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ON PROPER ACCRETIVE EXTENSIONS OF POSITIVE LINEAR RELATIONS

ПРО ВЛАСНІ АКРЕТИВНІ РОЗШИРЕННЯ ДОДАТНИХ ЛІНІЙНИХ ВІДНОШЕНЬ

A linear relation $\tilde{S}$ is called a proper extension of a symmetric linear relation S if $S \subset \tilde{S} \subset S^*$. As is well known, an arbitrary dissipative extension of a symmetric linear relation is proper.

In the paper criterions for accretive extension of a given positive symmetric linear relation to be proper are established.

Лінійне відношення $\tilde{S}$ називається властивим розширенням симетричного лінійного відношення S, якщо $S \subset \tilde{S} \subset S^*$. Як відомо, довільне дисипативне розширення симетричного лінійного відношення є властивим.

Одержані критерії того, що акретивне розширення даного додатного відношення є властивим.

1. Introduction. Let H be a complex Hilbert space and let $H^2 = H \oplus H$ be the set of all pairs $\langle u, u' \rangle$, $u, u' \in H$, with the inner product

$$(\langle u, u' \rangle, \langle v, v' \rangle) = (u, v) + (u', v'), \quad \langle u, u' \rangle, \langle v, v' \rangle \in H^2.$$

As is well known [1], a closed subspace $S \subseteq H^2$ is a linear relation (l.r.) or a multivalued linear operator. If T is a closed linear operator in H, then its graph $\mathrm{Gr}(T) = \{\langle u, Tu \rangle, u \in \mathcal{D}(T)\}$ is a l.r.

Basic concepts connected with l.r. can be found in [1]. In particular, $\mathcal{D}(S) = \{u \in H : \langle u, u' \rangle \in S$ for some $u' \in H\}$ is the domain of S, $S(u) = \{u' \in H : \langle u, u' \rangle \in S\}$, the subspace $S^* = H^2 \ominus JS$, where $J\langle x, x' \rangle = \langle -x', x \rangle$ for all $\langle x, x' \rangle \in H^2$, is called the adjoint of S.

A l.r. S will be called

a) symmetric if $S \subset S^*$;

b) selfadjoint if $S = S^*$;

c) positive if $(S(u), u) \geq 0$ for all $u \in \mathcal{D}(S)$;

d) dissipative if $\mathrm{Im}\,(S(u), u) \geq 0$ for all $u \in \mathcal{D}(S)$;

e) accretive if $\mathrm{Re}\,(S(u), u) \geq 0$ for all $u \in \mathcal{D}(S)$;

f) α-sectorial if S is accretive and $|\mathrm{Im}\,(S(u), u)| \leq \mathrm{tg}\,\alpha\, \mathrm{Re}\,(S(u), u)$ for all $u \in \mathcal{D}(S)$, where $\alpha \in [0, \pi/2)$;

g) m-accretive if both S and S^* are accretive;

h) $m-\alpha$-sectorial if S is m-accretive and α-sectorial.

A l.r. $\tilde{S}$ will be called a proper extension of a symmetric l.r. S if $S \subset \tilde{S} \subset S^*$.

It is well known [2] that an arbitrary dissipative extension of a symmetric l.r. is proper.

In this paper, criteria for an accretive or α-sectorial extension of a positive l.r. to be proper are established.

Assume that S is a positive l.r., the sesquilinear form $(S(u), v)$, $u, v \in \mathcal{D}(S)$, has the closure [1, 3] defined on a certain lineal $\mathcal{D}[S] \supseteq \mathcal{D}(S)$, its values are denoted by $S[u, v]$, $u, v \in \mathcal{D}[S]$, and $S[u] = S[u, u]$.

Let S_F and S_N be the Friedrichs and von Neumann positive selfadjoint extensions of S [1]. For an arbitrary positive selfadjoint extension $\tilde{S}$ of S, we have $\mathcal{D}[S] =$

$= \mathcal{D}[S_F] \subseteq \mathcal{D}[\tilde{S}] \subseteq \mathcal{D}[S_N]$, $\tilde{S}[u] = S_F[u] = S[u]$ for all $u \in \mathcal{D}[S]$, $S_N[u] \le \tilde{S}[u]$ for all $u \in \mathcal{D}[\tilde{S}]$ [4].

