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On Ψ Bounded Solutions for a Nonlinear Lyapunov Matrix Differential Equation on ℝ

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Dedicated to Professor Mihail Megan on the occasion of his 70th birthday

Abstract. Using Banach and Schauder - Tychonoff fixed point theorems, existence results for a nonlinear Lyapunov matrix differential equation on ℝ are given. The obtained results generalize and extend the results from [5] and [18].

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1 Introduction

The purpose of present paper is to provide sufficient conditions for the existence and uniqueness and existence of at least one $\Psi-$ bounded solution for the nonlinear Lyapunov matrix differential equation on ℝ

$$Z' = A(t)Z + ZB(t) + C(t) + F(t, Z) \tag{1.1}$$

with the help of Banach and Schauder-Tychonoff fixed point theorems.
We first establish two results in connection with the existence and uniqueness and existence of at least one $\Psi-$ bounded solution for the nonlinear matrix differential equation on ℝ of the form

$$Z' = A(t)Z + C(t) + F(t, Z). \tag{1.2}$$

Second, using vectorization operator and Kronecker product of matrices, we treat the same problems for the nonlinear Lyapunov matrix differential equation on $\mathbb{R}$ of the form (1.1).
History of problem. A classical result in connection with boundedness of solutions of systems of ordinary differential equations

$$x' = A(t)x + c(t) + f(t,x) \tag{1.3}$$

was given by Coppel [5] (Chapter V, section 2, Theorem 4). The problem of $\Psi-$ bounded solutions for systems of ordinary differential equations has been studied by many authors: [2], [3], [4], [8], [9], [12], [13], [14], [18] and for Lyapunov matrix differential equations, [6], [7], [10], [11], [16], [17].
The introduction of the matrix function Ψ in the study of solutions permits to obtain a mixed asymptotic behavior of the components of the solutions of the above equations.

2 Preliminaries

In this section we present some basic notations, definitions, hypotheses and results which are useful later on.
Let $\mathbb{R}^d$ be the Euclidean $d-$ dimensional space. For $x = (x_1, x_2, ..., x_d)^T \in \mathbb{R}^d$, let $\| x \| = \max\{| x_1|, | x_2|, ..., | x_d|\}$ be the norm of x (here, ^{T} denotes transpose).
Let $\mathbb{M}_{d\times d}$ be the linear space of all real $d \times d$ matrices.
For $A = (a_{ij}) \in \mathbb{M}_{d\times d}$, we define the norm $| A |$ by $| A | = \sup\limits_{\|x\|\leq 1} \| Ax \|$. It is well-known that $| A | = \max\limits_{1\leq i\leq d}\{\sum\limits_{j=1}^{d} | a_{ij}|\}$.
By a solution of the equation (1.1) we mean a continuous differentiable $d \times d$ matrix function satisfying the equation (1.1) for all $t \in \mathbb{R}$.
In equation (1.1) we assume that the coefficients are continuous functions.
Let $\Psi_i : \mathbb{R}_+ \longrightarrow (0, \infty)$, $i = 1, 2, ..., d$, be continuous functions and

$$\Psi = \text{diag}\ [\Psi_1, \Psi_2, \cdots \Psi_d].$$

A matrix P is said to be a projection if $P^2 = P$.

Definition 2.1. ([12], [8]) *A function $\varphi : R \longrightarrow R^d$ is said to be $\Psi-$ bounded on R if $\Psi(t)\varphi(t)$ is bounded on R (i.e. there exists $m > 0$ such that $\| \Psi(t)\varphi(t) \| \leq m$, for all $t \in R$).*
Otherwise, is said that the function φ is *$\Psi-$ unbounded on R.*

Definition 2.2. ([10]) *A matrix function* $M : \mathbb{R} \longrightarrow \mathbb{M}_{d\times d}$ *is said to be* $\Psi-$ *bounded on* R *if the matrix function* $\Psi(t)M(t)$ *is bounded on* R *(i.e. there exists* $m > 0$ *such that* $\mid \Psi(t)M(t) \mid \leq m$*, for all* $t \in R$*).*
Otherwise, is said that the matrix function M is $\Psi-$ *unbounded on* R.

We now describe a few definitions and properties in connection with Kronecker product of matrices and vectorization operator.

Definition 2.3. *([1]) Let* $A = (a_{ij}) \in M_{m\times n}$ *and* $B = (b_{ij}) \in M_{p\times q}$*. The Kronecker product of* A *and* B*, written* $A\otimes B$*, is defined to be the partitioned matrix*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Obviously, $A \otimes B \in \mathbb{M}_{mp\times nq}$.

We next show the important rules of calculation of the Kronecker product.

Lemma 2.1. *([1]) The Kronecker product has the following properties and rules, provided that the dimension of the matrices are such that the various expressions exist:*

1). $A \otimes (B \otimes C) = (A \otimes B) \otimes C$;
2). $(A \otimes B)^T = A^T \otimes B^T$;
3). $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$;
4). $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;
5). $A \otimes (B + C) = A \otimes B + A \otimes C$;
6). $(A + B) \otimes C = A \otimes C + B \otimes C$;
7). $I_d \otimes A = \begin{pmatrix} A & O & \cdots & O \\ O & A & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & A \end{pmatrix}$;
8).$(A(t) \otimes B(t))' = A'(t) \otimes B(t) + A(t) \otimes B'(t)$; $('$ *denotes the derivative* $\frac{d}{dt}$*).*

Proof. See in [1]. □

Definition 2.4. ([15]) *The application* $\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow \mathbb{R}^{mn}$*, defined by*

$$\mathcal{V}ec(A) = (a_{11}, a_{21}, \cdots, a_{m1}, a_{12}, a_{22}, \cdots, a_{m2}, \cdots, a_{1n}, a_{2n}, \cdots, a_{mn})^T,$$

where $A = (a_{ij}) \in \mathbb{M}_{m\times n}$*, is called the vectorization operator.*

Lemma 2.2. *([10]) The vectorization operator*

$$\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow \mathbb{R}^{mn},\ A \longrightarrow \mathcal{V}ec(A),$$

is a linear and one-to-one operator. In addition, $\mathcal{V}ec$ and $\mathcal{V}ec^{-1}$ are continuous operators.

