ON PRODUCTS OF PROFINITE GROUPS[1]

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0. In this note we consider pro-P-G groups, where P is a property of finite groups and G is a profinite group of operators. We define a free pro-P-G group and a free pro-P-G product of two given groups, and investigate their basic properties and the relation between them.

Free pro-P groups arise naturally as Galois groups. For example, the group of the p-closure of a p-adic field not containing the p^{th} roots of unity is free pro-p [6], and the group of the solvable closure of the abelian closure of the rationals is free pro-solvable [3]. Often a given Galois group is the semidirect product of two known groups, N, and G, so it is determined by the action of G on N. Koch [5] has studied an important class of cases where N is free pro-p-G.

In §1 we establish some facts about the operation of one profinite group on another. In §2 we define free pro-P-G group, Q-H-ification, and free pro-P-G products, and establish their basic properties and relationships. In §3 we deduce some information about the p-Sylows of a product from knowledge of the p-Sylows of the factors. The proofs of 3.1 and 3.4 are derived from a private communication from O. Kegel.

For the basic facts on profinite groups, in particular the notion of order and the Sylow theorems, we refer the reader to [7].

Except where otherwise indicated the word "homomorphism" and its relatives will imply continuity and "subgroup" will imply closure.

1. Let U be a profinite group and let $A = A_U$ be the set of all automorphisms of U. A is naturally an (abstract) group; to topologize it we take any fundamental system of neighborhoods of the identity, $\{U_i \mid i \in I\}$, in U and let a f.s.n.i. in A be $\{A(U_i) \mid i \in I\}$, where

$$A(U_i) = \{\sigma \in A \mid \sigma(x)x^{-1} \in U_i \quad \text{for all} \quad x \in U\}.$$

This topology is the topology of uniform convergence and is hence independent of the particular f.s.n.i. choosen for U. The topological group A is totally disconnected, (and hence so is any subgroup). A, (and hence any subgroup), is complete with respect to the uniform structure it acquires as a topological group; one need only check that the uniform limit of isomorphisms is a homomorphism and is invertible. A may, however, not be compact.

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LEMMA 1.1. *A subgroup, B, of A, is compact if and only if U admits a* f.s.n.i. *consisting of B-invariant normal subgroups.*

Proof. Suppose there is such a f.s.n.i., $\{U_i \mid i \in I\}$. Since each U_i is B-invariant any $\sigma \in B$ induces an automorphism σ_i of U/U_i. This gives maps $B \rightarrow A_{U/U_i}$ with kernels $B(U_i)$ $(= \{\sigma \in B \mid \sigma(x)x^{-1} \in U_i, \text{all } x \in U\})$. Since the kernels are open the maps are continuous, hence homomorphisms. For every i, $B(U_i)$ is of finite index in B, hence any ultrafilter contains a coset of $B(U_i)$. Hence any untrafilter in B is Cauchy and therefore converges.

Conversely suppose B is a compact subgroup of A. We need only show that any open normal subgroup, U', of U contains a B-invariant open normal subgroup, U''. $B(U')$ is an open neighborhood of the identity in B. Hence $C = \{\sigma \in B \mid \sigma(U') = U'\}$, which contains $B(U')$, is open, hence of finite index in B. Therefore there are only finitely many distinct $\sigma(U')$ for $\sigma \in B$, and their intersection, U'', which is B-invariant, is again open in U.

We say that a set $S \subset U$ generates U if U is the only subgroup of U containing S. In this case there is an (algebraic) homomorphism from the (discrete) free group on a set isomorphic to S into U, whose image is dense. We call U finitely generated if there is a finite such S.

We leave the proofs of the following to the reader: (Use the corresponding facts for discrete groups)

LEMMA 1.2. 1. *A subgroup U', of finite index, is finitely generated if and only if U is.*

2. *If U is finitely generated there are only finitely many subgroups of any given finite index n.*

THEOREM 1.3. *If U is finitely generated then A_U is compact.*

Proof. By 1.2.2 any open subgroup has only finitely many images under A_U. Hence their intersection is still open. The result now follows from 1.1.

The following will be needed later:

LEMMA 1.4. *If U has a* f.s.n.i. *consisting of characteristic subgroups (taken into themselves by any endomorphism) then any epimorphism of U to itself is an isomorphism.*

Proof. Let $\sigma : U \rightarrow U$ be onto. If $\sigma(x) = 1$, $x \neq 1$, let U' be a characteristic open subgroup such that $x \notin U'$. Then σ induces $\sigma' : U/U' \rightarrow U/U'$, a homomorphism of finite groups which is onto but not one to one.

