)^{H-1/2}$. Along the same line as for Lemma 5.7, it is readily checked that

$$\begin{aligned}
&C \int_s^t \left(\hat{L}(t,u)f(u) + \int_u^t \bigl(f(r) - f(u)\bigr)\partial_r \hat{L}(r,u)\,dr \right)^2 du \\
&\quad \leq \int_s^t \left(K(t,u)f(u) + \int_u^t \bigl(f(r) - f(u)\bigr)\partial_r K(r,u)\,dr \right)^2 du,
\end{aligned}$$

by using a scaling argument and the equivalence in distribution of the two processes $\int_0^t K(t+1,s+1)\,dW_s$ and $d_H \int_0^t (t-s)^{H-1/2}\,dW_s$.

Step 2. Some fractional calculus arguments show that the following lower bound holds true:

$$\begin{aligned}
\int_0^1 \left(\hat{L}(1,u)f(u) + \int_u^1 \bigl(f(r) - f(u)\bigr)\partial_r \hat{L}(r,u)\,dr \right)^2 du &\geq C \int_0^1 f(u)^2\,du \\
&\geq C \min_{[0,1]} f^2,
\end{aligned}$$

which can be rescaled to get the lower bound of our claim. □

As a last preliminary step before the proof of our bivariate density bound, let us mention that we shall express some of our Malliavin derivatives bounds in terms of Hölder norms on the interval $[s,t]$. However, it will be more convenient to work with Besov norms rather than Hölder's because Besov norms are smooth in the Malliavin calculus sense. This is why we introduce the following quantities: if Y is a process which is γ-Hölder, $1/2 < \gamma < H$, set

$$\mathcal{N}^{s,t}_{\gamma,p}(Y) = \int_s^t \int_s^t \frac{|Y_v - Y_u|^{2p}}{|v-u|^{2\gamma p+2}} \, du \, dv,$$

where $\gamma < H$ and $p > 0$. Then from the Besov–Hölder embedding, we have

$$\|Y\|_{s,t,\gamma} \le C\big(\mathcal{N}^{s,t}_{\gamma,p}(Y)\big)^{1/2p}, \qquad 0 \le s \le t \le 1.$$

From the Garsia–Rodemich–Rumsey inequality in Carnot groups (see [20]) or Theorem 7.34 in [3], this embedding extends to the rough paths case. More precisely, if Y is a γ-rough path with lift $\mathbf{Y}$, then

$$\|\mathbf{Y}\|_{s,t,\gamma} \le C\big(\mathcal{N}^{s,t}_{\gamma,p}(\mathbf{Y})\big)^{1/2p}, \qquad 0 \le s \le t \le 1,$$

where now,

$$\mathcal{N}^{s,t}_{\gamma,p}(\mathbf{Y}) = \int_s^t \int_s^t \frac{|\Gamma_{u,v}|^{2p}}{|v-u|^{2\gamma p+2}} \, du \, dv,$$

with

$$\Gamma_{s,t} = \sum_{k=1}^{[1/\gamma]} \left\| \int_{\Delta^k[s,t]} dY^{\otimes k} \right\|^{1/k}.$$

With this notation in mind, using the interpolation inequalities we just proved and arguing as in Section 4 we obtain then the following estimates.

PROPOSITION 5.9. *Let $\varepsilon \in (0,1)$, and consider $H \in (1/4, 1)$. Recall that the Malliavin matrix Γ_F of a random variable F with derivatives taken with respect to the Wiener process W are defined by* (42). *Then there exist constants $C, r > 0$ such that for $\varepsilon \le s \le t \le 1$ the following bounds hold true for $\gamma < H$*:

$$\big\|\Gamma^{-1}_{X^x_t - X^x_s, s}\big\|^n_{n,2^{n+2},s} \le \frac{C}{(t-s)^{2nH}} \mathbb{E}_s^{n/2^{n+2}}\big[\big(1 + \mathcal{N}^{0,1}_{\gamma,p}(\mathbf{M})\big)^r\big],$$

$$\big\|D\big(X^x_t - X^x_s\big)\big\|^n_{n,2^{n+2},s} \le C(t-s)^{nH} \mathbb{E}_s^{n/2^{n+2}}\big[\big(1 + \mathcal{N}^{0,1}_{\gamma,p}(\mathbf{M})\big)^r\big],$$

where

$$M = \big(B, \hat{Y}, X^x, \mathbf{J}, \mathbf{J}^{-1}\big),$$

with $\hat{Y}_t = \int_0^t (t-s)^{H-1/2} \, d\hat{W}_s$ where $\hat{W}$ is a Brownian motion independent from W.

