On properties of Parisi measures

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Mixed p-spin model

- Configuration space: $\Sigma_N = \{-1,+1\}^N$
- Energy:

$$-H_N(\boldsymbol{\sigma}) = \sum_{p\geq 2} \frac{\beta_p}{N^{(p-1)/2}} \sum_{i_1,\ldots,i_p=1}^{N} g_{i_1,\ldots,i_p}\sigma_{i_1}\cdots\sigma_{i_p},$$

  where $g_{i_1,\ldots,i_p}$ are i.i.d. standard Gaussian.
- Gibbs measure: G_N
- Free energy: $F_N = \frac{1}{N}\mathbb{E}\log\sum_{\boldsymbol{\sigma}}\exp(-H_N(\boldsymbol{\sigma}))$.

- Here,

$$\mathbb{E}H_N(\boldsymbol{\sigma}^1)H_N(\boldsymbol{\sigma}^2) = N\xi(R(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)),$$

  where $R(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2) = \frac{\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2}{N}$ and $\xi(x) = \sum_{p\geq 2}\beta_p^2 x^p$.

Theorem (Parisi' 79, Talagrand '06, Panchenko '10)

$$\lim_{N\to\infty} F_N = \inf_{\mu\in M[0,1]} \mathcal{P}(\mu),$$

where

- $M[0,1] = \{\textit{all prob. meas. on } [0,1]\}$.
- $\mathcal{P}(\mu) := \Phi_\mu(0,0) - \frac{1}{2}\int_0^1 q\xi''(q)\mu([0,q])dq$.
- $\Phi_\mu(q,x)$ *is the solution to the Parisi PDE,*

$$\partial_q\Phi_\mu(q,x) = -\frac{\xi''(q)}{2}\left(\partial_{xx}\Phi_\mu(q,x) + \mu([0,q])(\partial_x\Phi_\mu(q,x))^2\right)$$
$$\Phi_\mu(1,x) = \log\cosh x.$$

- How many minimizers?
- What is the role played by any such minimizers?

The minimizer of the Parisi functional: Predictions

We call any measure that minimizes the previous variational problem a *Parisi measure*. We denote it by μ_P.

It is expected that

(a) μ_P is unique.

(b) The limit law of the overlap under the measure $\mathbb{E}G_N^{\otimes 2}$ is given by μ_P.

(c) For i.i.d. sampled $(\boldsymbol{\sigma}^\ell)_{\ell\geq 1}$ from G_N, they are ultrametric and the joint law is determined by μ_P.

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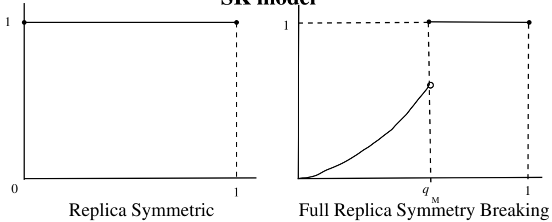
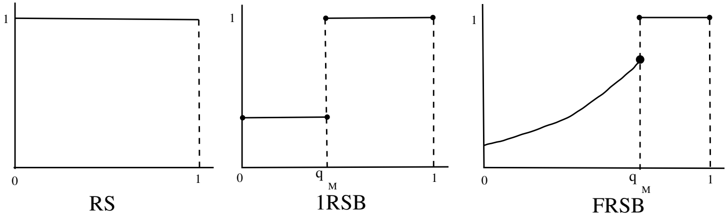
Quantitative behavior of μ_P: predictions

Let $\alpha(q) = \mu_P([0,q])$.

SK model



pure p-spin model



Physicists' predictions: summary

Any Parisi measure should satisfy the following:

(1) The origin is contained in the support of the Parisi measure at any temperature.

(2) One expects FRSB behavior at low temperature.

(3) Any Parisi measure has a jump discontinuity at q_M at any temperature.

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Quantitative behavior of μ_P: known results

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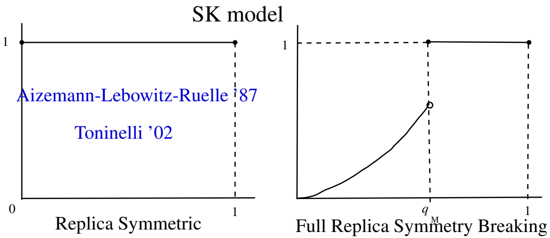

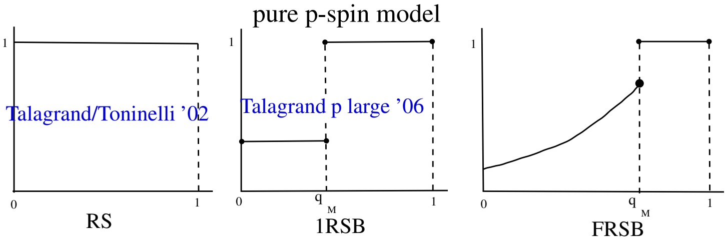

Our results

Theorem (Uniqueness, Auffinger-Chen '14)

At any temperature $(\beta_p)_{p\geq 2}$, the Parisi measure is unique.