COROLLARY 1.5. *If U is finitely generated pro-p then any epimorphism is an isomorphism.*

Proof. Let $U_1 = U$, $U_{i+1} = U_i^p[U_i, U_i]$. Then the U_i are characteristic. Since U is finitely generated they are open. It is well known that their intersection is trivial; hence they are a f.s.n.i.

Remarks. 1. Corollary 1.5 extends easily to the case of a pronilpotent U with finitely generated p-Sylows. The automorphism group of such a U is a direct product of compact groups by 1.3. On the other hand if the number of generators of a p-Sylow is not a bounded function of p, U is not finitely generated, so the hypothesis of 1.3 is not necessary for compactness of A.

2. The property of being finitely generated is not usually inherited by A_U, hence does not guarantee the compactness of A_{A_U}. The total completion of the integers is an example.

2. If G is a given profinite group we call a profinite group U a G-group if there is given a homomorphism of G into A_U. By 1.1. such a group can be written as a projective limit of a system of finite G-groups and G-homomorphisms. Conversely the limit of any such system is a G-group in a natural way.

If P is a property of finite groups preserved under passage to subgroups, quotients and finite direct sums we let C'_P be the category of all finite P-groups and homomorphisms and C_P the category of all pro-P groups (projective limits of P-groups) and homomorphisms.

We may combine the above concepts and consider the category whose objects are pro-P groups which are also G-groups and whose maps are G-homomorphisms. We call these groups pro-P-G, and denote the category by C_{PG}.

If U is pro-P-G and $S \subset U$ we say that S generates U (as a G-group) if U is the smallest G-invariant subgroup of U containing S. We call S a set of generators for U if, in addition, any open subgroup $U' \subset U$ contains almost all (all but finitely many elements) of S. (Thus the induced topology on S is discrete.) We say that U is free pro-P-G on $S \subset U$ if

1. U is pro-P-G,
2. any open $U' \subset U$ contains almost all of S,
3. for any pro-P-G group V and any function $f : S \to V$ such that any open $V' \subset V$ contains almost all of $f(S)$, there is a unique G-homomorphism $\phi : U \to V$ extending f.

(The uniqueness part of 3 implies that S generates U, hence, by 2, is a set of generators.)

This definition is equivalent to that given in [5] for the case considered there.

PROPOSITION 2.1. *For any discrete set T there is a group U which is free pro-P-G on a set S, homeomorphic to T. If U' is free pro-P-G on S', also homeomorphic to T, then a homeomorphism (correspondance) $S \cong S'$ induces an isomorphism $U \cong U'$.*

Proof. The uniqueness is standard. A construction in the case where G is trivial is given in [3]. (The countability of S is used only to insure the separability of the result.) If G is finite let $S' = G \times S$ and let U be free pro-P on S'. The action of G on S' induces an action on U which makes U free pro-P-G on $\{1\} \times S$. If $\phi : G_i \to G_j$ is a homomorphism of finite groups, $\phi \times 1 : G_i \times S \to G_j \times S$ gives rise to a map ψ on the corresponding free pro-P groups U_i and U_j. So if $G = \lim G_i$ we let $U = \lim U_i$ where U_i is free pro-P-G_i on $\{1_{G_i}\} \times S$. The identification of the appropriate copy of S in U and the proof that U is free on this set are routine, since any pro-P-G group, V, is the limit of pro-P-G_i groups.

COROLLARY 2.2. *Any pro-P-G group is the image of a free pro-P-G group.*

Proof. This follows from the existence of a set of generators, which can be shown by a simple modification of the ideas of [1].

If P and Q are properties of the type described above such that $Q \Rightarrow P$ and we are given a fixed homomorphism of profinite groups $\pi : G \to H$ and $U \in C_{PG}$, we call (U', φ) a Q-H-ification of U if

1. $U' \in C_{QH}$,
2. $\varphi : U \to U'$ is a G-map (G acting on U' through π)
3. If (U'', ψ) satisfies 1 and 2 there is a unique H-map $\eta : U' \to U''$ such that $\psi = \eta\varphi$.

