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ON PROPERTIES OF SEMIPOSITIVE CONES AND SIMPLICIAL CONES*

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Abstract. For a given nonsingular $n \times n$ matrix A, the cone $S_A = \{x : Ax \geq 0\}$, and its subcone K_A lying on the positive orthant, called as semipositive cone, are considered. If the interior of the semipositive cone K_A is not empty, then A is named as semipositive matrix. It is known that K_A is a proper polyhedral cone. In this paper, it is proved that S_A is a simplicial cone and properties of its extremals are analyzed. An one-one relation between simplicial cones and invertible matrices is established. For a proper cone K in $\mathbb{R}^n$, $\pi(K)$ denotes the collection of $n \times n$ matrices that leave K invariant. For a given minimally semipositive matrix (no column-deleted submatrix is semipositive) A, it is shown that the invariant cone $\pi(K_A)$ is a simplicial cone.

Key words. Extremal, Proper cone, Polyhedral cone, Simplicial cone, Semipositive matrix, Minimally semipositive matrix.

AMS subject classifications. 15A09, 15B48.

1. Introduction. An $m \times n$ real matrix A is called a semipositive (SP) matrix if there exists a positive vector x such that Ax is also a positive vector, that is, there exists a positive vector which is mapped to a positive vector under A. Non-negative matrices, M-matrices, P-matrices, and positive definite matrices are some important classes of semipositive matrices. SP matrices first appeared in literature as S-matrices named after Stiemke, due to Fiedler and Pták [5]. Motivated by the role of SP matrices in a variety of theoretical and practical problems, mainly in linear complementarity problems, an extensive research is carried in two directions. One is the algebraic properties and the other is cone theoretic properties of SP matrices, which can be reviewed in [3, 4, 7, 8, 11, 13, 15, 17].

Let $\mathbb{R}^{m\times n}$ denote the all $m \times n$ real matrices. For simplicity, we write $\mathbb{R}^n$ for $\mathbb{R}^{n\times 1}$, and $\mathbb{R}^n_+$ stands for all non-negative vectors in $\mathbb{R}^n$. The sign '$\geq$ ($>$)' represents the entry wise inequality of matrices or vectors. For $A \in \mathbb{R}^{m\times n}$, the aim of this paper is to study various geometric properties of its semipositive cone $K_A = \{x \in \mathbb{R}^n_+ : Ax \in \mathbb{R}^m_+\}$. If the interior of K_A is non-empty, then A is a semipositive matrix. The geometric properties of K_A generate an interest to review the class of semipositive matrices and their applications.

For an $n \times n$ real matrix A, in [15] author proved that K_A is a closed, pointed and solid cone in $\mathbb{R}^n$. In paricular, author provides a precise structure of the cone K_A and shows that K_A is a proper polyhedral cone in $\mathbb{R}^n$. Further the result has been proved for non-square matrix A in [13]. They also studied the possible inclusion of semipositive cone K_A into other K_B, for any two semipositive matrices A and B. In this paper, for any $m \times n$ real matrix A the cone $S_A = \{x \in \mathbb{R}^n : Ax \in \mathbb{R}^m_+\}$ containing K_A, is considered and various properties of the cone parallel to that of K_A are studied. Furthermore, given a cone K with specific properties, an attempt is made to classify the matrices A for which either $K = S_A$, or K_A. Again it is known that any two proper cones $K_1 \subseteq \mathbb{R}^m$ and $K_2 \subseteq \mathbb{R}^n$, generate a proper cone $\pi(K_1, K_2) = \{A \in \mathbb{R}^{m\times n} : AK_1 \subseteq K_2\}$ in $\mathbb{R}^{m\times n}$. If the cones $K_1 = K_2 = K$, then the cone $\pi(K, K) = \pi(K)$ is the collection of matrices that leave

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K invariant. In [12], authors proved that $\pi(K_1, K_2)$ is a proper polyhedral cone, if K_1 and K_2 both are polyhedral cones.