Theorem (Auffinger-Chen '13)

We have that

- $0 \in supp\mu_P$.
- *If* $\beta_2^2 < 1$, *then* 0 *is an isolated point.*

Example (Pure p-spin model with $p \geq 3$)

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Our results: towards FRSB

Theorem (Auffinger-Chen '13)

Suppose that

$$\xi(1) > \frac{1}{3}\sqrt{\xi'(1)}2^{\frac{\xi'(1)}{\xi(1)}+5}.$$

Then the Parisi measure is neither RS nor 1RSB.

Example (Pure p-spin model with $p \geq 2$)

Neither RS nor 1RSB if $\beta_p > 2^{p+5}\sqrt{p}/3$.

Theorem (Regularity, Auffinger-Chen '13)

- *Suppose that $u_\ell^-, u_\ell^+ \in \mathrm{supp}\,\mu_P$ for all $\ell \geq 1$ satisfy $u_\ell^- \uparrow u_0$ and $u_\ell^+ \downarrow u_0$. Then μ_P is continuous at u_0.*
- *If $(a,b) \subset \mathrm{supp}\,\mu_P$ for some $0 \leq a < b \leq 1$, then $\alpha(r) := \mu_P([0,r])$ is infinitely differentiable on $[a,b)$.*

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Our results: top of the support

Theorem (Auffinger-Chen '13)

Suppose that ξ satisfies $1/\sqrt{2} < \beta_2 \leq 3/2\sqrt{2}$ and

$$\frac{\xi'''(1)}{6} + \frac{2}{3}\sqrt{\xi''(1)} \leq 1.$$

Then the Parisi measure has a jump discontinuity at q_M.

Example (SK model)

If $1/\sqrt{2} < \beta_2 \leq 3/2\sqrt{2}$, then q_M is a jump discontinuity.

Whether there are infinitely many points in the support of μ_P or not is still unclear.

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Recall Parisi's formula says

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z_N = \inf_{\mu\in M[0,1]}\mathcal{P}(\mu),$$

where

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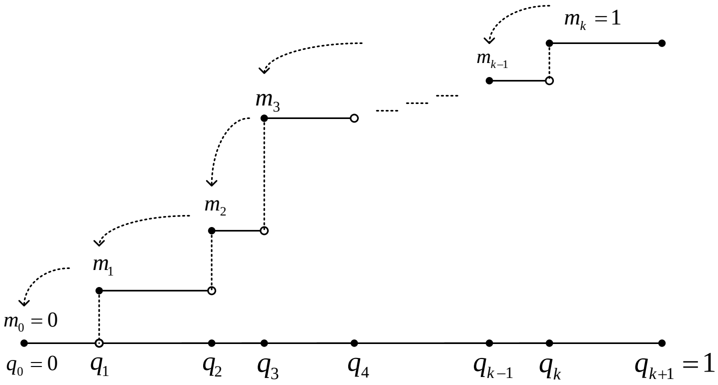
How to solve the Parisi PDE?

- Consider $\mu \in M([0,1])$,



- $\Phi(1,x) = \log\cosh x$.
- Using the Hopf-Cole transformation, for $q \in [q_p, q_{p+1})$,

$$\Phi(q,x) = \frac{1}{m_p}\log \mathbb{E}\exp m_p \Phi\left(q_{p+1}, x + z\sqrt{\xi'(q_{p+1}) - \xi'(q)}\right).$$

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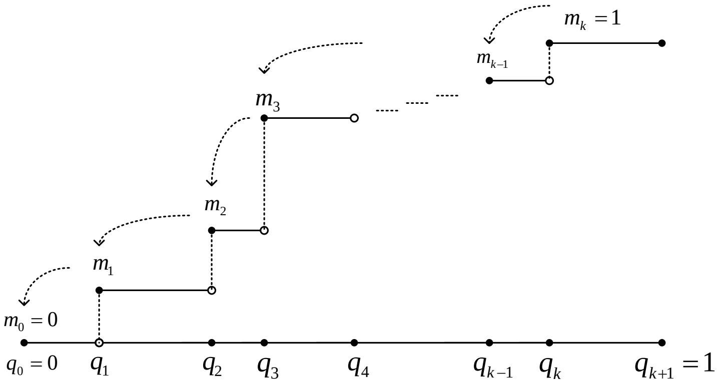