Assume that $\omega[u]$ is a positive functional on the lineal $\mathcal{D}$ and $\mathcal{D}_0 \subset \mathcal{D}$. For $u \in \mathcal{D}$, we set

$$\langle \omega[u] \rangle_{\mathcal{D}_0} = \inf\{\omega[u - u_0], u_0 \in \mathcal{D}_0\}.$$

If Θ is an α-sectorial l.r., then the quadratic forms

$$\mathrm{Re}\,(\Theta(u), u), \quad \mathrm{Re}\,[(1 \pm i \operatorname{ctg}\alpha)(\Theta(u), u)] = \mathrm{Re}\,(\Theta(u), u) \mp \operatorname{ctg}\alpha\, \mathrm{Im}\,(\Theta(u), u)$$

are positive on $\mathcal{D}(\Theta)$.

We will prove the following theorems:

Theorem 1. *Let* S *be a positive l.r. and let* $\tilde{S}$ *be an accretive extension of* S. *The following statements are equivalent:* 1) $\tilde{S} \subset S^*$; 2) $\mathcal{D}[\tilde{S}] \subseteq \mathcal{D}[S_N]$ *and* $\mathrm{Re}\,(\tilde{S}(v), v) \ge S_N[v]$ *for all* $v \in \mathcal{D}(\tilde{S})$; 3) $|(S(u), v)|^2 \le (S(u), u)\,\mathrm{Re}(\tilde{S}(v), v)$ *for all* $u \in \mathcal{D}(S)$, $v \in \mathcal{D}(\tilde{S})$.

Theorem 2. *Let* S *be a positive l.r. and let* $\Theta \subset S^*$ *be* m*-accretive. The following statements are equivalent:* 1) $\Theta \supset S$; 2) $\mathcal{D}(\Theta) \subseteq \mathcal{D}[S_N]$ *and* $\mathrm{Re}(\Theta(v), v) \ge S_N[v]$ *for all* $v \in \mathcal{D}(\Theta)$; 3) $|(S(u), v)|^2 \le (S(u), u)\,\mathrm{Re}(\Theta(v), v)$ *for all* $u \in \mathcal{D}(S)$, $v \in \mathcal{D}(\Theta)$.

Theorem 3. *Suppose that* S *is a positive l.r. and* Θ *is an* α*-sectorial extension of* S. *The following statements are equivalent:*

1) $\Theta \subset S^*$;

2) $\langle \mathrm{Re}\,[(1 - i\operatorname{ctg}\alpha)(\Theta(v), v)] \rangle_{\mathcal{D}(S)} + \langle \mathrm{Re}\,[(1 + i\operatorname{ctg}\alpha)(\Theta(v), v)] \rangle_{\mathcal{D}(S)} = 2\langle \mathrm{Re}\,(\Theta(v), v) \rangle_{\mathcal{D}(S)}$ *for all* $v \in \mathcal{D}(\Theta)$;

3) *the sesquilinear form*

$$\omega[u, v] = (\Theta(u), v) - S_N[u, v]$$

is α*-sectorial on* $\mathcal{D}(\Theta)$.

2. Preliminaries. A) Let S be a l.r. and let

$$\mu(S) = \{\langle u + u', u - u' \rangle, \langle u, u' \rangle \in S\}$$

be a fractional-linear transformation (f.-l.t.).

It possesses the properties $\mu(\mu(S)) = S$, $\mu(S^*) = (\mu(S))^*$, $\mu(S_1) \subseteq \mu(S_2)$ if $S_1 \subseteq S_2$.

One can easily check that S is accretive (positive) if and only if $\mu(S) = \mathrm{Gr}(T)$, where T is a contraction (Hermitian contraction) and S is m-accretive (positive selfadjoint) if and only if T is defined on H (selfadjoint contraction).