PROPOSITION 2.3. *Given P, Q and $\pi : G \to H$ as above, there is a Q-H-ification (U', φ). If (U'', ψ) is another such there is a unique H-isomorphism $\eta : U' \to U''$ such that $\psi = \eta\varphi$.*

Proof. The uniqueness is routine. We sketch the construction.

First form a Q-G-ification (U_1, φ_1) by letting U_1 be the limit of all finite Q-G-quotients, U_i, of U and φ_1 be the limit of the maps $U \to U_i$. Then form a Q-$\pi(G)$-ification (U_2, ϕ_2) of U_1 by letting U_2 be the quotient of U_1 by the normal subgroup generated by all $x^{-1}x^g$, $x \in U_1$, $g \in \ker(\pi)$, and φ_2 the canonical map.

Finally, to Q-H-ify U_2, let U_2 have a set, S, of generators. Let V be free pro-Q-$\pi(G)$ on $R \cong S$ and W free pro-Q-H on $T \cong R$ and let $\alpha : V \to U_2$, $\beta : V \to W$ be the G-maps induced by the homeomorphisms of R, S and T (again letting G act on W through π). Let N be the smallest H-invariant subgroup of W containing $\beta(\ker(\alpha))$, let $U_3 = W/N$ and let $\gamma : W \to U_3$ be the canonical map. Let φ_3 be the unique map making $\varphi_3\alpha = \gamma\beta$. We leave to the reader the diagram chasing needed to verify that $(U_3, \varphi_3\varphi_2\varphi_1)$ is a Q-H-ification.

The following properties of Q-H-ifications are routine to verify:

LEMMA 2.4. 1. *If $\pi : G \to G'$, $\pi' : G' \to G''$, $P'' \Rightarrow P'$, $P' \Rightarrow P$, then if (U', φ) is a P'-G'-ification of $U \in C_{PG}$ and (U'', φ') a P''-G''-ification of U', $(U'', \varphi'\varphi)$ is a P''-G''-ification of U.*

2. *If π is onto then so is φ.*
3. *φ is one to one if and only if $U \in C_Q$ and* $\ker(\pi)$ *acts trivially on U.*
4. *If U is free pro-P-G on S then U' is free pro-Q-H on $\phi(S)$.*

Since C_{QH} is a subcategory of C_{PG} the construction, for each $U \in C_{PG}$ of a Q-H-ification (U', φ), provides a functor $\mathfrak{F} : C_{PG} \to C_{QH}$ and a natural transformation Φ from the identity functor $\mathfrak{I}$ to $\mathfrak{F}$, defined by $\mathfrak{F}(U) = U'$ for all $U \in C_{PG}$; for $\alpha : U \to V$, $\mathfrak{F}(\alpha) = \alpha'$, the unique H-map from U' to V' making the obvious diagram commutative, and $\Phi(U) = \varphi$. If we consider the category of such pairs $(\mathfrak{F}', \Phi')$, the above results shown that $(\mathfrak{F}, \Phi)$ is an initial object.

If we are given P, Q and $\pi : G \to H$ as above and U, $V \in C_{PG}$, we define a free pro-Q-H product of U and V to be a triple (W, φ, ψ) such that

1. $W \in C_{QH}$,
2. $\varphi : U \to W$, $\psi : V \to W$ are G-maps (usual G-action on W),
3. if (W', φ', ψ') satisfies 1 and 2 there is a unique H-map $\eta : W \to W'$ such that $\eta\varphi = \varphi'$ and $\eta\psi = \psi'$.

Proposition 2.5. *For any P, Q and $\pi : G \to H$ as above there is a free pro-Q-H product (W, φ, ψ) of U and V. If (W', φ', ψ') is another such, there is a unique H-isomorphism $\eta : W \to W'$ such that $\eta\varphi = \varphi'$ and $\eta\psi = \psi'$.*

Proof. Again uniqueness is routine. We construct the product first in the case $P = Q$ and π is an isomorphism. Let S and T be sets of generators for U and V respectively, let $S' \cong S$, $T' \cong T$ be disjoint sets of the corresponding cardinalities, let $R' = S' \cup T'$, let X, Y, Z be free pro-P-G on S', T', R', respectively, and let $\alpha : X \to U$, $\beta : Y \to V$, $\gamma : X \to Z$, $\delta : Y \to Z$ be G-maps induced by $S' \cong S$, $T' \cong T$, $S' \to R'$, $T' \to R'$, respectively. Let $N \subset Z$ be the smallest normal G-subgroup of Z containing $\gamma(\ker(\alpha))$ and $\delta(\ker(\beta))$, let $W = Z/N$ and ε be the canonical map. There are unique G-maps $\varphi : U \to W$, $\psi : V \to W$ such that $\varphi\alpha = \varepsilon\gamma$, $\psi\beta = \varepsilon\delta$; the proof that (W, φ, ψ) is a free pro-P-G product is elementary diagram chasing.