Proof. See Lemma 2, [10]. □

Remark 2.1. *Obviously, a function $F : \mathbb{R} \longrightarrow \mathbb{M}_{d\times d}$ is a continuous (differentiable) matrix function on R if and only if the function $f : R \longrightarrow \mathbb{R}^{d^2}$, defined by $f(t) = \mathcal{V}ec(F(t))$, is a continuous (differentiable) vector function on R.*

We recall that the vectorization operator $\mathcal{V}ec$ has the following properties as concerns the calculations.

Lemma 2.3. *([15]) If $A, B, M \in \mathbb{M}_{n\times n}$, then*

1). $\mathcal{V}ec(AMB) = (B^T \otimes A) \cdot \mathcal{V}ec(M)$;
2). $\mathcal{V}ec(MB) = (B^T \otimes I_n) \cdot \mathcal{V}ec(M)$;
3). $\mathcal{V}ec(AM) = (I_n \otimes A) \cdot \mathcal{V}ec(M)$;
4). $\mathcal{V}ec(AM) = (M^T \otimes A) \cdot \mathcal{V}ec(I_n)$.

Proof. See [15], Chapter 2. □

The following lemmas play a vital role in the proofs of main results of present paper.

Lemma 2.4. *([10]) The matrix function $Z(t)$ is a solution on R of (1.1) if and only if the vector function $z(t) = \mathcal{V}ec(Z(t))$ is a solution of the differential system*

$$z' = \left(I_d \otimes A(t) + B^T(t) \otimes I_d\right) z + c(t) + f(t, z), \tag{2.1}$$

where $c(t) = \mathcal{V}ec(C(t))$ and $f(t, z) = \mathcal{V}ec\left(F(t, Z)\right)$, on the same interval R.

Proof. See Lemma 7, [10]. □

Definition 2.5. *([10]) The above system (2.1) is called "corresponding Kronecker product system associated with (1.1)".*

Lemma 2.5. *([10]). For every matrix function $M : R \longrightarrow \mathbb{M}_{d\times d}$,*

$$\frac{1}{d} \mid \Psi(t)M(t) \mid \leq \parallel \left(I_d \otimes \Psi(t)\right) \mathcal{V}ec\left(M(t)\right) \parallel_{\mathbb{R}^{d^2}} \leq \mid \Psi(t)M(t) \mid, \forall t \geq 0. \tag{2.2}$$

Proof. See Lemma 4, [10]. □

Lemma 2.6. *The solutions of (1.1) are* $\Psi-$ *bounded on* R *if and only if the solutions of the differential system (2.1) are* $I_d \otimes \Psi-$ *bounded on* R.

Proof. It results from above Lemma (2.5). □

Lemma 2.7. *([10]). Let X(t) and Y(t) be a fundamental matrices for the equations*

$$Z' = A(t)Z \tag{2.3}$$

$$Z' = ZB(t) \tag{2.4}$$

respectively.
Then, the matrix $Z(t) = Y^T(t) \otimes X(t)$ *is a fundamental matrix for the linear differential system*

$$z' = \left(I_d \otimes A(t) + B^T(t) \otimes I_d\right) z \tag{2.5}$$

(i.e. for homogeneous differential system associated with (2.1).

Proof. See Lemma 6, [10]. □

3 $\Psi-$ bounded solutions for the matrix differential equation (1.2)

The purpose of this section is to provide sufficient conditions for the existence and uniqueness and existence of at least one $\Psi-$ bounded solution on $\mathbb{R}$ for the equation (1.2).

Theorem 3.1. *Suppose that:*
1). There exist supplementary projections P_-, P_0, $P_+ \in \mathbb{M}_{d\times d}$ *and a positive constant* K *such that the fundamental matrix* $X(t)$ *for (2.3) satisfies the condition*

$$\begin{aligned}&\textstyle\int_{-\infty}^{t} \mid \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s) \mid ds+\\ &+ \mid \textstyle\int_0^t \mid \Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s) \mid ds \mid + \\ &+ \textstyle\int_t^{\infty} \mid \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s) \mid ds \le K,\end{aligned} \tag{3.1}$$

for all $t \ge 0$;
2). The continuous function $F : R \times \mathbb{M}_{d\times d} \to \mathbb{M}_{d\times d}$ *satisfies* $F(t, O) = O$ *and the Lypschitz condition*

$$\mid \Psi(t)\left(F(t, Z_1) - F(t, Z_2)\right) \mid \le \gamma \mid \Psi(t)\left(Z_1 - Z_2\right) \mid,$$

for $t \in R$, $Z_1, Z_2 \in \mathbb{M}_{d\times d}$ *with* $\mid \Psi(t)Z_1 \mid \leq \rho$, $\mid \Psi(t)Z_2 \mid \leq \rho$ *for* $t \in R$, *(*$\rho > 0$ *is given), where* γ *is a positive constant such that* $\gamma K < 1$;
3). The continuous function $C : R \to \mathbb{M}_{d\times d}$ *is* $\Psi-$ *bounded on* R *such that*

$$\mid C \mid_\Psi \; = \; \sup_{t\in R} \mid \Psi(t)C(t) \mid \leq \frac{\rho(1-\gamma K)}{K}.$$

Then, the equation (1.2) has a unique $\Psi-$ *bounded solution Z(t) on R for which* $\mid \Psi(t)Z(t) \mid \leq \rho$, *for all* $t \in R$.