We now describe the paper as follows: Section 2 narrates the definitions and some preliminary results. Section 3 collects cone theoretical results of S_A, for some given non-singular matrices. In case A is either SP or MSP matrix, then properties of the semipositive cone K_A and its extremals are discussed. Furthermore, the relation between K_A and $K_{A^{-1}}$ is established in terms of the number of extremals. For a given polyhedral or simplicial cone K, Section 4 is devoted to examine matrices A for which either $S_A = K$ or the semipositive cone $K_A = K$, and the uniqueness of such matrices. Section 5 contains the properties of the cone $\pi(K_A)$ for given minimally semipositive matrix A. Finally, the results are summarized in Section 6.

2. Notations and preliminaries. Throughout this paper, we consider real matrices. We refer a matrix $A \in \mathbb{R}^{m\times n}$ as non-negative (positive) if all its entries are non-negative (positive) and in that case, we write $A \geq (>)0$. Similarly, we denote a non-negative (positive) vector as $x \geq (>)0$, if all its entries are non-negative (positive). The i-th column vector of the identity matrix is denoted by e_i with size determined by the context.

An $m \times n$ matrix A is called a *semipositive* (SP) matrix if there exists a vector $x \geq 0$ such that $Ax > 0$. As $x \to Ax$ is a continuous mapping, it is equivalent to say that A is SP if there exists a vector $x > 0$ such that $Ax > 0$. The vector x is referred to as a *semipositivity vector* of A. The set of all SP matrices has two disjoint subclasses, namely *minimally semipositive* (MSP) matrices and *redundantly semipositive* (RSP) matrices. For a semipositive matrix A if none of the column deleted submatrix of A is semipositive, then it is known as MSP matrix, otherwise it is called as RSP matrix. We refer to [7, 17, 8, 4, 3] for some basic properties of SP, MSP and RSP matrices.

Some preliminary facts about SP matrices are listed below:

THEOREM 2.1. [7, 8, 17] *If $A \in \mathbb{R}^{m\times n}$ is an SP matrix, following results hold:*

(i) *If $n \geq 2$ and A has a positive (negative) column, then A is RSP.*
(ii) *If A is of full column rank, then the pseudo-inverse $A^{\dagger}$ of A is SP.*
(iii) *If A is MSP, then* $\operatorname{rank}(A) = n$.
(iv) *Square MSP matrices are inverse non-negative matrices and non-square matrices are those matrices having a non-negative left inverse.*

THEOREM 2.2. [17] *Let $A \in \mathbb{R}^{n\times n}$. Then A is MSP if and only if*

$$Ax \geq 0 \;\Rightarrow\; x \geq 0.$$

We now discuss some geometric properties, which will be required in the succeeding sections.

DEFINITION 2.3. [1, 2] For any subset S of $\mathbb{R}^n$, the set *generated by* S consists of all finite non-negative linear combinations of elements of S, and we denote the set by S^G. A set $K \in \mathbb{R}^n$ is said to be a *cone* if $K = S^G$ for some subset S. A subcone of a cone K is any cone contained in K.

A cone $K \in \mathbb{R}^n$ is called *polyhedral cone* if there exists a finite set S such that $K = S^G$. Equivalently, polyhedral cones are the image of the nonnegative orthant under a linear mapping. A set is called *convex*, if it contains the line segment joining any two of its points.

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A convex cone K in $\mathbb{R}^n$ is called

(i) *pointed* if $K \cap (-K) = \{0\}$,
(ii) *solid* if K has non-empty interior,
(iii) *reproducing* if $K - K = \mathbb{R}^n$.