To construct the free pro-Q-H product for general Q and H one can either Q-H-ify U and V and take the free pro-Q-H product, or first take the free pro-P-Q product and then Q-H-ify.

Note. If we omit the hypothesis $Q \Rightarrow P$ then we can use the same definitions of Q-H-ification and free pro-Q-H product, but they will be simply the R-H-ification and free pro-R-H product, where R is the conjunction of P and Q.

The following properties follow directly from the definition:

Lemma 2.6. 1. *If U' is the smallest H-subgroup of W containing $\phi(U)$ then (U', φ) is a Q-H-ification of U.*

2. *$\varphi(U)$ and $\psi(V)$ generate W as an H-group.*

3. *If U and V are free pro-P-G on S and T, respectively, then W is free pro-Q-H on $\varphi(S) \cup \psi(T)$.*

4. *If $U = \lim_I U_i$, $V = \lim_J V_j$ and $(W_{ij}, \varphi_{ij}, \psi_{ij})$* are free pro-$Q$-$H$ products of U_i and V_j, then $W = \lim_{I \times J} W_{ij}$, together with the obvious maps, is a free pro-Q-H product of U and V.

Remarks. 1. By a fairly standard abuse of language we shall sometimes refer to W itself as the free pro-Q-H product, the maps being understood, and write it as $U *_{QH} V$. It is easy to see that

$$(U *_{QH} U') *_{QH} U'' = U *_{QH} (U' *_{QH} U'').$$

2. When $P = Q$ and $\pi : G \cong H$ then (w, φ, ψ) is the categorical sum (or coproduct) in C_{PG}.

The following shows a relationship between the notions of free products and free operator groups:

PROPOSITION 2.7. *Suppose property P is, in addition, preserved under exact sequences. Let $U, V \in C_P$, $\pi : \{1\} \to V$, and let (U', φ'') be a P-V-ification of U. Let W' be the semidirect product, defined, as a set, as $U' \times V$ with the product topology; multiplication is defined by*

$$(u, v)(u', v') = (uv(u'), vv').$$

Define $\varphi' : U \to W'$, $\psi' : V \to W'$ by *$\varphi'(u) = (\varphi''(u), 1)$, $\psi'(v) = (1, v)$. Then (W', φ', ψ') is a free pro-P product of U and V.*

Proof. W' is clearly compact and totally disconnected. Since the action of V on U' is continuous, the multiplication in W' is, hence W' is a profinite group. From the exact sequence

$$\{1\} \to U' \to W' \to W'/U' \to \{1\},$$

$W'/U' \cong V$, we see W' is pro-P.

If (W, φ, ψ) is a free pro-P product of U and V, the maps φ', ψ' induce $\theta : W \to W'$ such that $\theta\varphi = \varphi'$, $\theta\psi = \psi'$,

To construct an inverse to θ note that V acts, through ψ, on W by inner automorphism. Hence $\varphi : U \to W$ induces a V-map $\eta : U' \to W$ such that $\eta\varphi'' = \varphi$. Define $\theta' : W' \to W$ by $\theta'(u, v) = \eta(u)\psi(v)$. It is easy to check that $\theta\theta'$ and $\theta'\theta$ are the identity maps.

Remark. It follows easily from this that if W is a free pro-P product of pro-P groups U and V, and X is the kernel of the map $W \to V$ induced by the identity on V and the trivial map on U, then the pair $(X, U \to X)$ is a V-ification of U under $\{1\} \to V$.

LEMMA 2.8. *Let U and V be profinite and let (W, φ, ψ) be a free pro-p product of U and V. For any $x, y \in W$ define $\varphi_x : U \to W$, $\psi_y : V \to W$* by *$\varphi_x(u) = x^{-1}\varphi(u)x$, $\psi_y(v) = y^{-1}\psi(v)y$.* Then *$(W, \varphi_x, \psi_y)$ is also a free pro-p product of U and V.*

Proof. First assume that U, V, and hence W are finitely generated. The maps φ_x, ψ_y induce $\theta : W \to W$ such that $\theta\varphi = \varphi_x$, $\theta\psi = \psi_y$. The map θ is onto by Proposition 23 bis of [7], hence an isomorphism by 1.5. Since any U and V are limits of finitely generated groups the general result follows from 2.6.4.