PROOF. Taking into account the interpolation inequalities of Lemmas 5.7 and 5.8, the bound

$$\|\Gamma^{-1}_{X^x_t - X^x_s, s}\|^n_{n, 2^{n+2}, s} \le \frac{C}{(t-s)^{2nH}} \mathbb{E}_s^{n/2^{n+2}}\big[\big(1 + \mathcal{N}^{0,1}_{\gamma,p}(\mathbf{M})\big)^r\big]$$

follows along the same lines as in Section 4. We now turn to the upper bound for the Malliavin derivative. Again, we use the method by Inahama [25]. Set

$$\Theta_1(t) = \mathbf{J}_t \int_s^t K_t^*\big(\mathbf{J}^{-1} V(X)\big)(u)\, d\hat{W}(u),$$

where

$$K_t^*\big(\mathbf{J}^{-1} V(X)\big)(v) = \begin{cases} \displaystyle\int_v^t \partial_u K(u,s) \mathbf{J}_u^{-1} V_j(X_u)\, du, & H > 1/2, \\ \displaystyle K(t,v)\mathbf{J}_v^{-1} V_j(X_v) + \int_v^t \big(\mathbf{J}_r^{-1} V_j(X_r) - \mathbf{J}_s^{-1} V_j(X_s)\big) \partial_r K(r,s)\, dr, & H \le 1/2. \end{cases}$$

As in Inahama [25], we have

$$\|D(X^x_t - X^x_s)\|_{L^2_s} \le C \hat{\mathbb{E}}\big(|\Theta_1(t)|^2\big)^{1/2}.$$

From the previous lemmas, we can estimate

$$\hat{\mathbb{E}}\big(|\Theta_1(t)|^2\big)^{1/2} \le C \hat{\mathbb{E}}\big(|\tilde{\Theta}_1(t)|^2\big)^{1/2},$$

where

$$\tilde{\Theta}_1(t) = \mathbf{J}_t \int_s^t \hat{L}_t^*\big(\mathbf{J} V(X)\big)(u)\, d\hat{W}(u),$$

with, as before, $\hat{L}(t,s) = (t-s)^{H-1/2}$. We can now write $\tilde{\Theta}$ as a rough paths integral,

$$\tilde{\Theta}_1(t) = \mathbf{J}_t \int_s^t \mathbf{J}_u^{-1} V(X_u)\, d\hat{Z}(u),$$

where

$$\hat{Z}(u) = \int_s^u (s-v)^{H-1/2}\, d\hat{W}(v).$$

The advantage of working with the kernel $(s-v)^{H-1/2}$ is that it is translation invariant, so it is easily seen that we have in distribution with respect to $\hat{\mathbb{P}}$ (i.e., W is fixed),

$$\tilde{\Theta}_1(t) = (t-s)^H \mathbf{J}_t \int_0^1 \mathbf{J}^{-1}_{s+(t-s)u} V(X_{s+(t-s)u})\, d\hat{Y}(u),$$

where $\hat{Y}$ is an independent copy of the process Y defined by (47). Using rough paths theory, as in Section 4, we get an upper bound of the form $(1+\mathcal{N}_{\gamma,p}^{0,1}(\mathbf{M}))^r$ for the integral $\int_0^1 \mathbf{J}_{s+(t-s)u}^{-1} V(X_{s+(t-s)u})\,d\hat{Y}(u)$. Thus, we get

$$\begin{aligned}\mathbb{E}_s\big(\|D(X_t^x - X_s^x)\|_{L_s^2}^n\big)^{1/n} &\le C(t-s)^H \mathbb{E}_s\big(\hat{\mathbb{E}}\big((1+\mathcal{N}_{\gamma,p}^{0,1}(\mathbf{M}))^{2r}\big)^{n/2}\big)^{1/n}\\ &\le C(t-s)^H \mathbb{E}_s\big((1+\mathcal{N}_{\gamma,p}^{0,1}(\mathbf{M}))^{rn}\big)^{1/n}.\end{aligned}$$

Higher order derivatives are treated similarly. □

We are finally ready for the proof of condition (A2).