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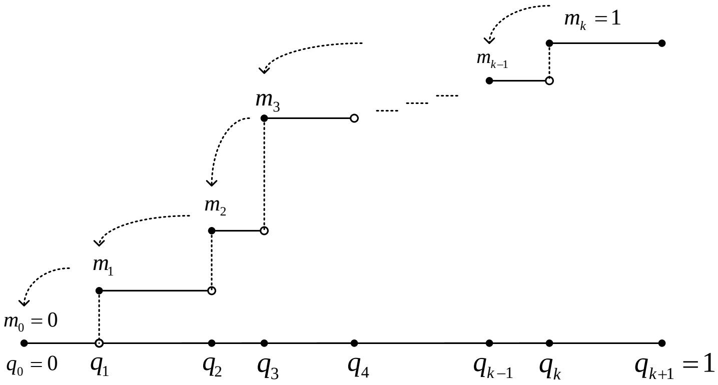

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Know results

- Convexity along one-sided direction if we have stochastic dominance.
  - Panchenko '05
  - Chen '13
- Suppose that f is convex. Is the following convex?

$$m \mapsto \frac{1}{m} \log \mathbb{E} \exp mf(Z).$$

- Approaches for Gaussian inequalities:
  - Ornstein-Uhlenbeck semi-group
  - Maximum principle in PDE
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Variational representation for Φ_μ

Let $\mu \in M[0,1]$. For $0 \le s \le t \le 1$, define

- $D[s,t]$: all prog. meas. $u = (u(w))_{s \le w \le t}$ w.r.t. W with $\|u\|_\infty \le 1$.
- $C_\mu(u,x) = \Phi_\mu\left(t, x + \int_s^t \mu([0,w])\xi''(w)u(w)dw + \int_s^t \xi''(w)^{1/2}dW(w)\right)$.
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$$\Phi_\mu(s,x) = \max_{u \in D[s,t]} F_\mu(u,x).$$

The maximum is attained by $u_\mu^*(w) = \partial_x \Phi_\mu(w, X(w))$, *where* $(X(w))_{s \le w \le t}$ *satisfies*

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Proof of convexity of $\mathcal{P}(\mu)$

Let $s = 0$ and $t = 1$.

- $D[0,1]$: all prog. meas. $u = (u(w))_{0 \le w \le 1}$ w.r.t. W with $\|u\|_\infty \le 1$.
- $C_\mu(u,x) = \log\cosh\left(x + \int_0^1 \mu([0,w])\xi''(w)u(w)dw + \int_0^1 \xi''(w)^{1/2}dW(w)\right)$.
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Let $\mu_\theta := (1-\theta)\mu_0 + \theta\mu_1$. Then

$$C_{\mu_\theta}(u,x) \le (1-\theta)C_{\mu_0}(u,x) + \theta C_{\mu_1}(u,x),$$
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Proof of convexity of $\mathcal{P}(\mu)$

Let $s = 0$ and $t = 1$.

- $D[0,1]$: all prog. meas. $u = (u(w))_{0 \le w \le 1}$ w.r.t. W with $\|u\|_\infty \le 1$.
- $C_\mu(u,x) = \log\cosh\left(x + \int_0^1 \mu([0,w])\xi''(w)u(w)dw + \int_0^1 \xi''(w)^{1/2}dW(w)\right).$
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- Assume distinct $\mu_0, \mu_1 \in M[0,1]$ and $\theta \in (0,1)$.
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Key computation:

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In the simplest case

Theorem (Boué-Dupuis '98)

Let Z be standard Gaussian and $(W_t)_{0\le t\le 1}$ be a standard BM,

$$\log \mathbb{E} \exp f(Z) = \sup_v \mathbb{E}\left(f\left(W(1) + \int_0^1 v(s)ds \right) - \frac{1}{2}\int_0^1 v(s)^2 ds \right),$$

where the supremum is taken over all progressively measurable v w.r.t. W.

Using change of variables, $v = mu$,

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A Gaussian inequality

Theorem (Auffinger-Chen '14)

Let $m_0, m_1 > 0$ *and* $\theta \in [0,1]$. *If*

$$f((1-\theta)x + \theta y) \le (1-\theta)g(x) + \theta h(y),$$

then

$$\frac{1}{(1-\theta)m_0 + \theta m_1} \log \mathbb{E} \exp((1-\theta)m_0 + \theta m_1) f(Z)$$
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Equivalently, if $F, G, H \ge 0$ *and*

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Derivative of the Parisi functional

Consider the mixed p-spin model with external field h. Let $s = 0$ and $t = 1$. Write

$$\mathcal{P}(\mu) = \log 2 + \max_{u \in D[0,1]} \left(F_\mu(u, h) - \frac{1}{2} \int_0^1 \mu([0, w]) w \xi''(w) dw \right).$$