Proof. We prove this theorem by means of Banach fixed point theorem.
Consider the space

$$C_\Psi = \{Z : R \to \mathbb{M}_{d\times d} \mid Z \text{ is continuous and } \Psi - \text{bounded on } R\}.$$

C_Ψ is a Banach space with respect to the norm $\mid Z \mid_\Psi = \sup_{t\in R} \mid \Psi(t)Z(t) \mid$.
Let the ball $S_\rho = \{Z \in C_\Psi \mid\mid Z \mid_\Psi \leq \rho\}$.
For $Z \in C_\Psi$, define the operator T by

$$(TZ)(t) \; = \; \int_{-\infty}^{t} X(t)P_- X^{-1}(s)\left(C(s) + F(s, Z(s))\right) ds+$$

$$+ \int_0^t X(t)P_0X^{-1}(s)\left(C(s) + F(s, Z(s))\right) ds-$$

$$- \int_t^\infty X(t)P_+X^{-1}(s)\left(C(s) + F(s, Z(s))\right) ds.$$

From hypotheses, TZ exists and is continuous differentiable on R.
For $Z \in S_\rho$ and $t \in R$, we have

$$\mid \Psi(t)\left(TZ\right)(t) \mid = \mid \int_{-\infty}^{t} \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(C(s) + F(s, Z(s))\right) ds+$$

$$+ \int_0^t \Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(C(s) + F(s, Z(s))\right) ds-$$

$$- \int_t^\infty \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(C(s) + F(s, Z(s))\right) ds \mid\leq$$

$$\leq \; \int_{-\infty}^{t} \mid \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)\left(C(s) + F(s, Z(s))\right) \mid ds+$$

$$+ \mid \int_0^t \mid \Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)\left(C(s) + F(s, Z(s))\right) \mid ds \mid +$$

$$+ \int_t^\infty \mid \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)\left(C(s) + F(s, Z(s))\right) \mid ds \leq$$

$$\leq \; K \; \cdot \; \sup_{t\in R} \mid \Psi(s)\left(C(s) + F(s, Z(s))\right) \mid\leq$$

$$\leq \; K \; \cdot \; \left(\tfrac{\rho(1-\gamma K)}{K} + \gamma \mid Z \mid_\Psi\right) = \rho(1-\gamma K) + \gamma K\rho = \rho.$$

It follows that $TZ \in S_\rho$ and hence,

$$TS_\rho \subset S_\rho.$$

On the other hand, for Z_1, $Z_2 \in S_\rho$ and $t \in R$, we have

$$\mid \Psi(t)\left(\left(TZ_1\right)(t) - \left(TZ_2\right)(t)\right) \mid \ \ =$$

$$= \mid [\int_{-\infty}^{t} \Psi(t)X(t)P_- X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(C(s) + F(s, Z_1(s))\right) ds +$$

$$+ \int_0^t \Psi(t)X(t)P_0 X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(C(s) + F(s, Z_1(s))\right) ds -$$

$$- \int_t^\infty \Psi(t)X(t)P_+ X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(C(s) + F(s, Z_1(s))\right) ds] -$$

$$-[\int_{-\infty}^{t} \Psi(t)X(t)P_- X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(C(s) + F(s, Z_2(s))\right) ds +$$

$$+ \int_0^t \Psi(t)X(t)P_0 X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(C(s) + F(s, Z_2(s))\right) ds -$$

$$- \int_t^\infty \Psi(t)X(t)P_+ X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(C(s) + F(s, Z_2(s))\right) ds] \mid =$$

$$= \mid \int_{-\infty}^{t} \Psi(t)X(t)P_- X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(F(s, Z_1(s)) - F(s, Z_2(s))\right) ds +$$

$$+ \int_0^t \Psi(t)X(t)P_0 X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(F(s, Z_1(s)) - F(s, Z_2(s))\right) ds -$$

$$- \int_t^\infty \Psi(t)X(t)P_+ X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(F(s, Z_1(s)) - F(s, Z_2(s))\right) ds \mid \leq$$

$$\leq \int_{-\infty}^{t} \mid \Psi(t)X(t)P_- X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)\left(F(s, Z_1(s)) - F(s, Z_2(s))\right) \mid ds +$$

$$+ \mid \int_0^t \mid \Psi(t)X(t)P_0 X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)\left(F(s, Z_1(s)) - F(s, Z_2(s))\right) \mid ds \mid +$$

$$+ \int_t^\infty \mid \Psi(t)X(t)P_+ X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)\left(F(s, Z_1(s)) - F(s, Z_2(s))\right) \mid ds \leq$$

$$\leq \gamma \int_{-\infty}^{t} \mid \Psi(t)X(t)P_- X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)\left(Z_1(s) - Z_2(s)\right) \mid ds +$$

$$+ \gamma \mid \int_0^t \mid \Psi(t)X(t)P_0 X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)\left(Z_1(s) - Z_2(s)\right) \mid ds \mid +$$

$$+ \gamma \int_t^\infty \mid \Psi(t)X(t)P_+ X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)\left(Z_1(s) - Z_2(s)\right) \mid ds \leq$$

$$\leq \gamma K \sup_{t \in R} \mid \Psi(s)\left(Z_1(s) - Z_2(s)\right) \mid \ \ = \ \gamma K \mid Z_1 - Z_2 \mid_\Psi .$$

It follows that

$$| TZ_1 - TZ_2 |_{\Psi} \leq \gamma K \mid Z_1 - Z_2 |_{\Psi} .$$

Therefore, T is a contraction operator on S_ρ. Hence, by Banach fixed point theorem, T has a unique fixed point $Z \in S_\rho$. From $Z = TZ$, it follows that Z is continuous differentiable on R and then, for $t \in R$,

$Z'(t) = (TZ)'(t) =$

$= \int_{-\infty}^{t} X'(t)P_- X^{-1}(s)\left(C(s) + F(s, Z(s))\right) ds +$

$+ X(t)P_- X^{-1}(t)\left(C(t) + F(t, Z(t))\right) +$

$+ \int_{0}^{t} X'(t)P_0 X^{-1}(s)\left(C(s) + F(s, Z(s))\right) ds +$

$+ X(t)P_0 X^{-1}(t)\left(C(t) + F(s, Z(t))\right) -$

$- \int_{t}^{\infty} X'(t)P_+ X^{-1}(s)\left(C(s) + F(s, Z(s))\right) ds +$

$+ X(t)P_+ X^{-1}(t)\left(C(t) + F(t, Z(t))\right) =$

$= A(t)\left(TZ\right)(t) + X(t)\left(P_- + P_0 + P_+\right) X^{-1}(t)\left(C(t) + F(t, Z(t))\right) =$

$= A(t)Z(t) + C(t) + F(t, Z(t)).$

Thus, Z(t) is a solution of equation (1.2).