B) Assume that A is an Hermitian contraction defined on the subspace $\mathcal{D}(A) \subset H$. M. G. Krein in [5] described the set of all selfadjoint conrtactive (sc) extensions of A as the operator segment $[A_\mu, A_M]$, where A_μ and A_M are the so-called hard and soft sc-extensions of A, i.e., unique sc-extensions possessing the properties: for all $f \in H$

$$\inf\{((I + A_\mu)(f - \varphi), f - \varphi),\ \varphi \in \mathcal{D}(A)\} = 0, \tag{1}$$

$$\inf\{((I - A_M)(f - \varphi), f - \varphi),\ \varphi \in \mathcal{D}(A)\} = 0. \tag{2}$$

A linear operator T defined on H is called a quasiselfadjoint contractive (qsc) extension of a Hermitian contraction A if

$$T \supset A, \quad T^* \supset A, \quad \|T\| \le 1.$$

In [6, 7], it was obtained that the formula

$$T = (A_M + A_\mu)/2 + (A_M - A_\mu)^{1/2} X (A_M - A_\mu)^{1/2}/2 \tag{3}$$

establishes a bijective correspondence between the set of all qsc-extensions of A and the set of all contractions X in the space $\mathfrak{N}_0 = \overline{(A_M - A_\mu)H}$ and, if $A_\mu = A_M$, then A has a unique qsc-extension (the symbol $B^{1/2}$ denotes the positive square root of the positive selfadjoint operator B).

Let

$$\mathfrak{N} = H \ominus \mathfrak{D}(A), \quad H_0 = \overline{(I + A_M)H}, \quad \mathfrak{M} = \{\varphi \in H_0 : (I + A_M)^{1/2}\varphi \in \mathfrak{N}\}$$

and let $(I + A_M)_{\mathfrak{N}}$ be the "shorted operator" [5, 8]. Then, for all $f \in H$,

$$((I + A_M)_{\mathfrak{N}} f, f) = \inf\{((I + A_M)(f - \varphi), f - \varphi), \ \varphi \in \mathfrak{D}(A)\} =$$

$$= ((I + A_M)^{1/2} P_{\mathfrak{M}} (I + A_M)^{1/2} f, f),$$

where $P_{\mathfrak{M}}$ is the orthogonal projection onto $\mathfrak{M}$. From (1), $(I + A_M)_{\mathfrak{N}} = A_M - A_\mu$.

Consequently, $(A_M - A_\mu)^{1/2} = UP_{\mathfrak{M}}(I + A_M)^{1/2}$, where U is the unitary operator from $\mathfrak{M}$ onto $\mathfrak{N}_0$.

Hence, (3) implies the followi g descriptions of qsc-extensions:

$$T = A_M + (I + A_M)^{1/2}(Y - I) P_{\mathfrak{M}} (I + A_M)^{1/2}/2, \tag{4}$$

where Y is an arbitrary contraction in $\mathfrak{M}$.

C) Let $\mathbf{S}$ be a positive l.r. Then $\mu(\mathbf{S}) = \mathrm{Gr}(A)$, where A is an Hermitian contraction, $\mathfrak{D}(A) = (\mathbf{S} + I)\mathfrak{D}(\mathbf{S})$. In [5, 1], it was established that the following equalities hold:

$$\mu(\mathbf{S}_F) = \mathrm{Gr}(A_\mu), \quad \mu(\mathbf{S}_N) = \mathrm{Gr}(A_M).$$

Put $A_M^0 = A_M | H_0$, $(I + A_M^0)^{-1/2}$ to be the inverse of $(I + A_M)^{1/2}$ in H_0. Since $\mathbf{S}_N = \{\langle (I + A_M)f, (I - A_M)f\rangle, f \in H\}$, we get, for $v = (I + A_M)f$,

$$(\mathbf{S}_N(v), v) = ((I - A_M)f\rangle, (I + A_M)f) = -\|(I + A_M)f\|^2 + 2\|(I + A_M)^{1/2}f\|^2 =$$

$$= -\|v\|^2 + 2\|(I + A_M^0)^{-1/2}v\|^2.$$

Therefore, $\mathfrak{D}[\mathbf{S}_N] = (I + A_M)^{1/2}H = (I + A_M^0)^{1/2}H_0$ and

$$\mathbf{S}_N[v] = -\|v\|^2 + 2\|(I + A_M^0)^{-1/2}v\|^2 \tag{5}$$

for all $v \in \mathfrak{D}[\mathbf{S}_N]$.