Idea: Define $f(x) = \max_{y \in A} g(x, y)$. Assume that $\exists y(x)$ such that $f(x) = g(x, y(x))$. Then

$$\begin{aligned} f'(x) &= \partial_x g(x, y(x)) + y'(x) \partial_y g(x, y(x)) \\ &= \partial_x g(x, y(x)). \end{aligned}$$

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Recall $u_\mu^*(w) = \partial_x \Phi_\mu(w, X(w))$, where $(X(w))_{0 \le w \le 1}$ satisfies

$$X(w) = x + \int_0^w \alpha(r)\xi''(r)\partial_x \Phi_\mu(r, X(r))dr + \int_0^w \xi''(r)^{1/2} dB(r).$$

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Let $\mu_0 \in M[0,1]$. For $\mu \in M[0,1]$, define $\mu_\theta = (1-\theta)\mu_0 + \theta\mu$, $\forall \theta \in [0,1]$. Then

$$\frac{d}{d\theta}\mathcal{P}(\mu_\theta)\Big|_{\theta=0} = \frac{1}{2}\int_0^1 \xi''(r)(\mu([0,r]) - \mu_0([0,r]))(\mathbb{E}u_{\mu_0}^*(r)^2 - r)dr.$$

TFAE:

- *μ_0 is the Parisi measure.*
- *$\frac{d}{d\theta}\mathcal{P}(\mu_\theta)\big|_{\theta=0} \ge 0$ for all $\mu \in \mathcal{M}$.*
- *$\frac{d}{d\theta}\mathcal{P}(\mu_\theta)\big|_{\theta=0} \ge 0$ for all $\mu = \delta_q$ with $q \in [0,1]$.*

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First conclusion

Theorem (Chen '14)

For all q in the support of μ_P,

$$\mathbb{E}\partial_x \Phi_{\mu_P}(q, X(q))^2 = q,$$
$$\xi''(q)\mathbb{E}\partial_{xx}\Phi_{\mu_P}(q, X(q))^2 \leq 1,$$

where for all $s \in [0,1]$,

$$X(s) = h + \int_0^s \mu_P([0,r])\xi''(r)\partial_x \Phi_{\mu_P}(w, X(w))dw + \int_0^s \xi''(w)^{1/2} dW(w).$$

Assume SK, i.e., $\xi(x) = \beta^2 x^2$. Then

- Toninelli '02: If $\mu_P = \delta_q$ for some $q \in [0,1]$, then

$$\mathbb{E}\tanh^2(\beta z\sqrt{2q} + h) = q,$$
$$\beta^2 \mathbb{E}\frac{1}{\cosh^4(\beta z\sqrt{2q} + h)} \leq 1 \text{ (AT line)}.$$

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Second conclusion

For $k \geq 0$, set

$$M_k[0,1] = \{\text{atomic } \mu \in M[0,1] \text{ with at most } k+1 \text{ atoms}\}$$

Theorem (Chen '14)

If $\inf_{\mu \in M_{k+1}[0,1]} \mathcal{P}(\mu) = \mathcal{P}(\mu_0)$ *for some* $\mu_0 \in M_k[0,1]$, *then* μ_0 *is the Parisi measure.*

New result on positivity of the overlap

Positivity holds in the following situations:

- Strong positivity: mixed even p-spin model and $h \neq 0$. (Talagrand)
- Weak positivity: generic mixed p-spin model, $\beta_p \neq 0, \forall p \geq 2$.

Theorem (Chen '14)

Let $q^* = \min supp \mu_P$. *Suppose that* ξ *is convex on* $[-1,1]$ *and*

$$\frac{\xi''(s)}{\xi''(s) + \xi''(-s)}$$

is nondecreasing for $s \in (0,1]$. *If* ξ *is not even, then*

$$\mathbb{E} G_N^{\otimes 2}((\sigma^1, \sigma^2) : R(\sigma^1, \sigma^2) < q^* - \varepsilon) \leq K \exp\left(-\frac{N}{K}\right)$$

Note: We do not require $h \neq 0$.

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Positivity holds in different nature

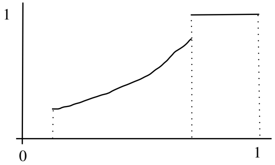

Figure: Positivity of the SK model with $h \neq 0$

Example

$\xi(x) = \beta_{2p}^2 x^{2p} + \beta_{2p+1}^2 x^{2p+1}$ on $[-1, 1]$, where $(2p+1)\beta_{2p+1}^2 < (2p-1)\beta_{2p}^2$.

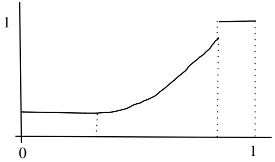

Figure: Positivity of the $(4+5)$-spin model with $h = 0$.