In conclusion, the equation (1.2) has a unique $\Psi-$ bounded solution $Z(t)$ on R for which $\mid \Psi(t)Z(t) \mid \leq \rho$, for all $t \in R$. □

Remark 3.1. Theorem generalizes the Theorem 4 ([5], Ch. 5, s. 2) and Theorem 2.1, [18] from systems of differential equations to matrix differential equations and extents them for case $P_0 \neq 0$.

The next simple example is an illustration of Theorem.

Example 3.1. Consider the equation (1.2) with

$$A(t) = \text{diag}\,[-2t, 1, -1],\ C(t) = \text{diag}\,[\frac{\alpha}{1+t^2}, \alpha e^{-t}\sin t, \alpha e^{t}\cos t]$$

and

$$F(t, Z) = \text{diag}\,[\ln\left(1 + a \mid z_{11} \mid\right), \sin a z_{22}, \text{arctg}\ a z_{33}],$$

where $Z = (z_{ij}) \in \mathbb{M}_{3\times 3}$ and a,α are real constants such that

$$0 < a < \left[1 + \int_0^1 e^{s^2} ds\right]^{-1}.$$

Then, X(t) = diag $[e^{t^2}, e^t, e^{-t}]$, $t \in R$, is a fundamental matrix for (2.3).
Consider Ψ(t) = diag $[1, e^t, e^{-t}]$, $t \in R$.
There exist supplementary projections

$$P_- = \text{diag } [0,0,1],\ P_0 = \text{diag } [1,0,0],\ \text{and } P_+ = \text{diag } [0,1,0]$$

such that:

- $\Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s) = \text{diag } [0,0,e^{-2(t-s)}]$ and then

$$\int_{-\infty}^{t} \mid \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s) \mid ds \leq \int_{-\infty}^{t} e^{-2(t-s)}ds = \tfrac{1}{2};$$

- $\Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s) = \text{diag } [0,e^{2(t-s)},0]$ and then

$$\int_{t}^{\infty} \mid \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s) \mid ds \leq \int_{t}^{\infty} e^{2(t-s)}ds = \tfrac{1}{2};$$

- $\Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s) = \text{diag } [e^{-t^2+s^2},0,0]$ and then

$$\mid \int_{0}^{t} \mid \Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s) \mid ds \mid = \mid \int_{0}^{t} e^{-t^2+s^2}ds \mid =$$

$$= \int_{0}^{|t|} e^{-t^2+s^2}ds \leq 1 + \int_{0}^{1} e^{s^2}ds.$$

and then, the condition (3.1) is satisfied with $K = 1 + \int_0^1 e^{s^2}ds$.
After that, for $t \in R$ and for Z', $Z'' \in \mathbb{M}_{3\times 3}$, we have

$$\mid \Psi(t)\left(F(t,Z') - F(t,Z'')\right) \mid \leq$$

$$\leq \max\{a \mid z'_{11} - z''_{11} \mid, ae^t \mid z'_{22} - z''_{22} \mid, ae^{-t} \mid z'_{33} - z''_{33} \mid\} =$$

$$= a \cdot \max\{\mid z'_{11} - z''_{11} \mid, e^t \mid z'_{22} - z''_{22} \mid, e^{-t} \mid z'_{33} - z''_{33} \mid\} =$$

$$= a \cdot \mid \Psi(t)\left(Z' - Z''\right) \mid .$$

and then, the condition 2) of Theorem is satisfied.
At least, for the matrix $C(t)$ we have that

$$\mid C \mid \ = \sup_{t\in R} \mid \Psi(t)C(t) \mid = \mid \alpha \mid .$$

From Theorem, it follows that for $\rho \geq \frac{|\alpha|K}{1-aK}$, the equqtion (1.2) has a unique $\Psi-$ bounded solution $Z(t)$ for which $\mid \Psi(t)Z(t) \mid \leq \rho$, for all $t \in R$.

Theorem 3.2. *Suppose that:*
1). There exist supplementary projections P_-, $P_+ \in \mathbb{M}_{d\times d}$ *and a positive constants* K_1, K_2, α *and* β *such that the fundamental matrix X(t) for (2.3) satisfies the conditions*

$$| \Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s) | \leq K_1 e^{-\alpha(t-s)}, \ \textit{for } s \leq t$$

$$| \Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s) | \leq K_2 e^{-\beta(s-t)}, \ \textit{for } t \leq s$$

2). The continuous function $F : R \times \mathbb{M}_{d\times d} \to \mathbb{M}_{d\times d}$ *satisfies the condition*

$$| \Psi(t)F(t,Z) | \leq \gamma \mid \Psi(t)Z |,$$

for $t \in R$, $Z \in \mathbb{M}_{d\times d}$ *with* $| \Psi(t)Z | \leq \rho$ *for* $t \in R$ *(*$\rho > 0$ *is given), where* γ *is a positive constant such that* $\gamma\left(\frac{K_1}{\alpha} + \frac{K_2}{\beta}\right) < 1$;
3). The continuous function $C : R \to \mathbb{M}_{d\times d}$ *is* $\Psi-$ *bounded on R such that*

$$| C |_\Psi = \sup_{t\in R} | \Psi(t)C(t) | \leq \frac{\rho\left[1 - \gamma(\frac{K_1}{\alpha} + \frac{K_2}{\beta})\right]}{\frac{K_1}{\alpha} + \frac{K_2}{\beta}}.$$

Then, the equation (1.2) has at least one $\Psi-$ *bounded solution* $Z(t)$ *on R for which* $| \Psi(t)Z(t) | \leq \rho$*, for all* $t \in R$.