PROOF THAT CONDITION (A2) HOLDS WITH $\beta = n$. In all the proof the range of the parameters s, t will be $\varepsilon < s \le t \le 1$ where $0 < \varepsilon < 1$. Also C will denote a deterministic constant that varies from line to line but which is independent from s, t (however it may depend on other parameters like n, p, V_i, ε).

Consider the joint probability density function of the $2n$-dimensional random vector (X_t^x, X_s^x) with $s < t$ denoted $p_{s,t}(z_1, z_2)$ (the fact that it exists as a smooth function is a consequence of Proposition 5.9). We then write

$$p_{s,t}(z_1, z_2) = \hat{p}_{s,t-s}(z_1, z_2 - z_1) \qquad \text{for } z_1, z_2 \in \mathbb{R}^n,$$

where $\hat{p}_{s,t-s}(\cdot,\cdot)$ denotes the density of the random vector $(X_s^x, X_t^x - X_s^x)$. We now bound the function $\hat{p}_{s,t-s}$, which shall be expressed as

$$\begin{aligned}\hat{p}_{s,t-s}(\xi_1, \xi_2) &= \mathbb{E}\big[\delta_{\xi_1}(X_s^x)\delta_{\xi_2}(X_t^x - X_s^x)\big] \qquad \text{for } \xi_1, \xi_2 \in \mathbb{R}^n,\\ &= \mathbb{E}\big[\delta_{\xi_1}(X_s^x)\mathbb{E}_s\big[\delta_{\xi_2}(X_t^x - X_s^x)\big]\big].\end{aligned}$$

The idea is now to bound $M_{st} = \mathbb{E}_s[\delta_{\xi_2}(X_t^x - X_s^x)]$ by using first the conditional integration by parts formula in Proposition 5.6 and then the Cauchy–Schwarz inequality. We obtain

$$|M_{s,t}| \le C \|\Gamma_{X_t^x - X_s^x, s}^{-1}\|_{n,2^{n+2},s}^n \|D(X_t^x - X_s^x)\|_{n,2^{n+2},s}^n \mathbb{E}_s^{1/2}[\mathbf{1}_{(X_t^x - X_s^x > \xi_2)}]. \tag{48}$$

Thus, owing to Proposition 5.9 we obtain

$$\begin{aligned}&\hat{p}_{s,t-s}(\xi_1,\xi_2)\\ &\quad\le \frac{C}{(t-s)^{nH}} \mathbb{E}\big[\delta_{\xi_1}(X_s^x)\mathbb{E}_s^{n/2^{n+1}}\big[(1+\mathcal{N}_{\gamma,p}^{0,1}(\mathbf{M}))^r\big]\mathbb{E}_s^{1/2}[\mathbf{1}_{(X_t^x - X_s^x > \xi_2)}]\big].\end{aligned} \tag{49}$$

Furthermore, it is readily checked that

$$|X_t^x - X_s^x| \le C|t-s|^\gamma \mathcal{N}_{\gamma,2p}^{1/2p}(\mathbf{M}),$$

and thus, for q arbitrarily large, we have

$$\mathbb{E}_s[\mathbf{1}_{(X_t^x - X_s^x > \xi_2)}] \le C\bigg(1 \wedge \frac{|t-s|^{\gamma q}}{\xi_2^q}\mathbb{E}_s\big[\mathcal{N}_{\gamma,2p}^q(\mathbf{M})\big]\bigg).$$

Plugging this inequality into (49), we end up with

$$\hat{p}_{s,t-s}(\xi_1, \xi_2) \leq \frac{C}{(t-s)^{nH}} \mathbb{E}\left[\delta_{\xi_1}(X_s^x)\Psi_1\left(1 \wedge \frac{|t-s|^{\gamma q}}{\xi_2^q}\Psi_2\right)\right], \tag{50}$$

where Ψ_1 and Ψ_2 are two random variables which are smooth in the Malliavin calculus sense. We can now integrate (50) safely by parts in order to regularize the term $\delta_{\xi_1}(X_s^x)$, which finishes the proof. □

5.3. *Lower bound on hitting probabilities.* We now apply Theorem 5.1, which yields the lower bound of Theorem 1.6.

