On Propositional QBF Expansions and Q-Resolution

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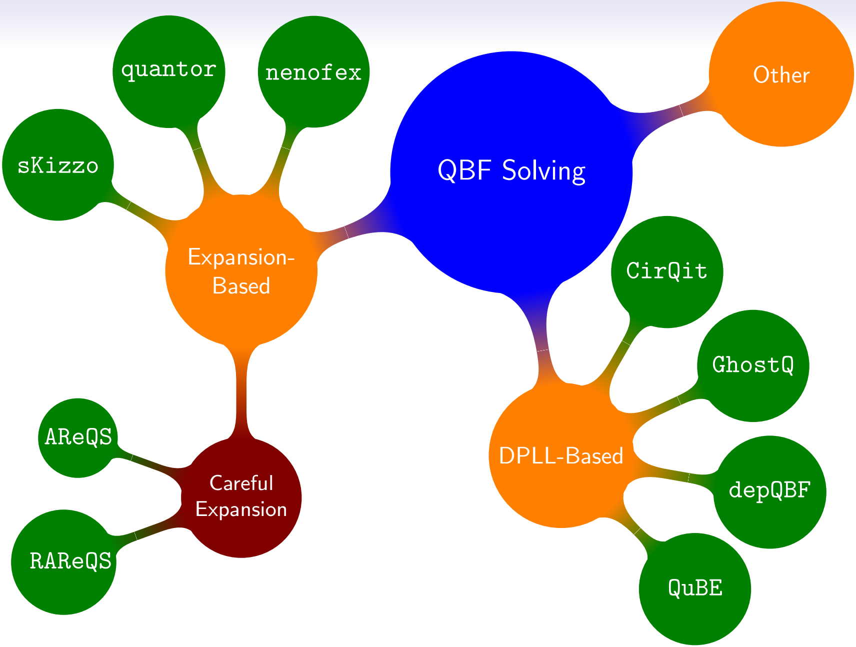
QBF Solving
Other
Expansion-Based
quantor
nenofex
sKizzo
Careful Expansion
AReQS
RAReQS
DPLL-Based
CirQit
GhostQ
depQBF
QuBE

Quantified Boolean Formula (QBF)

- an extension of SAT with quantifiers

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$$\forall \mathcal{U}_1 \exists \mathcal{E}_2 \ldots \forall \mathcal{U}_{2N-1} \exists \mathcal{E}_{2N} .\ \phi$$

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Solving

- DPLL — Q-Resolution (`QuBE`, `depqbf`, etc.)
- Expansion — ?? (`Quantor`, `sKizzo`, `Nenofex`)
  - "Careful" expansion (`AReQS`,`RAReQS`)

Q-resolution

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Tautologous resolvents are generally unsound!

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Fresh variables in order to keep prenex form

$$\exists e_1 \forall u_2 \exists e_3.\ (\bar{e}_1 \vee e_3) \wedge (\bar{e}_3 \vee e_1) \wedge (u_2 \vee e_3) \wedge (\bar{u}_2 \vee \bar{e}_3)$$

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$$\begin{aligned}\exists e_1 e_3^{u_2/0} e_3^{u_2/1}.\quad & (\bar{e}_1 \vee e_3^{u_2/0}) \wedge (\bar{e}_3^{u_2/0} \vee e_1) \wedge \\ & (\bar{e}_1 \vee e_3^{u_2/1}) \wedge (\bar{e}_3^{u_2/1} \vee e_1) \wedge \\ & e_3^{u_2/0} \wedge \\ & \bar{e}_3^{u_2/1}\end{aligned}$$

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Only certain values may be needed:

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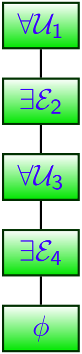
$$\exists e^{u/0} e^{u/1}.\ e^{u/0} \wedge \bar{e}^{u/0} \wedge e^{u/1} \quad \text{(false full)}$$
$$\exists e^{u/0}.\ e^{u/0} \wedge \bar{e}^{u/0} \quad \text{(false partial)}$$

Recursive Partial Expansion



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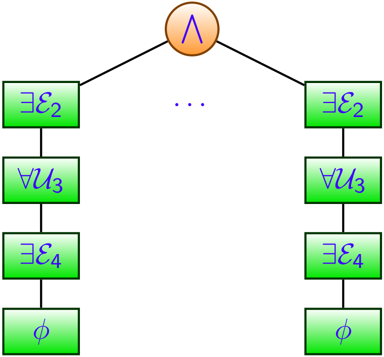