A closed, pointed and solid convex cone is called a *proper cone*. The *dual* of a cone $K \in \mathbb{R}^n$ is denoted by K^* and is defined as

$$K^* = \left\{x \in \mathbb{R}^n : x^T y \geq 0 \text{ for all } y \in K\right\}.$$

A proper cone K always generates a partial order in $\mathbb{R}^n$ via $y \overset{K}{\leq} x$ if and only if $x - y \in K$. A vector x in $\mathbb{R}^n$ is an *extremal* of K in $\mathbb{R}^n$ if $0 \overset{K}{\leq} y \overset{K}{\leq} x$ implies that $y = \alpha x$, for some $\alpha \geq 0$. If x is an extremal vector of cone K, then $\{x\}^G$ is called an *extremal ray* of K, and in that case it can be verified that $x \in \partial K$, the boundary of K. If a cone K has exactly n extremals, then the cone K is called a *simplicial cone*. Equivalently, K is a simplicial cone if $K = B\mathbb{R}^n_+$, for some non-singular matrix B of size n.

Let $K_1 \in \mathbb{R}^n$ and $K_2 \in \mathbb{R}^m$ be two proper cones. Then the set $\pi(K_1, K_2)$ in $\mathbb{R}^{m\times n}$ is defined as

$$\pi(K_1, K_2) = \{A \in \mathbb{R}^{m\times n} : A(K_1) \subseteq K_2\}.$$

It is well known that $\pi(K_1, K_2)$ is a proper cone in $\mathbb{R}^{m\times n}$ (see [1]). In case, $m = n$ and $K_1 = K_2 = K$, for simplicity we write $\pi(K)$ for $\pi(K, K)$.

For a non-zero vector $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, we define

(a) a *hyperplane* as the set $\{x \in \mathbb{R}^n : \; a^T x = b\}$,
(b) a (closed) *halfspace* as the set $\{x \in \mathbb{R}^n : \; a^T x \leq b\}$.

Note that a hyperplane divides the Euclidean space $\mathbb{R}^n$ into two halfspaces.

We now state a few basic results related to cones, which are used in the following sections.

Theorem 2.4. [1] *Let K be a non-empty set in $\mathbb{R}^n$. Then following results hold.*

(i) *K is a polyhedral cone if and only if it is the intersection of finitely many closed half spaces, each containing the origin on its boundary.*
(ii) *If K is a polyhedral cone, then K is a closed convex cone.*
(iii) *A nonempty subset K of $\mathbb{R}^n$ is a polyhedral cone if and only if its dual K^* is a polyhedral cone.*

Theorem 2.5. [1, 16] *A proper cone is generated by its extremals.*

Theorem 2.6. [1] *Let K_1 and K_2 be proper cones, in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then*

$$\pi(K_1, K_2) = \left\{A \in \mathbb{R}^{m\times n} : \; A(K_1 \setminus \{0\}) \subseteq \operatorname{int} K_2\right\}.$$

Definition 2.7. [13, 15] For any $A \in \mathbb{R}^{m\times n}$, the *semipositive cone* of A is denoted by K_A, and is defined as

$$K_A = \left\{x \in \mathbb{R}^n_+ : Ax \in \mathbb{R}^m_+\right\}.$$

Note that K_A is a cone and if $\operatorname{int}(K_A) \neq \emptyset$, then A is a SP matrix and any vector in $\operatorname{int}(K_A)$ is a semipositivity vector of A.

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A few properties of semipositive cones, mainly due to [13], are given below:

THEOREM 2.8. [13, 15] *If $A \in \mathbb{R}^{m\times n}$, then K_A is proper polyhedral cone.*

THEOREM 2.9. [13] *If $A \in \mathbb{R}^{n\times n}$ is invertible, then $K_A = \mathbb{R}^n_+ \cap A^{-1}(\mathbb{R}^n_+)$ and $A(K_A) = K_{A^{-1}}$.*

DEFINITION 2.10. For any $A \in \mathbb{R}^{m\times n}$, define

$$S_A = \left\{x \in \mathbb{R}^n : Ax \in \mathbb{R}^m_+\right\}.$$

Note that S_A is also a cone in $\mathbb{R}^n$, which is not necessarily a proper cone and $K_A = S_A \cap \mathbb{R}^n_+$.