Proof. We prove this treorem by means of Schauder-Tychonoff fixed point theorem.
For this, let C_Ψ denote the set of all matrix functions $Z(t)$ which are continuous and $\Psi-$bounded on R, and S_ρ be the subset formed by those functions $Z(t)$ such that $| Z |_\Psi = \sup_{t\in R} | \Psi(t)Z(t) | \leq \rho$.
For $Z \in C_\Psi$, define the operator T by

$$(TZ)(t) = \int_{-\infty}^{t} X(t)P_-X^{-1}(s)\left(C(s) + F(s,Z(s))\right)ds -$$

$$- \int_t^{\infty} X(t)P_+X^{-1}(s)\left(C(s) + F(s,Z(s))\right)ds,$$

This operator have the following two properties:
i). T is continuous, in the sense that if $Z_n \in S_\rho$ $(n = 1, 2, ...)$ and $Z_n \to Z$ uniformly on every compact subinterval J of R, then $TZ_n \to TZ$ uniformly on every compact subinterval J of R.
Indeed, let $Z_n \in S_\rho$ $(n = 1, 2, ...)$ and $Z_n \to Z$ uniformly on every compact subinterval $J = [p, q]$ of R. For an arbitrary small $\varepsilon > 0$, choose $\tau > 0$ so large that

$$\tau > \max\{-\frac{1}{\alpha}\ln\frac{\alpha\varepsilon}{8\rho\gamma K_1}, -\frac{1}{\beta}\ln\frac{\beta\varepsilon}{8\rho\gamma K_2}\}.$$

Since $F(t,Z)$ is uniformly continuous for $t \in [p-\tau, q+\tau]$ and $\mid \Psi(t)Z \mid \leq \rho$, it follows that the sequence $U_n(t) = \Psi(t)\left(F(t,Z_n(t)) - F(t,Z(t))\right)$ tends to zero uniformly on $[p-\tau, q+\tau]$. Thus, there exists $n_0 \in \mathbb{N}$ such that $\mid U_n(t) \mid < \frac{\varepsilon}{4\tau\max\{K_1,K_2\}}$, for $n \geq n_0$ and $t \in [p-\tau, q+\tau]$.
For $t \in J$ and $n \geq n_0$, consider

$$\mid \Psi(t)\left(\left(TZ_n\right)(t) - \left(TZ\right)(t)\right) \mid =$$

$$=\mid [\int_{-\infty}^{t} \Psi(t)X(t)P_{-}X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(C(s)+F(s,Z_n(s))\right)ds-$$

$$-\int_{t}^{\infty} \Psi(t)X(t)P_{+}X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(C(s)+F(s,Z_n(s))\right)ds]-$$

$$-[\int_{-\infty}^{t} \Psi(t)X(t)P_{-}X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(C(s)+F(s,Z(s))\right)ds-$$

$$-\int_{t}^{\infty} \Psi(t)X(t)P_{+}X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(C(s)+F(s,Z(s))\right)ds] \mid =$$

$$=\mid \int_{-\infty}^{t} \Psi(t)X(t)P_{-}X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(F(s,Z_n(s))-F(s,Z(s))\right)ds-$$

$$-\int_{t}^{\infty} \Psi(t)X(t)P_{+}X^{-1}(s)\Psi^{-1}(s)\Psi(s)\left(F(s,Z_n(s))-F(s,Z(s))\right)ds \mid \leq$$

$$\leq \int_{-\infty}^{t} \mid \Psi(t)X(t)P_{-}X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)\left(F(s,Z_n(s))-F(s,Z(s))\right) \mid ds+$$

$$+\int_{t}^{\infty} \mid \Psi(t)X(t)P_{+}X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)\left(F(s,Z_n(s))-F(s,Z(s))\right) \mid ds \leq$$

$$\leq K_1\int_{-\infty}^{t} e^{-\alpha(t-s)} \mid U_n(s) \mid ds + K_2\int_{t}^{\infty} e^{-\beta(s-t)} \mid U_n(s) \mid ds =$$

$$= K_1\int_{-\infty}^{t-\tau} e^{-\alpha(t-s)} \mid U_n(s) \mid ds + K_1\int_{t-\tau}^{t} e^{-\alpha(t-s)} \mid U_n(s) \mid ds+$$

$$+K_2\int_{t+\tau}^{\infty} e^{-\beta(s-t)} \mid U_n(s) \mid ds + K_2\int_{t}^{t+\tau} e^{-\beta(s-t)} \mid U_n(s) \mid ds \leq$$

$$\leq 2\rho\gamma(K_1\int_{-\infty}^{t-\tau} e^{-\alpha(t-s)}ds+K_2\int_{t+\tau}^{\infty} e^{-\beta(s-t)}ds) + \max\{K_1,K_2\}\int_{t-\tau}^{t+\tau} \mid U_n(s) \mid ds <$$

$$< 2\rho\gamma\left(K_1\cdot\frac{e^{-\alpha\tau}}{\alpha} + K_2\cdot\frac{e^{-\beta\tau}}{\beta}\right) + \max\{K_1,K_2\}\cdot\frac{\varepsilon}{4\tau\max\{K_1,K_2\}}\cdot 2\tau < \varepsilon.$$

This shows that $TZ_n \to TZ$ uniformly on every compact subinterval of R.
Thus, T is continuous.
ii). the functions in the image set TS_ρ are equicontinuous and bounded at every point of J.
Indeed, from $Z \in S_\rho$, we have

$$| \Psi(t)\,(TZ)\,(t) \,| =$$

$$=\Big| \int_{-\infty}^{t} \Psi(t)X(t)P_{-}X^{-1}(s)\Psi^{-1}(s)\Psi(s)\,(C(s)+F(s,Z(s)))\,ds-$$

$$-\int_{t}^{\infty} \Psi(t)X(t)P_{+}X^{-1}(s)\Psi^{-1}(s)\Psi(s)\,(C(s)+F(s,Z(s)))\,ds \,\Big| \leq$$

$$\leq \int_{-\infty}^{t} | \Psi(t)X(t)P_{-}X^{-1}(s)\Psi^{-1}(s) || \Psi(s)\,(C(s)+F(s,Z(s))) |\,ds+$$

$$+\int_{t}^{\infty} | \Psi(t)X(t)P_{+}X^{-1}(s)\Psi^{-1}(s) || \Psi(s)\,(C(s)+F(s,Z(s))) |\,ds \leq$$