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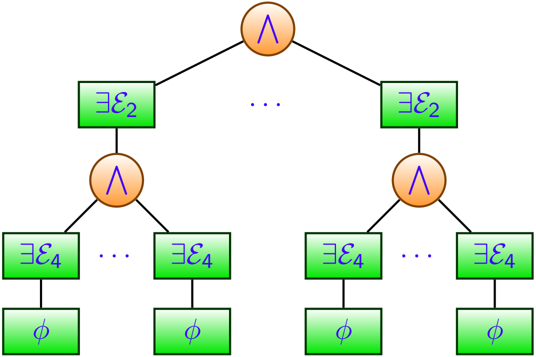

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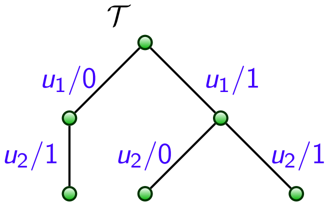
(1) Expansion tree $\mathcal{T}$: for each block of variables it tells us how to expand it.



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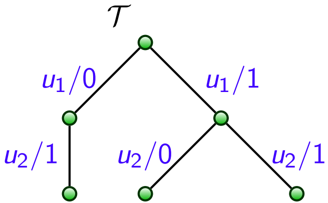

(2) Propositional Resolution Refutation π of expansion resulting from the expansion tree $\mathcal{T}$.

Performing Expansion

- For a clause $C = e_i \vee u \vee e_k$, for $\tau = \tau_1, \ldots, \tau_n$

$$
\begin{array}{lcll}
\mathscr{E}(\tau_1, \ldots, \tau_n,\ C) & = & e_i^{\tau_1, \ldots, \tau_{i/2}} \vee e_k^{\tau_1, \ldots, \tau_{k/2}} & \text{if } u[\tau] = 0 \\
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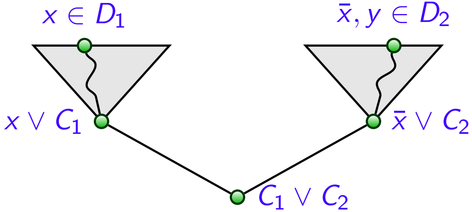
- For an expansion tree $\mathcal{T}$ and a matrix ϕ consider the union of clauses $\mathscr{E}(\tau, C)$ for all branches $\tau \in \mathcal{T}$ and $C \in \phi$.

From Tree Q-resolution to ∀Exp+Res



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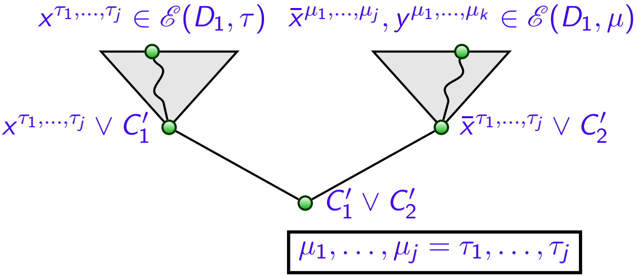

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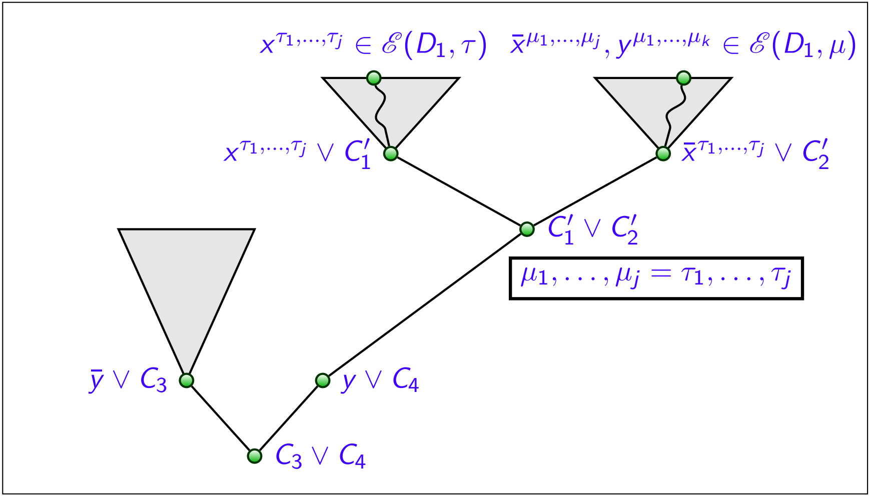