3. Properties of S_A and semipositive cone K_A .

In [13], it is proved that K_A is a polyhedral cone, for a given matrix A. In this section, we study various properties of the cones S_A and K_A in $\mathbb{R}^n$, for a given non-singular matrix $A \in \mathbb{R}^{n\times n}$, and analyze their relation if A is either SP or MSP matrix. We also discuss the extremals of these cones in relevance to the inverse of A.

THEOREM 3.1. *For an invertible matrix $A \in \mathbb{R}^{n\times n}$, S_A is a proper polyhedral cone.*

Proof. As any $z \in \mathbb{R}^n_+$ can be written as $z = A(A^{-1}z) \geq 0$, so $A(S_A) = \mathbb{R}^n_+$, that is, $S_A = A^{-1}(\mathbb{R}^n_+)$, and hence, S_A is a polyhebdral cone, being image of nonnegative orthant under the linear map A^{-1}.

Again A is invertible implies S_A is pointed. The interior of S_A is

$$\mathrm{int}(S_A) = \{x \in \mathbb{R}^n : Ax > 0\}.$$

As $A^{-1}y \in \mathrm{int}(S_A)$ for any positive vector y, so S_A is a solid cone. Thus S_A is a proper polyhedral cone. ∎

THEOREM 3.2. *For an invertible matrix $A \in \mathbb{R}^{n\times n}$, the extremals of the simplicial cone S_A have the form $\alpha_i A^{-1}e_i$ for some $\alpha_i > 0$, $i = 1, 2, \ldots, n$.*

Proof. From the proof of Theorem 3.1, we have that $A(S_A) = \mathbb{R}^n_+$. Since A is an $n \times n$ invertible matrix, S_A is a simplicial cone.

Choose n-distinct vectors $x_1, x_2, \ldots, x_n$ in S_A, whose image under A lies in different axes of $\mathbb{R}^n_+$. Theorem 3.1 assures the existence of such vectors. Without loss of generality, we assume that for each i, Ax_i lies in the ith-axis, that is, $Ax_i = \alpha_i e_i$, for some positive scalar α_i.

Let $0 \overset{S_A}{\leq} y \overset{S_A}{\leq} x_i$, so that $y \in S_A$ and $x_i - y \in S_A$. As $A(S_A) = \mathbb{R}^n_+$, $0 \leq Ay \leq Ax_i = \alpha_i e_i$. Hence, $Ay = \beta e_i = \gamma_i Ax_i$ and $\gamma_i = \frac{\beta}{\alpha_i} \geq 0$. Since A is invertible, $y = \gamma_i x_i$. Therefore, x_i is an extremal of S_A.

We now prove that $x_1, x_2, \ldots, x_n$ are the only extremals of S_A. Let $x \in S_A$ for which Ax has at least two positive entries. Write $Ax = b_{n_1}e_{n_1} + \cdots + b_{n_k}e_{n_k}$, where $k \geq 2$ and each b_{n_j} $(1 \leq j \leq k)$ is a positive scalar with $1 \leq n_j \leq n$. Choose y such that $Ay = b_i e_i$. Then $0 \overset{S_A}{\leq} y \overset{S_A}{\leq} x$ but y is not expressible as a positive scalar multiple of x. Thus, x is not an extremal of S_A. Thus, the result follows. ∎

The following example illustrates the fact that the semipositive cone K_A may not be simplicial, for an invertible matrix A.

EXAMPLE 3.3. Consider the invertible matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{5} & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

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It can be verified that $[1,0,\frac{1}{2}]^T,\ [1,0,0]^T,\ [0,1,0]^T$ and $[0,1,\frac{1}{5}]^T$ are extremals of K_A, and so K_A is not a simplicial cone.