$$\leq K_1 \int_{-\infty}^{t} e^{-\alpha(t-s)} | \Psi(s)\,(C(s)+F(s,Z(s))) |\,ds+$$

$$+K_2 \int_{t}^{\infty} e^{-\beta(s-t)} | \Psi(s)\,(C(s)+F(s,Z(s))) |\,ds \leq$$

$$\leq K_1 \int_{-\infty}^{t} e^{-\alpha(t-s)} \left(| \Psi(s)C(s) | +\gamma \,| \Psi(s)Z(s) |\right) ds+$$

$$+K_2 \int_{t}^{\infty} e^{-\beta(s-t)} \left(| \Psi(s)C(s) | +\gamma \,| \Psi(s)Z(s) |\right) ds \leq$$

$$\leq \left(\frac{\rho\left[1-\gamma(\frac{K_1}{\alpha}+\frac{K_2}{\beta})\right]}{\frac{K_1}{\alpha}+\frac{K_2}{\beta}} + \gamma\rho \right) \left(\frac{K_1}{\alpha}+\frac{K_2}{\beta}\right) = \rho.$$

Hence, the functions in the image set TS_ρ are uniformly bounded at every point of J.
On the other hand, we have

$$(TZ)'\,(t) =$$

$$= \int_{-\infty}^{t} X'(t)P_{-}X^{-1}(s)\,(C(s)+F(s,Z(s)))\,ds+$$

$$+X(t)P_{-}X^{-1}(t)\,(C(t)+F(t,Z(t)))+$$

$$-\int_{t}^{\infty} X'(t)P_{+}X^{-1}(s)\,(C(s)+F(s,Z(s)))\,ds+$$

$$+X(t)P_{+}X^{-1}(t)\,(C(t)+F(t,Z(t))) =$$

$$= A(t)\,(TZ)\,(t) + X(t)\,(P_{-}+P_{+})\,X^{-1}(t)\,(C(t)+F(t,Z(t))) =$$

$$= A(t)\,(TZ)\,(t) + C(t) + F(t,Z(t)),$$

which shows that $(TZ)\,(t)$ is a solution of equation $W' = A(t)W + C(t) + F(t,Z(t))$.

It follows that the derivatives $(TZ)'(t)$ are uniformly bounded on any compact subinterval J of R. Thus, the functions in TS_ρ are echicontinuous on any compact subinterval J of R.
From i) and ii), all the conditions of the Schauder-Tychonoff theorem are satisfied. Hence, the operator T has at least one fixed point Z(t) in S_ρ. But the fixed point of T is just the solution of the integral equation

$$Z = TZ$$

in S_ρ, i.e. , of the matrix differential equation (1.2), with the required properties. □

Remark 3.2. In a particular case, our result reduces to Theorem 2.2 obtained in [18].
Indeed, if

$$F(t,Z) = \begin{pmatrix} f_1(t,z) & f_1(t,z) & \cdots & f_1(t,z) \\ f_2(t,z) & f_2(t,z) & \cdots & f_2(t,z) \\ \vdots & \vdots & \vdots & \vdots \\ f_d(t,z) & f_d(t,z) & \cdots & f_d(t,z) \end{pmatrix}, C(t) = \begin{pmatrix} c_1(t) & c_1(t) & \cdots & c_1(t) \\ c_2(t) & c_2(t) & \cdots & c_2(t) \\ \vdots & \vdots & \vdots & \vdots \\ c_d(t) & c_d(t) & \cdots & c_d(t) \end{pmatrix}$$

it is easy to see that the solutions of the equation (1.2) is

$$Z(t) = \begin{pmatrix} z_1(t) & z_1(t) & \cdots & z_1(t) \\ z_2(t) & z_2(t) & \cdots & z_2(t) \\ \vdots & \vdots & \vdots & \vdots \\ z_d(t) & z_d(t) & \cdots & z_d(t) \end{pmatrix},$$

where $z(t) = (z_1(t), z_2(t), \cdots, z_d(t))^T$ is the solution of the equation (1.3) with

$$c(t) = (c_1(t), c_2(t), \cdots, c_d(t))^T \text{ and } f(t,z) = (f_1(t,z), f_1(t,z), \cdots, f_1(t,z))^T.$$

In this case, the solution $z(t)$ is $\Psi-$bounded on R iff the corresponding solution $Z(t)$ is $\Psi-$bounded on R.
Thus, the Theorem generalizes the result from [18], from systems of differential equations to matrix differential equations.

4 $\Psi-$ bounded solutions for the Lyapunov matrix differential equation (1)

The purpose of this section is to provide sufficient conditions for the existence and uniqueness and existence of at least one $\Psi-$ bounded solution on $\mathbb{R}$ for the Lyapunov matrix differential equation (1.1).

Theorem 4.1. *Suppose that:*
1). There exist supplementary projections P_-, P_0, $P_+ \in \mathbb{M}_{d\times d}$ and a positive constant K such that the fundamental matrices X(t) and Y(t) for (2.3) and (2.4) respectively satisfy the condition

$$\int_{-\infty}^{t} \mid \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)\right) \mid ds+$$

$$+ \mid \int_0^t \mid \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_0X^{-1}(s)\Psi^{-1}(s)\right) \mid ds \mid +$$

$$+ \int_t^{\infty} \mid \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)\right) \mid ds \leq K,$$

for all $t \geq 0$;
2). The continuous function $F : R \times \mathbb{M}_{d\times d} \to \mathbb{M}_{d\times d}$ satisfies the Lypschitz condition

$$\mid \Psi(t)\left(F(t,Z_1) - F(t,Z_2)\right) \mid \leq \frac{\gamma}{d} \mid \Psi(t)\left(Z_1 - Z_2\right) \mid,$$

for $t \in R$, $Z_1, Z_2 \in \mathbb{M}_{d\times d}$ with $\mid \Psi(t)Z_1 \mid \leq \rho$, $\mid \Psi(t)Z_2 \mid \leq \rho$ for $t \in R$, ($\rho > 0$ is given), where γ is a positive constant such that $\gamma K < 1$;
3). The continuous matrix function $C : R \to \mathbb{M}_{d\times d}$ is $\Psi-$ bounded on R such that

$$\mid C \mid_{\Psi} \; = \; \sup_{t\in R} \mid \Psi(t)C(t) \mid \leq \frac{\rho(1-\gamma K)}{K}.$$

Then, the Lyapunov matrix differential equation (1.1) has a unique $\Psi-$ bounded solution Z(t) on R for which $\mid \Psi(t)Z(t) \mid \leq \rho d$, for all $t \in R$.