THEOREM 5.10. *Let X_t^x denote the solution to equation* (19) *where B is a fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$ and where the vector fields $V_1, \dots, V_d$ satisfy Hypothesis* 1.3. *Fix $0 < a < b \leq 1$ and $M > 0$. Then there exists a positive constant $c = c(a, b, H, M, n)$ such that for all compact sets $A \subseteq [-M, M]^n$,*

$$\mathbb{P}\big(X_t^x([a,b]) \cap A \neq \varnothing\big) \geq c\,\mathrm{Cap}_{n-(1/H)}(A).$$

PROOF. Since we have already proved that hypothesis (A2) holds with $\beta = n$, it suffices to verify hypotheses (A1) of Theorem 5.1. First of all, observe that, owing to Theorem 1.4, the density of our process $p_t(y)$ is strictly positive and continuous in y. Moreover, our nondegeneracy conditions on V yield the uniform continuity of the density on $[a, b] \times [-M, M]^n$ for any strictly positive M (see, e.g., [14]). Therefore, it holds that for all $z \in [-M, M]^n$

$$\int_a^b p_t(z)\,dt \geq \inf_{|z|\leq M} \int_a^b p_t(z)\,dt = C(a, b, M) > 0,$$

which proves that (A1) holds true. □

As a consequence of Theorem 5.10 and Corollary 5.3, we have the following result on hitting points for the process X_t^x.

COROLLARY 5.11. *Under the hypotheses of Theorem* 5.10, *if $n < \frac{1}{H}$, the process X_t^x hits points in $\mathbb{R}^n$ with positive probability.*

5.4. *Upper bounds on hitting probabilities.* As in the last subsection, we provide a general result that gives sufficient conditions on a continuous stochastic process in order to obtain an upper bound for the hitting probabilities of the process in terms of the Hausdorff measure. The proof follows along the same lines as in [16], Theorem 3.1, but for the sake of completeness we sketch the main steps.

Given $\alpha \geq 0$, the α-dimensional Hausdorff measure of a set A in $\mathbb{R}^n$ is defined as

$$\mathcal{H}_\alpha(A) = \lim_{\epsilon \to 0^+} \inf\left\{\sum_{i=1}^{\infty}(2r_i)^\alpha : A \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i), \sup_{i\geq 1} r_i \leq \epsilon\right\}, \tag{51}$$

where $B(x, r)$ denotes the open (Euclidean) ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$. When $\alpha < 0$, we define $\mathcal{H}_\alpha(A)$ to be infinite.

Let us now consider a continuous stochastic process $(u_t, t \geq 0)$ in $\mathbb{R}^n$, and for all positive integers N and $H \in (0, 1)$, set $t_k^{N,H} := k2^{-N/H}$, and $I_k^{N,H} = [t_k^{N,H}, t_{k+1}^{N,H}]$.

THEOREM 5.12. *Fix $0 < a < b$, $\beta > 0$, and $M > 0$. Suppose that there exists $H \in (0, 1)$ and $c_H > 0$ such that for all $z \in [-M, M]^n$, $\epsilon > 0$, large N and $I_k^{N,H} \subseteq [a, b]$,*

$$\mathbb{P}\big(u\big(I_k^{N,H}\big) \cap B(z, \epsilon) \neq \varnothing\big) \leq c_H \epsilon^\beta. \tag{52}$$

Then there exists a positive constant $C = C(a, b, \beta, M, H, n)$ such that for all Borel sets $A \subset [-M, M]^n$,

$$\mathbb{P}\big(u\big([a, b]\big) \cap A \neq \varnothing\big) \leq C\mathcal{H}_{\beta-(1/H)}(A).$$

REMARK 5.13. Because of the inequalities between capacity and Hausdorff measure, the right-hand side of Theorem 5.12 can be replaced by $C\,\mathrm{Cap}_{\beta-(1/H)-\epsilon}(A)$ (cf. [26], page 133).