THEOREM 3.4. *For an invertible SP matrix $A \in \mathbb{R}^{n\times n}$, if x is an extremal of K_A, then Ax is an extremal of $K_{A^{-1}}$. Moreover, if p is an extremal of $K_{A^{-1}}$, then $A^{-1}p$ is an extremal of K_A.*

Proof. From Theorem 2.9, we have $A(K_A) = K_{A^{-1}} = K$ (say), and hence, $Ax \in K$. Let $0 \overset{K}{\leq} y \overset{K}{\leq} Ax$. Then $y \in K$, and so we can choose $z \in K_A$ with $y = Az$. Again $Ax - y \in K$ implies that $A(x-z) \in K$, that is, $x - z \in K_A$. Thus, $0 \overset{K_A}{\leq} z \overset{K_A}{\leq} x$, and therefore, $z = cx$ for some $c > 0$, or equivalently, $y = cAx$. Hence Ax is an extremal of $K = K_{A^{-1}}$.

Let $0 \overset{K_A}{\leq} q \overset{K_A}{\leq} A^{-1}p$. Then $A^{-1}p - q \in K_A$, that is, $p - Aq \in K$, so that $0 \overset{K}{\leq} Aq \overset{K}{\leq} p$. Therefore, $p = \alpha Aq$ for some $\alpha > 0$, or equivalently, $q = \frac{1}{\alpha}A^{-1}p$. Hence $A^{-1}p$ is an extremal of K_A. $\square$

COROLLARY 3.5. *For an invertible SP matrix A, the number of extremals of K_A and $K_{A^{-1}}$ are same.*

In Example 3.3, we observe that K_A may not be a simplicial cone for an invertible matrix A. We consider the question as to whether there is a subclass of invertible matrices, for which K_A is a simplicial cone. This question is answered in the next theorem.

THEOREM 3.6. *Any matrix $A \in \mathbb{R}^{n\times n}$ is minimally semipositive if and only if $K_A = S_A$ and K_A is a simplicial cone.*

Proof. Let A be a minimally semipositive matrix. Then $A^{-1} \geq 0$, and hence, for $y \in S_A$, $y = A^{-1}(Ay) \geq 0$, that is, $S_A = K_A$. So K_A is a simplicial cone by Theorem 3.2.

Conversely, assume that $K_A = S_A$ is a simplicial cone and so K_A is a solid cone. Hence A is a semipositive matrix. Let $x \in \mathbb{R}^n$ and $Ax \geq 0$. Then $x \in S_A = K_A$, and thus, $x \geq 0$, which shows that A is minimally semipositive by Theorem 2.2. $\square$

Theorem 3.6 leads to the following conclusion.

COROLLARY 3.7. *Let $A \in \mathbb{R}^{n\times n}$. Then*

(i) *A is an SP matrix if and only if $\operatorname{int} S_A$ intersects $\mathbb{R}^n_+$, and*
(ii) *A is an MSP matrix if and only if S_A lies entirely in $\mathbb{R}^n_+$.*

COROLLARY 3.8. *For a square RSP matrix A, there exists a vector x with at least one negative entry such that $Ax > 0$.*

REMARK 3.9. In Theorem 3.6, the condition $K_A = S_A$ cannot be dropped. Take

$$A = \begin{bmatrix} 0 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 2 & -1 \end{bmatrix}.$$

Note that $K_A \neq S_A$ as $[1,1,-1]^T \in S_A$. It can be checked that K_A is simplicial with extremals $[1,0,0]^T$, $[0,1,0]^T$ and $[0,1,1]^T$, but A is not a minimally semipositive matrix being a singular matrix.

In Theorem 3.6, we notice that K_A is a simplicial cone if A is an MSP matrix, and we now find out the extremals K_A.