Proof. From Lemma 2.4, one know that Z(t) is a solution of (1.1) iff the vector function $z(t) = \mathcal{V}ec(Z(t))$ is a solution of the corresponding Kronecker product system associated with (1.1), i.e. of the differential system (2.1).
From Lemma (2.7), one know that $U(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the differential system (2.5).
Now, the hypotheses of the Theorem ensure the hypotheses of Theorem 3.1 (variant for systems) for the system (2.1). Indeed:

i). Since

$$\left(Y^T(t)\left(Y^T\right)^{-1}(s)\right)\otimes\left(\Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)\right)=$$

$$=\left(I\otimes\Psi(t)\right)\cdot\left(Y^T(t)\otimes X(t)\right)\cdot\left(I\otimes P_-\right)\cdot\left(\left(Y^T\right)^{-1}(s)\otimes X^{-1}(s)\right)\cdot\left(I\otimes\Psi^{-1}(s)\right)$$

(see Lemma 2.1) and similarly for P_0 and P_+, the hypothesis 1) of Theorem 3.1 is satisfied;

ii). Since

$$\parallel (I\otimes\Psi(t))\cdot(f(t,z_1)-f(t,z_2))\parallel_{R^{d^2}}=$$

$$=\parallel (I\otimes\Psi(t))\cdot\mathcal{V}ec\,(F(t,Z_1)-F(t,Z_2))\parallel_{R^{d^2}}\leq$$

$$\leq\mid \Psi(t)\,(F(t,Z_1)-F(t,Z_2))\mid\leq\tfrac{\gamma}{d}\mid\Psi(t)\,(Z_1-Z_2)\mid\leq$$

$$\leq\gamma\parallel (I\otimes\Psi(t))\cdot\mathcal{V}ec\,(Z_1-Z_2)\parallel_{R^{d^2}}=$$

$$=\gamma\parallel (I\otimes\Psi(t))\cdot(z_1-z_2)\parallel_{R^{d^2}},$$

for all z_1, z_2 with $\parallel (I\otimes\Psi(t))\cdot z_i \parallel_{R^{d^2}}=\parallel (I\otimes\Psi(t))\cdot\mathcal{V}ec\,(Z_i)\parallel_{R^{d^2}}\leq\mid \Psi(t)Z_i\mid\leq\rho$,
(see Lemmas 2.1 and 2.5) and $\gamma K<1$, the hypothesis 2) of Theorem 3.1 is satisfied;

iii). Since

$$\parallel c\parallel_{R^{d^2}}=\ \mid C\mid\ =\ \sup_{t\in R}\mid (I\otimes\Psi(t))\,\mathcal{V}ec\,(C(t))\mid\leq$$

$$\leq\ \sup_{t\in R}\mid\Psi(t)C(t)\mid\leq\tfrac{\rho(1-\gamma K)}{K},$$

the hypothesis 3) of Theorem 3.1 is satisfied.

At this stage we appeal to Theorem 3.1 to deduce that the system (2.1) has a unique $I\otimes\Psi(t)-$ bounded solution $z(t)$ on R for which $\parallel (I\otimes\Psi(t))\cdot z(t)\parallel_{R^{d^2}}\leq\rho$.

From Lemma 2.4 again, the matrix function $Z(t)=\mathcal{V}ec^{-1}(z(t))$ is unique solution of (1.1) on R such that (see Lemma 2.5) $\mid\Psi(t)Z(t)\mid\leq\rho d$, for all $t\in R$. □

Remark 4.1. The Theorem extends the Theorem 2.1, [18] and Theorem 3.1 above to Lyapunov matrix differential equation (1.1).

Theorem 4.2. *Suppose that:*

1). There exist supplementary projections P_-, $P_+\in\mathbb{M}_{d\times d}$ and a positive *constants K_1, K_2, α and β such that the fundamental matrices X(t) and Y(t) for (2.3) and (2.4) respectively satisfy the condition*

$$\mid \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)\right) \mid \leq K_1 e^{-\alpha(t-s)}, \text{ for } s \leq t$$

$$\mid \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)\right) \mid \leq K_2 e^{-\beta(s-t)}, \text{ for } t \leq s$$

2). The continuous function $F : R \times \mathbb{M}_{d\times d} \to \mathbb{M}_{d\times d}$ satisfies the condition

$$\mid \Psi(t)F(t,Z) \mid \leq \frac{\gamma}{d} \mid \Psi(t)Z \mid,$$

for $t \in R$, $Z \in \mathbb{M}_{d\times d}$ with $\mid \Psi(t)Z \mid \leq \rho$ for $t \in R$ ($\rho > 0$ is given), where γ is a positive constant such that $\gamma\left(\frac{K_1}{\alpha} + \frac{K_2}{\beta}\right) < 1$;
3). The continuous matrix function $C : R \to \mathbb{M}_{d\times d}$ is $\Psi-$ bounded on R such that

$$\mid C \mid_{\Psi} \;=\; \sup_{t\in R} \mid \Psi(t)C(t) \mid \leq \frac{\rho\left[1 - \gamma(\frac{K_1}{\alpha} + \frac{K_2}{\beta})\right]}{\frac{K_1}{\alpha} + \frac{K_2}{\beta}}.$$

Then, the Lyapunov matrix differential equation (1.1) has at least one $\Psi-$ bounded solution Z(t) on R for which $\mid \Psi(t)Z(t) \mid \leq \rho d$.