3. We now investigate the relation between the p-Sylow subgroups of the factors and those of the product. In this section, the properties P and Q, in addition to being preserved under passage to subgroups and quotients, will be supposed preserved under exact sequences.

THEOREM 3.1. *Let U, $V \in C_P$ have finite orders m and n, respectively. Let X be the kernel of the natural map $U *_P V \to U \oplus V$. Then X is free pro-P on a set of $(m - 1)(n - 1)$ generators.*

Proof. Let F be the (discrete) free product of U and V. Its P-completion, W, together with the obvious maps $U \to W$, $V \to W$, satisfies the mapping properties for a free pro-P product. Let E be the kernel of the natural map $F \to U \oplus V$. E is free (discrete) on $(m - 1)(n - 1)$ generators, (see [4]) and the closure of E in W is just X. It therefore suffices to show that the topology induced on E by the P-topology of F is the same as the P-topology on E.

If D is a normal subgroup of F such that F/D is a finite P-group then $E/E \cap D$ is a finite P-group. Conversely if C is a normal subgroup of E such that E/C is a finite P-group then it has only finitely many conjugates in F. Let D be their intersection. Then E/D is a finite P-group and hence so is F/D.

COROLLARY 3.2. *The p-Sylows of X are free pro-p.*

Proof. This is immediate from results in [7].

COROLLARY 3.3. *If all p-groups are P-groups but not all P-groups are p-groups and $(m - 1)(n - 1) > 1$ then the p-Sylows of X are not finitely generated.*

Proof. It suffices to show that the p-Sylows of a free pro-P group on $h \geq 2$ generators are not finitely generated and for this it is enough to show that a free (discrete) group, F, on h generators has finite P-quotients whose p-Sylow subgroups have arbitrarily many generators.

Let q divide the order of some finite P-group, $q \neq p$. Then all q-groups are P-groups. F has subgroups of index q^r for any r, hence normal subgroups of index q^s where s may be made arbitrarily large. Let E be normal of index q^s. Then E^p is normal and of finite index and E/E^p, a p-Sylow of F/E^p has $(h - 1)q^s + 1$ generators, (see [4]), a number which may be made arbitrarily large.

COROLLARY 3.4. *If p is an odd prime such that all p-groups are P-groups but not all P-groups are p-groups,* and *U and V have non-trivial p-ifications then a p-Sylow* of *$U *_P$ V is not finitely generated.*

Proof. Since U and V can be mapped onto U' and V', each cyclic of order p and a p-Sylow of $U *_P V$ gets mapped onto a p-Sylow of $U' *_P V'$ it suffices to show that the latter is not finitely generated. Let X_p be a p-Sylow of $X = \ker U' *_P V' \to U' \oplus V'$ and W_P a p-Sylow of $W = U' *_P V'$ containing it. Consider the exact sequence

$$\{1\} \to X_p \to W_p \to U' \oplus V' \to \{1\}.$$

The result now follows from 3.3 and 1.2.1.

THEOREM 3.5. *Let Q be a property possessed by all finite p-groups. Let U_p, V_p and W_p be p-Sylow subgroups of profinite groups U, V, and $U *_Q V$ respectively. Then there are maps*

$$\alpha : U_p *_p V_p \to W_p \quad \text{and} \quad \beta : W_p \to U *_p V$$

such that $\beta\alpha$ is onto and if U_p and V_p have normal complements then $\beta\alpha$ is an isomorphism.

Proof. Any pro-p subgroup of $U *_Q V$ is conjugate to a subgroup of W_p. Therefore if φ, ψ are maps making $(U *_Q V, \varphi, \psi)$ a free pro-Q product then $x^{-1}\varphi(U_p)x$ and $y^{-1}\psi(V_p)y$ are in W_p for some x and y in $U *_Q V$. This induces maps $U_p \to W_p$ and $V_p \to W_p$, hence a map $\alpha : U_p *_p V_p \to W_p$.