D) Assume that $\tilde{\mathbf{S}}$ is an accretive l.r. Then $\mu(\tilde{\mathbf{S}}) = \mathrm{Gr}(\tilde{T})$, where $\tilde{T}$ is a contraction, $\mathfrak{D}(\tilde{T}) = (\tilde{\mathbf{S}} + I)\mathfrak{D}(\tilde{\mathbf{S}})$, and

$$\tilde{\mathbf{S}} = \{\langle (I + \tilde{T})f, (I - \tilde{T})f\rangle, f \in \mathfrak{D}(\tilde{T})\}.$$

Hence, for $v = (I + \tilde{T})f$, $f \in \mathfrak{D}(\tilde{T})$, we have

$$(\tilde{\mathbf{S}}(v), v) = -\|v\|^2 + 2(f, (I + \tilde{T})f). \tag{6}$$

E) Let Θ be a densely defined α-sectorial operator.

In accordance with [3], the Friedrichs $m-\alpha$-sectorial extension of Θ_F is the operator associated with the closure of the sesquilinear form $(\Theta u, v)$, $u, v \in \mathfrak{D}(\Theta)$, $\mathfrak{D}[\Theta] = \mathfrak{D}[\Theta_F]$.

If Θ is an α-sectorial l.r., then

$$\Theta = \mathrm{Gr}(\Theta) \oplus \langle 0, \Theta(0)\rangle,$$

where Θ is an α-sectorial closed operator (the operator part of Θ). Put $\mathfrak{H}_0 = \overline{\mathfrak{D}(\Theta)}$, π_0 to be the orthogonal projection onto $\mathfrak{H}_0$, $\Theta_0 = \pi_0\Theta$. Let Θ_{0F} be the Friedrichs extension of Θ_0 in $\mathfrak{H}_0$. Put

$$\Theta_F = \mathrm{Gr}(\Theta_{0F}) \oplus \langle 0, \mathfrak{H}_0^{\perp}\rangle,$$

where $\mathfrak{H}_0^{\perp} = H \ominus \mathfrak{H}_0$.

Clearly, $\Theta(0) \subseteq \mathfrak{H}_0^{\perp}$ and Θ_F is an $m-\alpha$-sectorial extension of Θ. We will call Θ_F the Friedrichs extension of Θ. It readily follows from the definition that

$$\mathfrak{D}[\Theta] = \mathfrak{D}[\Theta_F], \quad \Theta[u, v] = \Theta_F[u, v] = \Theta_{0F}[u, v], \quad u, v \in \mathfrak{D}[\Theta].$$

F) Let $\tilde{\mathbf{S}}$ be an $m-\alpha$-sectorial l.r. Then

$$\tilde{\mathbf{S}} = \mathrm{Gr}(\tilde{S}) \oplus \langle 0, \tilde{\mathbf{S}}(0)\rangle,$$

where $\tilde{S}$ is an $m-\alpha$-sectorial operator in the subspace $\mathfrak{H} = \overline{\mathfrak{D}(\tilde{\mathbf{S}})}$.

In accordance with [3], the operator $\tilde{S}$ has the representation

$$\tilde{S} = \tilde{S}_R^{1/2}(I + i\tilde{G})\tilde{S}_R^{1/2},$$

where $\tilde{S}_R$ is the positive selfadjoint operator associated with the positive form $b[u, v] = \left(\tilde{S}[u, v] + \overline{\tilde{S}[v, u]}\right)/2$, $G = G^*$, $\| G \| \le \mathrm{tg}\,\alpha$ is an operator in the subspace $\overline{\mathcal{R}(\tilde{\mathbf{S}}_R^{1/2})}$, and

$$\mathfrak{D}[\tilde{\mathbf{S}}] = \mathfrak{D}[\tilde{S}] = \mathfrak{D}(\tilde{\mathbf{S}}_R^{1/2}).$$

G) Let $\mathbf{S}_N$ be the von Neumann extension of the positive l.r. $\mathbf{S}$. Passing to the operator part S_N and using the relation established in [4], one can prove that, for all $v \in \mathfrak{D}[\mathbf{S}_N]$,

$$\sup\{|(\mathbf{S}(u), v)|^2/(\mathbf{S}(u), u),\ u \in \mathfrak{D}(\mathbf{S})\} = \| S_N^{1/2} v \|^2, \tag{8}$$

and $\mathfrak{D}[\mathbf{S}_N] = \mathfrak{D}[S_N^{1/2}]$ consists of all vectors v, for which the left-hand side of (8) is finite.