PROOF. When $\beta < \frac{1}{H}$, there is nothing to prove, so we assume that $\beta - \frac{1}{H} > 0$. Fix $\epsilon \in (0, 1)$ and $N \in \mathbb{N}$ such that $2^{-N-1} < \epsilon \leq 2^{-N}$, and write

$$\mathbb{P}\big(u\big([a, b]\big) \cap B(z, \epsilon) \neq \varnothing\big) \leq \sum_{k: I_k^{N,H} \cap [a,b] \neq \varnothing} \mathbb{P}\big(u\big(I_k^{N,H}\big) \cap B(z, \epsilon) \neq \varnothing\big),$$

where the number of k's involved in the sum is at most $2^{N/H}$. Then hypothesis (52) implies that for all large N and $z \in A$,

$$\mathbb{P}\big(u\big([a, b]\big) \cap B(z, \epsilon) \neq \varnothing\big) \leq \tilde{C} 2^{-N(\beta-(1/H))} \leq C\epsilon^{\beta-(1/H)}.$$

Finally, a covering argument completes the desired proof. □

By the definition of Hausdorff measure and as a consequence of Theorem 5.12, we have the following result on hitting points for the process u.

COROLLARY 5.14. *Under the hypotheses of Theorem 5.12, if $\beta > \frac{1}{H}$, the process u does not hit points in $\mathbb{R}^n$ a.s., that is,*

$$\mathbb{P}(\exists t > 0 : u_t = x) = 0 \qquad \textit{for all } x \in \mathbb{R}^n.$$

PROOF. If $\beta > \frac{1}{H}$, then $\mathcal{H}_{\beta-(1/H)}(\{x\}) = 0$ by the definition of Hausdorff measure, and the result follows from Theorem 5.12. □

The next result provides sufficient conditions that imply hypothesis (52) of Theorem 5.12. These conditions are easier to verify for nonlinear equations than hypothesis (52). The proof follows exactly as the proof of [16], Theorem 3.3, and is therefore omitted. It suffices to replace the parabolic metric $\Delta((t,x);(s,y)) = |t-s|^{1/2} + |x-y|$ therein by our fractional metric $|t-s|^{2H}$.

THEOREM 5.15. *Fix $0 < a < b$ and $M > 0$. Assume that the $\mathbb{R}^n$-valued stochastic process u satisfies the following two conditions*:

(i) *For any $t > 0$, the random vector u_t has a density $p_t(z)$ which is uniformly bounded over $z \in [-M, M]^n$ and $t \in [a, b]$.*

(ii) *For some $H \in (0, 1)$ and for all $p > 1$, there exists a constant $C = C(p, H, a, b)$ such that for any $s, t \in [a, b]$,*

$$\mathbb{E}\big[|u_t - u_s|^p\big] \le C|t-s|^{Hp}.$$

Then for any $\beta \in]0, n[$, condition (52) in Theorem 5.12 is satisfied for such β.

Let us now apply this general theory to the n-dimensional process solution to equation (19).

THEOREM 5.16. *Let X_t^x denote the solution to equation (19) where B is a fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$ and the vector fields satisfy Hypothesis 1.3. Fix $0 < a < b \le 1$, $M > 0$ and $\eta > 0$. Then there exists a positive constant $C = C(a, b, H, M, n, \eta)$ such that for all Borel sets $A \subseteq [-M, M]^n$,*

$$\mathbb{P}\big(X_t^x([a, b]) \cap A \neq \varnothing\big) \le C\mathcal{H}_{n-(1/H)-\eta}(A).$$

REMARK 5.17. Because of the inequalities between capacity and Hausdorff measure, the right-hand side of Theorem 5.12 can be replaced by $C\,\mathrm{Cap}_{n-(1/H)-\eta'}(A)$ (cf., [26], page 133).

As a consequence of Theorem 5.16 and Corollary 5.14, we have the following result on hitting points for the process X_t^x.

COROLLARY 5.18. *Under the hypotheses of Theorem 5.16, if $n > \frac{1}{H}$, the process X_t^x does not hit points in $\mathbb{R}^n$ a.s.*

PROOF OF THEOREM 5.16. It suffices to check that conditions (i) and (ii) of Theorem 5.15 hold true for the solution to our equation (19). Condition (i) follows straightforwardly from our results in Section 4. Condition (ii) follows from standard estimates of rough paths theory (see, e.g., [20], Corollary 10.39). □

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