THEOREM 3.10. *For a square minimally semipositive matrix A, extremals of K_A are the columns of A^{-1}.*

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Proof. As A is MSP, so $A^{-1} \geq 0$ by Theorem 2.1, and hence, $K_{A^{-1}} = \mathbb{R}^n_+$, which implies that $\{e_1, e_2, \ldots, e_n\}$ is the set of the extremals of $K_{A^{-1}}$. Since $A(K_A) = K_{A^{-1}}$, so by Theorem 3.4, $\{A^{-1}e_1, A^{-1}e_2, \ldots, A^{-1}e_n\}$ is the set of extremals of K_A, and thus, the result follows. ∎

4. Simplicial cones and related matrices. In Section 3, we studied the geometric structure of S_A and K_A for a given matrix A with specific property. In this section, we explore the possibilities if the converse is also true. More precisely, given a simplicial or polyhedral cone K, our aim is to find out the classes of matrices A for which either $K = S_A$ or $K = K_A$. We also discuss the uniqueness of such matrices.

In Theorem 3.2, we have seen that S_A is a simplicial cone, for invertible matrix A. Following theorem gives an affirmative answer if the converse of Theorem 3.2 is true.

Theorem 4.1. *Let K be a simplicial cone in $\mathbb{R}^n$. We then have:*

(a) *There exists an invertible matrix A such that $S_A = K$.*
(b) *If K contains a positive vector, then there exists an invertible SP matrix A such that $K = S_A$.*
(c) *If $K \subseteq \mathbb{R}^n_+$, then there exists a MSP matrix A such that $K = K_A$.*

Proof. (a) As K is simplicial, so by definition we can write $K = B\mathbb{R}^n_+$, where B is invertible and the columns of B are extremals of K. Taking $B = A^{-1}$, we get $S_A = K$.

(b) Let p be a positive vector in K. By Part (a), there exists an invertible matrix A such that $S_A = K$. It suffices to show that A is SP.

Since p is positive, we can always choose an open ball $B(p, \epsilon)$ contains only positive vectors. As S_A is a closed, so there exists q such that $q \in B(p, \epsilon) \cap \operatorname{int} S_A$. So $q > 0$ and $Aq > 0$.

(c) Choose an invertible SP matrix A such that $S_A = K$, which is possible by (b). As K lies in the positive orthant, so $K = S_A = K_A$. We only need to show that A is an MSP matrix.

Let $y \in \mathbb{R}^n$ and $Ay \geq 0$. Then $y \in S_A = K$, and hence, by our assumption, $y \geq 0$. So, A is an MSP matrix by Theorem 2.2. ∎

In Theorem 4.1, we have seen that any simplicial cone K must have the form $K = S_A$, for some invertible matrix A. Following theorem provides the uniqueness of such matrix A up to post-multiplying by a positive diagonal matrix.

Theorem 4.2. *Let K be a simplicial cone in $\mathbb{R}^n$, and let A and B are two invertible matrices such that $K = S_A = S_B$. Then there exists a positive diagonal matrix D such that $A = BD$.*

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be the extremals of K. As $K = S_A = S_B$, from Theorem 3.2, we have positive scalar z_i and w_j such that

$$x_i = z_i A^{-1} e_i = w_j B^{-1} e_j, \qquad i, j = 1, 2, \ldots, n,$$

or

$$z_i B e_i = w_j A e_j, \qquad i, j = 1, 2, \ldots, n. \tag{4.1}$$

Choose $D_1 = \operatorname{diag}(z_1, z_2, \ldots, z_n)$ and $D_2 = \operatorname{diag}(w_1, w_2, \ldots, w_n)$. The equation (4.1), can be written $BD_1P = AD_2$, for some permutation matrix P. As D_1 and D_2 are positive diagonal matrices, so conclusion follows by considering the positive diagonal matrix $D = D_1PD_2^{-1}$. ∎

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THEOREM 4.3. *For every SP matrix A, there exists a MSP matrix B such that $K_B \subseteq K_A$.*