H) Let $\tilde{\mathbf{S}}$ be an $m-\alpha$-sectorial extension of the positive l.r. $\mathbf{S}$ and let $\tilde{S}$ be the operator part of $\tilde{\mathbf{S}}$. Using (7) for $v \in \mathfrak{D}(\tilde{S}_R^{1/2})$, $u \in \mathfrak{D}(\mathbf{S})$, we get

$$(\mathbf{S}(u), v) = (\tilde{\mathbf{S}}(u), v) = (\tilde{S}_R^{1/2}(I + i\tilde{G})\tilde{S}_R^{1/2}u, v) = (\tilde{S}_R^{1/2}u, (I - i\tilde{G})\tilde{S}_R^{1/2}v).$$

Denote by $\tilde{\pi}$ the orthogonal projection onto the subspace $\overline{(\tilde{\mathbf{S}}_R^{1/2})\mathfrak{D}(\mathbf{S})}$. Taking into account the above relation, we obtain, for all $v \in \mathfrak{D}[\tilde{\mathbf{S}}]$,

$$\sup\{|(\mathbf{S}(u), v)|^2/(\mathbf{S}(u), u),\ u \in \mathfrak{D}(\mathbf{S})\} = \| \tilde{\pi}(I - i\tilde{G})\tilde{S}_R^{1/2} v \|^2. \tag{9}$$

Therefore, $\mathfrak{D}[\tilde{\mathbf{S}}] \subseteq \mathfrak{D}[\mathbf{S}_N]$ and, from (8), (9),

$$\| S_N^{1/2} v \|^2 = \| \tilde{\pi} (I - i\tilde{G}) \tilde{S}_R^{1/2} v \|^2 \quad \text{for all } v \in \mathfrak{D}[\tilde{\mathrm{S}}]. \tag{10}$$

I) The following lemma will be used in the proof of the Theorem 3:

Lemma. *Suppose that* F *is a selfadjoint contraction in* H, $\mathfrak{L}$ *is a subspace in* H, *and* $P_{\mathfrak{L}}$ *is the orthogonal projection onto* $\mathfrak{L}$. *The following statements are equivalent:*

1) $\mathfrak{L}$ *reduces* F;

2) $(I-F)_{\mathfrak{L}} + (I+F)_{\mathfrak{L}} = 2P_{\mathfrak{L}}$, *where* $(I \pm F)_{\mathfrak{L}}$ *are "shorted operators".*

Proof. Put $\mathfrak{L}^{\perp} = H \ominus \mathfrak{L}$, $\mathfrak{H}_{\pm} = H \ominus (I \pm F)^{1/2}\mathfrak{L}^{\perp}$, $P_{\pm}$ to be orthogonal projections onto $\mathfrak{H}_{\pm}$. By the definition [5, 8], $((I \pm F)_{\mathfrak{L}} f, f) = \inf\{((I \pm F)(f-\varphi), f-\varphi)$, $\varphi \in \mathfrak{L}^{\perp}\} = \| P_{\pm}(I \pm F)^{1/2} f \|$ for all $f \in H_+$.

1) $\Rightarrow$ 2). If $F\mathfrak{L} \subseteq \mathfrak{L}$, then $F\mathfrak{L}^{\perp} \subseteq \mathfrak{L}^{\perp}$. Therefore,

$$\| P_{\pm}(I \pm F)^{1/2} f \|^2 = \| (I \pm F)^{1/2} P_{\mathfrak{L}} f \|^2,$$

$$((I+F)_{\mathfrak{L}} f, f) + ((I-F)_{\mathfrak{L}} f, f) =$$

$$= \| (I+F)^{1/2} P_{\mathfrak{L}} f \|^2 + \| (I-F)^{1/2} P_{\mathfrak{L}} f \|^2 = 2\| P_{\mathfrak{L}} f \|^2, \quad f \in H.$$

2) $\Rightarrow$ 1). For all $f \in H$, we have

$$\| P_+(I+F)^{1/2} f \|^2 + \| P_-(I-F)^{1/2} f \|^2 = 2\| P_{\mathfrak{L}} f \|^2. \tag{11}$$

Substituting $f \in \mathfrak{L}$ in (11), we obtain

$$2\| f \|^2 = \| P_+(I+F)^{1/2} f \|^2 + \| P_-(I-F)^{1/2} f \|^2 \le$$

$$\le \| (I+F)^{1/2} f \|^2 + \| (I-F)^{1/2} f \|^2 = 2\| f \|^2.$$

Consequently, $P_{\pm}(I \pm F)^{1/2} f = (I \pm F)^{1/2} f$ for all $f \in \mathfrak{L}$. Hence, $F\mathfrak{L} \subseteq \mathfrak{L}$ Q.E.D.