Proof. From Lemma 2.4, one know that Z(t) is a solution of (1.1) iff the vector function $z(t) = \mathcal{V}ec(Z(t))$ is a solution of the corresponding Kronecker product system associated with (1.1), i.e. of the differential system (2.1).
From Lemma (2.7), one know that $U(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the differential system (2.5).
Now, the hypotheses of the Theorem ensure the hypotheses of Theorem 3.2 (variant for systems) for the system (2.1). Indeed:
i). Since

$$\left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)\right) =$$

$$= (I \otimes \Psi(t)) \cdot \left(Y^T(t) \otimes X(t)\right) \cdot (I \otimes P_-) \cdot \left(\left(Y^T\right)^{-1}(s) \otimes X^{-1}(s)\right) \cdot (I \otimes \Psi^{-1}(s))$$

(see Lemma 2.1) and similarly for P_+, the hypothesis 1) of Theorem 3.2 is satisfied;
ii). Since

$$\parallel (I \otimes \Psi(t)) \cdot f(t,z) \parallel_{R^{d^2}} = \parallel (I \otimes \Psi(t)) \cdot \mathcal{V}ecF(t,Z) \parallel_{R^{d^2}} \leq$$

$$\leq \mid \Psi(t)F(t,Z) \mid \leq \tfrac{\gamma}{d} \mid \Psi(t)Z \mid \leq \gamma \parallel (I \otimes \Psi(t)) \cdot \mathcal{V}ec\,(Z) \parallel_{R^{d^2}} =$$

$$= \gamma \parallel (I \otimes \Psi(t)) \cdot z \parallel_{R^{d^2}},$$

for $t \in R$ and $z \in R^{d^2}$, (see Lemmas 2.1 and 2.5) and $\gamma\left(\frac{K_1}{\alpha}+\frac{K_2}{\beta}\right) < 1$, the hypothesis 2) of Theorem 3.2 is satisfied;
iii). Since

$$\| c \|_{R^{d^2}} = \| \mathcal{V}ec\,(C(t)) \|_{R^{d^2}} = \sup_{t\in R} \mid (I \otimes \Psi(t))\, \mathcal{V}ec\,(C(t)) \mid \leq$$

$$\leq \sup_{t\in R} \mid \Psi(t)C(t) \mid \leq \frac{\rho\left(1-\gamma\left(\frac{K_1}{\alpha}+\frac{K_2}{\beta}\right)\right)}{\frac{K_1}{\alpha}+\frac{K_2}{\beta}},$$

the hypothesis 3) of Theorem 3.2 is satisfied.
At this stage we appeal to Theorem 3.2 to deduce that the system (2.1) has at least one $I \otimes \Psi(t)-$ bounded solution $z(t)$ on R for which $\| (I \otimes \Psi(t)) \cdot z(t) \|_{R^{d^2}} \leq \rho$.
From Lemma 2.4 again, the matrix function $Z(t) = Vec^{-1}(z(t))$ is a solution of (1.1) on R such that (see Lemma 2.5), $\mid \Psi(t)Z(t) \mid \leq \rho d$, for all $t \in R$. □

Remark 4.2. The Theorem extends the Theorem 2.2 [18] and Theorem 3.2 above to Lyapunov matrix differential equation (1.1).

The next simple example is an illustration of Theorem.

Example 4.1. Consider the nonlinear *Lyapunov matrix differential equation (1.1) with*

$$A(t) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix},\ B(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$C(t) = \begin{pmatrix} 0 & ae^{4t}\cos \mid t \mid \\ ae^{-3t}\sin t^2 & ae^{-3|t|} \end{pmatrix}$$

and

$$F(t,Z) = m \begin{pmatrix} \sin z_1 \sin t & z_2 \cos t \\ z_3 \sin z_4 & \frac{2}{\pi} z_4 \text{arctg } t \end{pmatrix},$$

where $t \in R$, $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \mathbb{M}_{2\times 2}$ and a, m are real constants, $0 <\mid m \mid < \frac{1}{4}$.
Then,

$$X(t) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \text{ and } Y(t) = \begin{pmatrix} e^{t} & 0 \\ 0 & 1 \end{pmatrix}$$

are fundamental matrices for (2.3) and (2.4) *respectively.*
Consider

$$\Psi(t) = \begin{pmatrix} e^{-4t} & 0 \\ 0 & e^{3t} \end{pmatrix},\ t \in R.$$

There exist supplementary projections

$$P_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } P_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

such that

$$\mid \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_-X^{-1}(s)\Psi^{-1}(s)\right) \mid = e^{-\alpha(t-s)}, \text{ for } s \leq t$$

and

$$\mid \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_+X^{-1}(s)\Psi^{-1}(s)\right) \mid = e^{-(s-t)}, \text{ for } t \leq s.$$

Thus, the condition 1) of Theorem is satisfied with $K_1 = K_2 = 1$ and $\alpha = \beta = 1$.
After that, for $t \in R$, $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \mathbb{M}_{2\times 2}$, we have

$$\mid \Psi(t)F(t,Z) \mid \;=\; \left| m \begin{pmatrix} e^{-4t}\sin z_1 \sin t & e^{-4t} z_2 \cos t \\ e^{3t} z_3 \sin z_4 & \frac{2}{\pi} e^{3t} z_4 \text{arctg } t \end{pmatrix} \right| \;\leq\; \mid m \mid\mid \Psi(t)Z \mid .$$

Thus, the condition 2) of Theorem is satisfied with $\gamma = 2 \mid m \mid$ and $d = 2$.
At least, for the matrix $C(t)$ we have that

$$\mid C \mid \;=\; \sup_{t\in R} \mid \Psi(t)C(t) \mid \;=\; \sup_{t\in R} \left| \begin{pmatrix} 0 & a\cos \mid t \mid \\ a \sin t^2 & ae^{-3|t|+3t} \end{pmatrix} \right| \;=\; 2 \mid a \mid .$$

Now, from Theorem 4.2, it follows that for all $\rho \geq \frac{8|a|}{1-4|m|}$, the equation (1.1) has at least one solution $Z(t)$ for which $\mid \Psi(t)Z(t) \mid \leq \rho$, for all $t \in R$.

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