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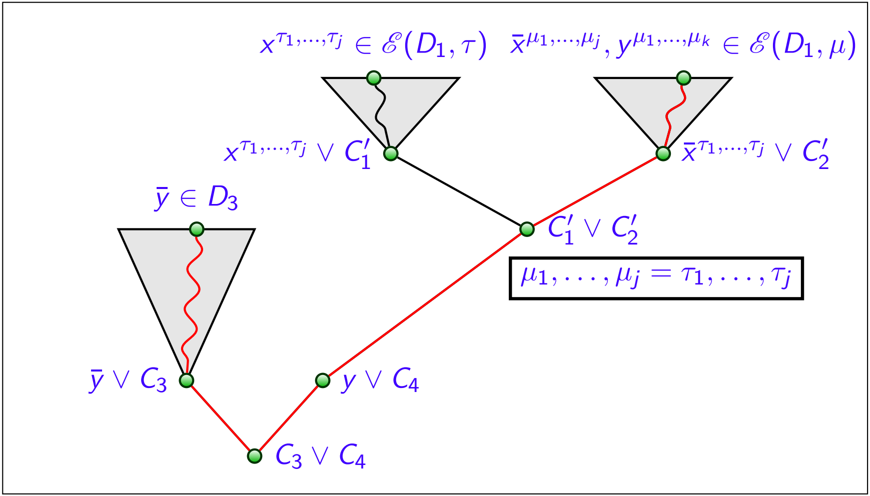

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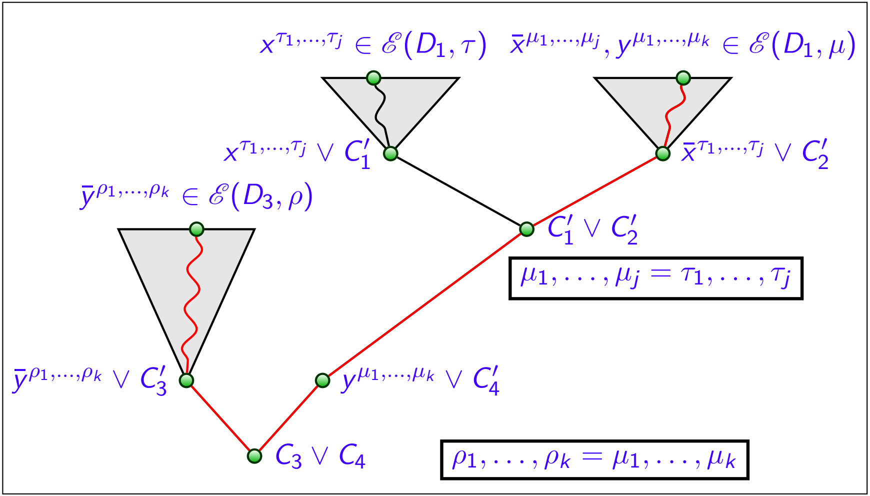
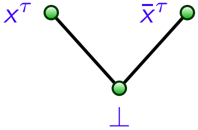


From Expansion Refutation to Q-resolution

- Why don't we just revert substitutions?



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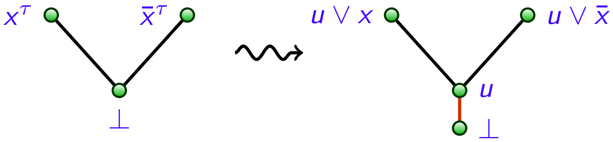

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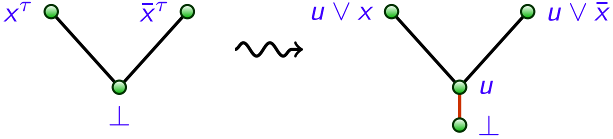

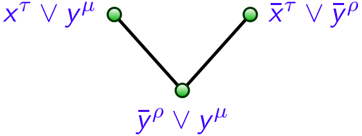

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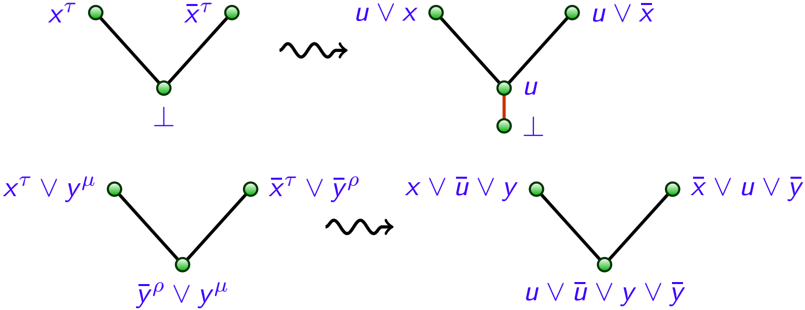