3. Proof of Theorem 1. Suppose that A and $\tilde{T}$ are f.-l.t. of S and $\tilde{\mathrm{S}}$ respectively. Then $\tilde{T}$ is a contractive extension of the Hermitian contraction A. Denote by $\tilde{H}$ the domain of $\tilde{T}$ an let $\tilde{P}$ be the orthogonal projection onto $\tilde{H}$.

1) $\Rightarrow$ 2). For all $f \in \tilde{H}$ and $\varphi \in \mathfrak{D}(A)$, we have the equality $(\tilde{T}f, \varphi) = (f, A\varphi)$. Therefore [9], there exists a contractive extension T of $\tilde{T}$ on H such that $T^* \supset A$. This means that T is a qsc-extension of A. Put $\Theta = \mu(\mathrm{Gr}(T))$. Then Θ is an m-accretive proper extension of S, $\mathfrak{D}(\Theta) = (I+T)H$, $\Theta \supseteq \tilde{\mathrm{S}}$. From (4),

$$I + T = (I + A_M)^{1/2}(I + 1/2(Y-I)P_{\mathfrak{M}})(I + A_M)^{1/2},$$

where Y is the contraction in $\mathfrak{M}$.

From (5) and (6), for $v = (I + \tilde{T})f = (I+T)f$, $f \in \tilde{H}$, we get

$$\mathrm{Re}(\tilde{\mathrm{S}}(v), v) - \mathrm{S}_N[v] = 2\,\mathrm{Re}((I+T)f, f) - 2\| (I + A_M^0)^{-1/2}(I+T) f \|^2 =$$

$$= 2\,\mathrm{Re}((I + 1/2(Y-I)P_{\mathfrak{M}})(I+A_M)^{1/2}, (I+A_M)^{1/2} f) -$$

$$- 2\| (I + 1/2(Y-I)P_{\mathfrak{M}})(I+A_M)^{1/2} f \|^2 =$$

$$= 1/2(\| P_{\mathfrak{M}}(I+A_M)^{1/2} f \|^2 - \| Y P_{\mathfrak{M}}(I+A_M)^{1/2} f \|^2) > 0.$$

2) $\Rightarrow$ 1). Relations (5) and (6) imply $(I + \tilde{T})\tilde{H} \subseteq (I + A_M)^{1/2} H$,

$$\mathrm{Re}\,(f, (I+\tilde{T})f) \geq \|(I+A_M^0)^{-1/2}(I+\tilde{T})f\|^2, \quad f \in \tilde{H}. \qquad (12)$$

Put $Q = \tilde{P}\tilde{T}$. Then Q is a contraction in $\tilde{H}$, $Q_R = (Q+Q^*)/2$ is a selfadjoint contraction in $\tilde{H}$.

Inequality (12) can be rewritten as

$$\|(I+Q_R)^{1/2}f\|^2 \geq \|(I+A_M^0)^{-1/2}(I+\tilde{T})f\|^2, \quad f \in \tilde{H}.$$

Hence, $(I+A_M^0)^{-1/2}(I+\tilde{T})f = W(I+Q_R)^{1/2}f$, $f \in \tilde{H}$, where W is a contraction.

Furthermore, we have, for all $f \in \tilde{H}$,

$$\|(I+Q_R)^{1/2}f\|^2 = \mathrm{Re}\,(f, (I+\tilde{T})f) = \mathrm{Re}\,(f, (I+A_M^0)^{1/2}W(I+Q_R)^{1/2}f) \leq$$

$$\leq \|(I+A_M)^{1/2}f\|\,\|(I+Q_R)^{1/2}f\|.$$

This implies $\|(I+Q_R)^{1/2}f\|^2 \leq \|(I+A_M)^{1/2}f\|^2$, $f \in \tilde{H}$, or $Q_R \leq \tilde{P}A_M|\tilde{H}$. For $\varphi \in \mathcal{D}(A)$, $(Q_R\varphi, \varphi) = \mathrm{Re}\,(\tilde{T}\varphi, \varphi) = (A\varphi, \varphi) = (\tilde{P}A_M\varphi, \varphi)$. Hence, $Q_R|\mathcal{D}(A) = \tilde{P}A$. Consequently, $Q^*|\mathcal{D}(A) = Q|\mathcal{D}(A) = \tilde{P}A$, i.e., Q is a qsc-extension in $\tilde{H}$ of the Hermitian contraction $\tilde{P}A$. This yields $\mathrm{Gr}(\tilde{T}) \subset (\mathrm{Gr}(A))^*$ and $\tilde{\mathbf{S}} \subset \mathbf{S}^*$.