Proof. Let A be any SP matrix and $p > 0$ be such that $Ap > 0$. Note that $p \in \operatorname{int} K_A$ and so we can find $\epsilon > 0$ such that $B(p, \epsilon) \subseteq \operatorname{int} K_A$. Choose n linearly independent vectors $y_1, y_2, \ldots, y_n$ (say) in $B(p, \epsilon)$. Set $\widetilde{A}^{-1} = [y_1, y_2, \ldots, y_n]$ so that by Theorem 3.2, $S_{\widetilde{A}}$ is a simplicial cone with $\alpha_1 y_1, \alpha_2 y_2, \ldots, \alpha_n y_n$ as its extremals with $\alpha_i > 0$, for all i. Hence by Theorem 4.1(c) there exists a MSP matrix B such that $K_B = S_{\widetilde{A}} = \{\alpha_1 y_1, \alpha_2 y_2, \ldots, \alpha_n y_n\}^G$. As $B(p, \epsilon) \subseteq \operatorname{int} K_A$, so $K_B \subseteq K_A$. ◻

The above theorem together with Theorem 3.6 reveals that every solid semipositive cone contains a simplicial cone.

THEOREM 4.4. *Let K be any proper polyhedral cone in $\mathbb{R}^n$ such that one of its extremals lie in the positive orthant. Then there exists a minimally semipositive matrix A such that K_A is a subcone of K.*

Proof. Let $x \in \operatorname{int} \mathbb{R}^n_+$ be an extremal of K. Choose an open ball $B(x, \epsilon) \subseteq \operatorname{int} \mathbb{R}^n_+$. As $x \in \partial K$, $B(x, \epsilon) \cap \operatorname{int} K \neq \phi$. Hence,

$$\operatorname{int} \mathbb{R}^n_+ \cap \operatorname{int} K \neq \phi.$$

Now let $V = K \cap \mathbb{R}^n_+$. Since K and $\mathbb{R}^n_+$ both are proper polyhedral cone and $\operatorname{int} V = \operatorname{int} \mathbb{R}^n_+ \cap \operatorname{int} K \neq \phi$, V is also a proper polyhedral cone. Hence $V = \{x_1, \ldots, x_k\}^G$. Since $V - V = \mathbb{R}^n$, so $\mathbb{R}^n = \operatorname{span}\{x_1, \ldots, x_k\}$. So, $k \geq n$ and we choose n-linearly independent extremals, say $\{x_1, \ldots, x_n\}$. Consider the simplicial subcone $\widetilde{K} = \{x_1, \ldots, x_n\}^G$ of V and hence of K. Hence by Theorem 4.1 there exists a minimally semipositive matrix A such that $K_A = \widetilde{K}$. ◻

From the above theorem, we can conclude that every polyhedral cone contains a simplicial cone.

5. Matrices leaving a cone invariant: $\pi(K)$**.** In [12], authors proved that for any two proper cones K_1 and K_2 in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, $\pi(K_1, K_2)$ is a proper cone. Also $\pi(K_1, K_2)$ is a proper polyhedral cone if so are K_1 and K_2. In this section, we discuss the geometric structure of $\pi(K_A)$, for a given MSP matrix A.

THEOREM 5.1. *If A is square MSP matrix, then $\pi(K_A)$ is a simplicial cone.*

Proof. Since A is a MSP matrix, K_A is a simplicial cone with $A^{-1}e_1, \ldots, A^{-1}e_n$ as extremals of K_A. For $i = 1, 2, \ldots, n$, set $\tilde{a_i} = A^{-1}e_i \geq 0$, as A is MSP. If b_i^T denotes the ith row of A, then

$$b_i^T \tilde{a_j} = \delta_{ij}.$$

For $p, q \in \{1, 2, \ldots, n\}$, we define B_{pq} (as in Lemma 5 of [12]) by,

$$B_{pq} = \tilde{a_p} b_q^T.$$

Note that $\mathcal{B} = \{B_{pq} : \ 1 \leq p, q \leq n\}$ forms a basis of $\mathbb{R}^{n \times n}$. Let $x \in K_A$ so that $x \geq 0$ and $Ax \geq 0$. Now

$$AB_{pq}x = A\tilde{a_p} b_q^T x = (b_q^T x) e_p \geq 0. \tag{5.2}$$

Hence $B_{pq} \in \pi(S_A) = \pi(K_A)$, by Theorem 3.6. We now prove that $\pi(K_A)$ is simplicial by showing that the extremals of $\pi(K_A)$ are the elements of $\mathcal{B}$.