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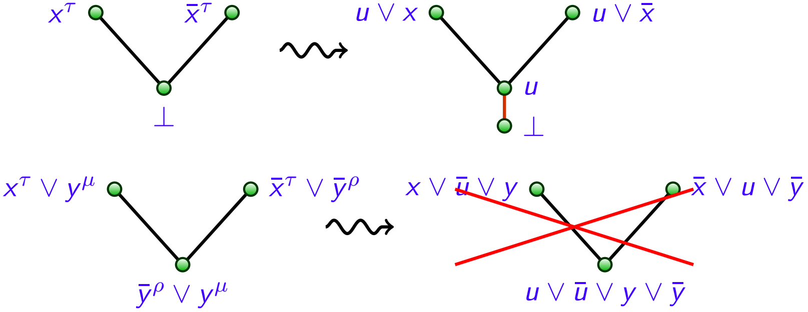

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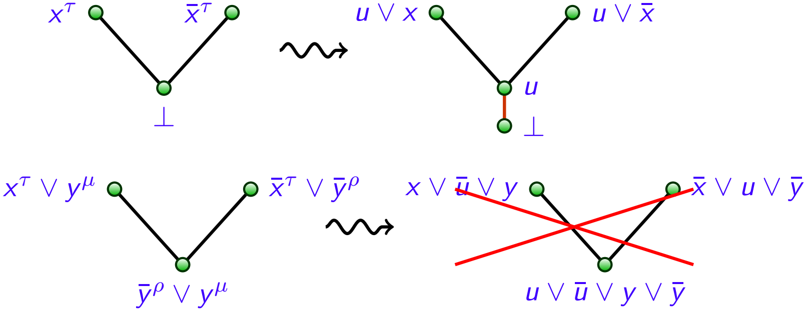
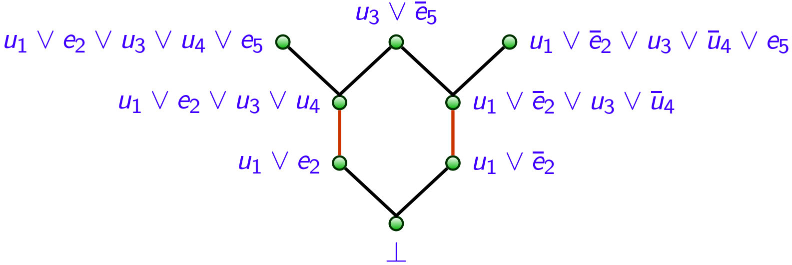


- Such a construction is possible if propositional resolution follows the order of the prefix, starting with the innermost levels.

What is hard for ∀Exp+Res



What Seems to Be Hard for Q-resolution

$\mathbf{x_i} \vee \mathbf{z} \vee \mathbf{C_i^1}$	$\mathbf{\bar{x}_i} \vee \mathbf{\bar{z}} \vee \mathbf{C_i^2}$
$x_1 \vee z \vee \bar{y}_1$	$\bar{x}_1 \vee \bar{z} \vee \bar{y}_1$
$x_2 \vee z \vee y_1$	$\bar{x}_2 \vee \bar{z} \vee \bar{y}_1$
$x_3 \vee z \vee \bar{y}_1$	$\bar{x}_3 \vee \bar{z} \vee y_1$
$x_4 \vee z \vee y_1$	$\bar{x}_4 \vee \bar{z} \vee y_1$

$\mathbf{z/0}$	$\mathbf{z/1}$
$x_1 \vee \bar{y}_1^{z/0}$	$\bar{x}_1 \vee \bar{y}_1^{z/1}$
$x_2 \vee y_1^{z/0}$	$\bar{x}_2 \vee \bar{y}_1^{z/1}$
$x_3 \vee \bar{y}_1^{z/0}$	$\bar{x}_3 \vee y_1^{z/1}$
$x_4 \vee y_1^{z/0}$	$\bar{x}_4 \vee y_1^{z/1}$

Figure : Example formula for $n = 1$

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- Such system simulates tree Q-resolution.
- Q-resolution can simulate a fragment of this system, when variables are resolved "inside out".
- We conjecture that the systems are incomparable. Showing such is the subject of future work.

Thank you for your attention!

Questions?