The map β is the restriction to W_p of that induced on $U *_Q V$ by the natural maps $U \to U *_p V$, $V \to U *_p V$. The images, in $U *_p V$, of $\varphi(U_p)$ and $\psi(V_p)$ are the same as the images of $\varphi(U)$ and $\psi(V)$, hence generate $U *_p V$. But $\beta\alpha(U_p *_p V_p)$ contains subgroups conjugate to these images; hence, since $U *_p V$ is pro-p, $\beta\alpha(U_p *_p V_p) = U *_p V$.

If U_p and V_p have normal complements then they are isomorphic to the p-ifications of U and V, respectively. Hence we may think of $\beta\alpha$ as taking $U_p *_p V_p$ to itself. The result now follows from 2.8

THEOREM 3.6. *Let Q be a property possessed by all p-groups and let U_p, V_p and W_p be p-Sylow subgroups of pro-Q groups U, V and $U *_Q V$ respectively. Then W_p is free pro-p if and only if U_p and V_p are.*

Proof. Since U and V are pro-Q the maps φ and ψ making $(U *_Q V, \varphi, \psi)$ a free pro-Q product are monomorphisms. Hence W_p contains subgroups isomorphic to each of U_p and V_p. Since subgroups of free pro-p groups are free pro-p (see [7]) we have the "only if" part.

Now from [7] we know that the freeness of a p-Sylow of a group X is equivalent to having p-cohomological dimension ≤ 1, and that this in turn holds if and only if, whenever maps $\eta : X \to Z$, $\varepsilon : Y \to Z$, of profinite groups are given with ε onto and $\ker(\eta)$ pro-p, there is $\delta : X \to Y$ such that $\eta = \varepsilon\,\delta$. In view of our definition of pro-Q product by mapping properties the "if" part is elementary diagram chasing.

Remark. This result has been extended by Brumer (unpublished) who showed that $c\,d_p(U *_Q V) = \max(c\,d_p\,U, c\,d_p\,V)$.

We have now shown that if p is odd and U and V are non-trivial pro-p groups (or, more generally, have non-trivial p-Sylows admitting normal complements) and P as in 3.4, then letting U_p, V_p and W_p denote p-Sylows of U, V and their free pro-p product, respectively, there is an exact sequence

$$\{1\} \to K \to W_p \to G \to \{1\},$$

where $G = U_p *_p V_p$ and K is free pro-p. To describe W_p completely it therefore suffices to describe the action of G on K induced by α. (We know K is not finitely generated.) We show that in the case where U_p and V_p are free pro-p K is a free pro-p-G group, or equivalently (by 2.7) that $W_p = \alpha(G) *_p K'$ for some free pro-p K'.

In this case W_p is free pro-p by 3.6. It is enough to show that if U_p and V_p are free pro-p on S and T respectively then W_p is free pro-p on $\alpha(S \cup T) \cup R$ for some R. Let $\bar{W}_p$ and $\bar{G}$ denote $W_p/(W_p)_2$ and G/G_2 respectively (see 1.5). The map π induces $\bar{\pi} : \bar{W}_p \to \bar{G}$; since $\alpha(G_2) \subset (W_p)_2$, α induces $\bar{\alpha} : \bar{G} \to \bar{W}_p$. Clearly $\bar{\pi}\bar{\alpha}$ is the identity. In the category of abelian profinite groups of exponent p every subgroup is complemented and every group is free. If $\bar{W}_p = \bar{\alpha}(\bar{G}) \oplus \bar{H}$ let $\bar{R}$ be a basis for $\bar{H}$ and choose a set, R, of representatives in W_p for $\bar{R}$. The fact that W_p is free pro-p on $\alpha(S \cup T) \cup R$ now follows from Proposition 23 bis of [7].

Remark. Some of the above results can be used to give results about fields. For example Iwasawa's result mentioned earlier, together with 3.4, gives the following:

If K is the solvable closure of $Q(\zeta_\infty)$, (all the roots of 1), let its Galois group be free pro-solvable on $S = \{\sigma_i \mid i = 1, 2, \cdots\}$. Let p be odd and let L be the fixed field of σ_1^p, σ_2^p, all σ_i, $i \neq 1, 2$, and their conjugates. Then there exist finite subextensions M and N of $L/Q(\zeta_\infty)$ with $M \supset N$ and $[M^{*p} \cap N^* : N^{*p}]$ arbitrarily large. To see this note that $G(L/Q(\zeta_\infty)$ is the free pro-solvable product of two groups of order p. The rest is 3.4 and Kummer theory.

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