Let $Y \in \pi(K_A)$. Choose scalars d_{rl} such that

$$Y = \sum_{r,l=1}^{n} d_{rl} B_{rl}. \tag{5.3}$$

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Now for any $x \in K_A$, we have $Yx \in K_A$ so that $Yx \geq 0$ and $A(Yx) \geq 0$. Then

$$b_i^T Y x \geq 0 \quad \text{for } i = 1, 2, \ldots, n. \tag{5.4}$$

Note that $\tilde{a_j} \in K_A$. Taking $x = \tilde{a_j}$ in equation (5.4) we obtain $b_i^T Y \tilde{a_j} \geq 0$, for $i, j = 1, 2, \ldots, n$. Hence equation (5.3) implies that

$$\sum_{r,l=1}^{n} d_{rl} b_i^T B_{rl} \tilde{a_j} \geq 0,$$

or

$$\sum_{r,l=1}^{n} d_{rl} b_i^T \tilde{a_r} b_l^T \tilde{a_j} \geq 0,$$

or

$$d_{ij} \geq 0.$$

So, from (5.3) we conclude that any $Y \in \pi(K_A)$, can be written as non-negative linear combination of elements of $\mathcal{B}$.

For simplicity, write $K = \pi(K_A)$. Let $0 \overset{K}{\leq} Y \overset{K}{\leq} B_{pq}$. Then

$$Y = \sum_{r,l=1}^{n} d_{rl} B_{rl} \quad \text{and} \quad B_{pq} - Y = \sum_{r,l=1}^{n} f_{rl} B_{rl}$$

for some $d_{rl}, f_{rl} \geq 0$ and $r, l \in \{1, 2, \ldots, n\}$. Now $B_{pq} = \sum_{r,l=1}^{n} (d_{rl} + f_{rl}) B_{rl}$ implies that $d_{rl} = f_{rl} = 0$, for $r \neq p$ and $l \neq q$. So $Y = d_{pq} B_{pq}$ and $d_{pq} \geq 0$, or equivalently, we say that elements of B_{pq} are extremals of $\pi(K_A)$. As each element in $\pi(K_A)$ is a non-negative linear combination of the extremals vectors in $\mathcal{B}$, so $\mathcal{B}$ is the set all extremals of $\pi(K_A)$. Therefore $\pi(K_A)$ is simplicial. $\square$

THEOREM 5.2. *If A is an SP matrix, then any matrix $B \in \operatorname{int} \pi(K_A)$ is also SP and $K_A \subseteq K_B$.*

Proof. Let u be a semipositivity vector of A. Note $u \in K_A \setminus \{0\}$. As $B \in \operatorname{int} \pi(K_A)$, from Theorem 2.6, we have $Bu \in \operatorname{int}(K_A)$, and hence, $Bu > 0$. This shows that B is SP and $u \in K_B$. Hence $K_A \subseteq K_B$. $\square$

6. Conclusion. In this paper, we have proved that the cone S_A is a simplicial cone, for a given nonsingular matrix A. We have also shown that S_A is a semipositive cone if A is a minimally semipositive matrix and the extremals of K_A are nothing but the columns of A^{-1}. Next, for a given simplicial cone K, we proved the existence of a nonsingular matrix A for which $S_A = K$ and the uniqueness of such a matrix is also discussed. Lastly, for a given matrix A, we have considered the cone $\pi(K_A)$ of all matrices that leave the semipositive cone K_A invariant. In particular, we have proved that if the matrix A is minimally semipositive matrix, then cone $\pi(K_A)$ must be a simplicial cone.

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