2) $\Leftrightarrow$ 3) is an immediate consequence of (8). Q.E.D.

4. Proof of Theorem 2. 1) $\Rightarrow$ 2) is a consequence of Theorem 1.

2) $\Rightarrow$ 1). Let T be a f.-l.t. of Θ. Then T is a contraction defined on H and $T^* \supset A$.

For $T_R = (T+T^*)/2$, using (5) and (6), we get

$$\|(I+T_R)^{1/2}f\|^2 \geq \|(I+A_M^0)^{-1/2}(I+T)f\|^2, \quad f \in H.$$

As before, this inequality implies $T_R \leq A_M$ and, in view of $T^* \supset A$, we have $T|\mathcal{D}(A) = A$. Thus, $\Theta \supset \mathbf{S}$.

2) $\Leftrightarrow$ 3) is a corollary of (8). Q.E.D.

5. Proof of Theorem 3. Consider the Friedrichs extension $\tilde{\mathbf{S}}$ of Θ. Then $\tilde{\mathbf{S}}$ is $m-\alpha$-sectorial, $\mathcal{D}[\tilde{\mathbf{S}}] = \mathcal{D}[\Theta] \subseteq \mathcal{D}[\mathbf{S}_N]$, and, for the operator part $\tilde{\mathbf{S}}$,

$$\tilde{\mathbf{S}} = \tilde{S}_R^{1/2}(I+i\tilde{G})\tilde{S}_R^{1/2}, \quad \tilde{G} = \tilde{G}^*, \quad \|\tilde{G}\| \leq \mathrm{tg}\,\alpha.$$

Put $\mathfrak{L} = \overline{\mathcal{R}(\tilde{S}_R^{1/2})} \ominus \tilde{S}_R^{1/2}\mathcal{D}(\mathbf{S})$, $\tilde{\pi}$, $P_{\mathfrak{L}}$ to be orthogonal projection onto $\mathfrak{L}^{\perp}$ and $\mathfrak{L}$ respectively.

A direct consequences of the definitions are the following relations for all $v \in \mathcal{D}(\Theta)$:

$$\langle \mathrm{Re}\,[(1 \pm i\,\mathrm{ctg}\,\alpha)(\Theta(v), v)]\rangle_{\mathcal{D}(\mathbf{S})} = ((I \mp \mathrm{ctg}\,\alpha\,\tilde{G})_{\mathfrak{L}}\,\tilde{S}_R^{1/2}v, \tilde{S}_R^{1/2}v), \qquad (13)$$

$$\langle \mathrm{Re}\,(\Theta(v), v)]\rangle_{\mathcal{D}(\mathbf{S})} = \|P_{\mathfrak{L}}\,\tilde{S}_R^{1/2}v\|^2. \qquad (14)$$

Besides, for $u \in \mathcal{D}(\mathbf{S})$,

$$\|\tilde{S}_R^{1/2}u\|^2 = \mathrm{Re}\,(\tilde{\mathbf{S}}(u), u) = (\tilde{\mathbf{S}}(u), u) = ((I+i\tilde{G})\,\tilde{S}_R^{1/2}u, \tilde{S}_R^{1/2}u),$$

Hence,

$$\tilde{\pi}\,\tilde{G}\,\tilde{\pi} = 0. \qquad (15)$$

1) $\Rightarrow$ 2). Since Θ is accretive, from Theorem 1,

$$\operatorname{Re}(\Theta(v), v) \geq \| S_N^{1/2} v \|^2 \quad \text{for all} \quad v \in \mathcal{D}(\Theta). \tag{16}$$

Since $\mathcal{D}(\Theta)$ is the core of the sesquilinear form $\Theta[u, v] = \tilde{S}[u, v]$, (16) implies that

$$\operatorname{Re} \tilde{S}[v] \geq \| S_N^{1/2} v \|^2 \quad \text{for all} \quad v \in \mathcal{D}[\Theta].$$

It follows from relation (10) that

$$\| \tilde{S}_R^{1/2} v \|^2 \geq \| \tilde{\pi}(I - i\tilde{G}) \tilde{S}_R^{1/2} v \|^2, \quad v \in \mathcal{D}[\Theta].$$

Therefore, the bounded selfadjoint operator

$$L = I - (I + i\tilde{G}) \tilde{\pi} (I - i\tilde{G})$$

acting in the subspace $\overline{\mathcal{R}(\tilde{S}_R^{1/2})}$ is positive. Using (15) for $\varphi = \tilde{S}_R^{1/2} u$, $u \in \mathcal{D}(S)$, we have $(L\varphi, \varphi) = \|\varphi\|^2 - \|\varphi\|^2 = 0$. Consequently, $L\tilde{\pi} = 0$. This yields $\tilde{G}\tilde{\pi} = 0$. Thus, $\tilde{G}\mathfrak{L} \subseteq \mathfrak{L}$.

Now the lemma implies

$$(I - \operatorname{ctg}\alpha\, \tilde{G})_{\mathfrak{L}} + (I + \operatorname{ctg}\alpha\, \tilde{G})_{\mathfrak{L}} = 2P_{\mathfrak{L}}.$$

Hence, in view of (13) and (14), for $v \in \mathcal{D}(\Theta)$,

$$\langle \operatorname{Re}[(1 - i\operatorname{ctg}\alpha)(\Theta(v), v)] \rangle_{\mathcal{D}(S)} + \langle \operatorname{Re}[(1 + i\operatorname{ctg}\alpha)(\Theta(v), v)] \rangle_{\mathcal{D}(S)} =$$
$$= 2\langle \operatorname{Re}(\Theta(v), v) \rangle_{\mathcal{D}(S)}. \tag{17}$$

2) $\Rightarrow$ 1). Let (17) be true for all $v \in \mathcal{D}(\Theta)$. Since $\mathcal{D}(\Theta)$ is the core of $\tilde{S}_R^{1/2}$, we have from (13), (14), and the lemma that $\tilde{G}\mathfrak{L} \subseteq \mathfrak{L}$.

Taking (15) into account, we get $\tilde{G}\tilde{\pi} = \tilde{\pi}\tilde{G} = 0$. Hence, from (10), for $v \in \mathcal{D}(\Theta)$,

$$S_N[v] = \| S_N^{1/2} v \|^2 = \| \tilde{\pi}(I - i\tilde{G}) \tilde{S}_R^{1/2} v \|^2 = \| \tilde{\pi} \tilde{S}_R^{1/2} v \|^2 \leq \operatorname{Re}(\Theta(v), v).$$

In accordance with Theorem 1, $\Theta \subset S^*$.

3) $\Rightarrow$ 1). If ω is an α-sectorial form, then

$$\operatorname{Re}(\Theta(v), v) \geq S_N[v] \quad \text{for all} \quad v \in \mathcal{D}(\Theta).$$

Furthermore, we apply Theorem 1.

1) $\Rightarrow$ 3). Since Θ is a proper accretive extension of S, from Theorem 1, $\mathcal{D}(\Theta) \subseteq \mathcal{D}[S_N]$. For all $u \in \mathcal{D}[S_N]$ and $u_0 \in \mathcal{D}(S)$, we have

$$S_N[u, u_0] = (u, S(u_0)).$$

Hence, it is easy to check that, for all $u \in \mathcal{D}(\Theta)$ and $u_0 \in \mathcal{D}(S)$, we have

$$\omega[u - u_0] = \omega[u]. \tag{18}$$

An immediate consequence of (8) is the relation

$$\inf\{ S_N[u - u_0],\, u_0 \in \mathcal{D}(S) \} = 0 \quad \text{for all} \quad u \in \mathcal{D}[S_N].$$

Therefore, for given $\varepsilon > 0$ and $u \in \mathcal{D}(\Theta)$, one can find $u_0 \in \mathcal{D}(S)$ such that $S_N[u - u_0] < \varepsilon$. Taking (18) into account, we obtain

$$\operatorname{Re}\omega[u] \pm \operatorname{ctg}\alpha \operatorname{Im}\omega[u] =$$