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On Pseudo Cyclic Ricci Symmetric Manifolds

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Abstract: The object of the present paper is to study concircularly symmetric $(PCRS)_n$, concircularly recurrent $(PCRS)_n$, decomposable $(PCRS)_n$. Among others it is shown that in a decomposable $(PCRS)_n$ one of the decompositions is Ricci flat and the other decomposition is cyclic parallel. The totally umbilical hypersurfaces of $(PCRS)_n$ are also studied.

***Key words*:** Concircularly symmetric manifold, Concircularly recurrent manifold, Decomposable manifold, Pseudo cyclic Ricci symmetric manifold, Totally umbilical hypersurfaces.

Mathematics Subject Classification 2000: 53B30, 53B50, 53C15, 53C25.

Yarı Devirli Ricci Simetrik Manifoldlar Üzerine

Abstract: Bu makalenin amacı, konsirkular simetrik $(PCRS)_n$, konsirkular tekrarlı $(PCRS)_n$, ayrışabilir $(PCRS)_n$ manifoldları incelemektir. $(PCRS)_n$ ayrışabilir manifoldunda ayrışımlardan birisinin Ricci düzlemsellik (flat), diğerinin de devirli paralellik olduğu gösterilmiştir. Aynı zamanda $(PCRS)_n$ nin tümüyle umbilik hiperyüzeyleri çalışılmıştır.

***Anahtar kelimeler*:** Konsirkular simetrik manifold, Konsirkular tekrarlı manifold, Ayrışabilir manifold, Yarı devirli Ricci simetrik manifold, Tümüyle umbilik hiperyüzeyler.

1. Introduction

A Riemannian manifold is Ricci symmetric if its Ricci tensor S of type (0,2) satisfies $\nabla S = 0$, where ∇ denotes the Riemannian connection. During the last five decades, the notion of Ricci symmetry has been weakened by many authors in several ways such as Ricci-recurrent manifolds [1], Ricci semi-symmetric manifolds [2], pseudo Ricci symmetric manifolds by M. C. Chaki [3]. A non-flat Riemannian manifold (M^n,g) is said to be pseudo Ricci symmetric [3] if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(Z,X) + A(Z)S(X,Y), \tag{1}$$

where A is a nowhere vanishing 1-form. Such an n-dimensional manifold is denoted by $(PRS)_n$.

Extending the notion of pseudo Ricci symmetric manifold, recently A. A. Shaikh and the present author [4] introduced the notion of *pseudo cyclic Ricci symmetric manifolds*. A Riemannian manifold $(M^n,g)(n>2)$ is said to be *pseudo cyclic Ricci symmetric* *manifold* if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following:

$$
\begin{aligned}
&(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\
&\quad = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X) \\
&\text{or} \\
&(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\
&\quad = 2A(Y)S(Z,X) + A(Z)S(X,Y) + A(X)S(Y,Z) \\
&\text{or} \\
&(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\
&\quad = 2A(Z)S(X,Y) + A(X)S(Y,Z) + A(Y)S(Z,X),
\end{aligned}
\tag{2}
$$

where A is a nowhere vanishing 1-form associated to the vector field ρ such that $A(X) = g(X, \rho)$ for all X. Such an n-dimensional manifold is denoted by $(PCRS)_n$. The $(PCRS)_n$ admitting semi-symmetric metric connection is also studied in [5]. The pseudo cyclic Ricci symmetric manifolds are also studied in [6, 7].

The object of the present paper is to study $(PCRS)_n$.The paper is organized as follows. Section 2 is devoted to the study of concircularly symmetric $(PCRS)_n$. It is shown that in a concircularly symmetric $(PCRS)_n$ with constant scalar curvature, $-r$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ. Section 3 deals with a study of concircularly recurrent $(PCRS)_n$. It is proved that in a concircularly recurrent $(PCRS)_n$ with constant scalar curvature, $-n$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ. In section 4, we study decomposable $(PCRS)_n$ and it is shown that in a decomposable $(PCRS)_n$, one of the decompositions is Ricci flat and the Ricci tensor of the other decomposition is cyclic parallel.

Recently Özen and Altay [8] studied the totally umbilical hypersurfaces of weakly and pseudosymmetric spaces. Again Özen and Altay [9] also studied the totally umbilical hypersurfaces of weakly concircular and pseudo concircular symmetric spaces. In this connection it may be mentioned that Shaikh, Roy and Hui [10] studied the totally umbilical hypersurfaces of weakly conharmonically symmetric spaces. Section 5 deals with the study of totally umbilical hypersurfaces of $(PCRS)_n$. It is proved that the totally geodesic hypersurface of a $(PCRS)_n$ is also a $(PCRS)_n$.

2. Concircularly Symmetric $(PCRS)_n$

A $(PCRS)_n$ is said to be concircularly symmetric if its concircular curvature tensor $\tilde{C}$, given by,

$$\tilde{C}(Y,Z,U,V) = R(Y,Z,U,V) - \frac{r}{n(n-1)} G(Y,Z,U,V), \tag{3}$$

where r is the scalar curvature of the manifold and the tensor G is defined by

$$G(Y,Z,U,V) = g(Z,U)\, g(Y,V) - g(Y,U)\, g(Z,V), \tag{4}$$

satisfies the relation

$$(\nabla_X \tilde{C})(Y,Z,U,V) = 0. \tag{5}$$

Let us consider a concircularly symmetric $(PCRS)_n$. Then by virtue of (3), it follows from (5) that

$$(\nabla_X R)(Y,Z,U,V) - \frac{dr(X)}{n(n-1)} G(Y,Z,U,V) = 0. \tag{6}$$

Let $\{e_i : i = 1, 2, \ldots, n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = V = e_i$ in (6) and taking summation over i, $1 \leq i \leq n$, we get

$$(\nabla_X S)(Z,U) = \frac{dr(X)}{n} g(Z,U). \tag{7}$$

Using (7) in (2), we obtain

$$\begin{aligned} &2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X) \\ &\quad = \frac{1}{n}[dr(X)g(Y,Z) + dr(Y)g(Z,X) + dr(Z)g(X,Y)]. \end{aligned} \tag{8}$$

Taking contraction of (8) over Y and Z, we get

$$A(QX) + rA(X) = \frac{n+2}{2n} dr(X), \tag{9}$$

where Q is the Ricci-operator i.e., $g(QX,Y) = S(X,Y)$ for all X, Y.

We now suppose that the scalar curvature r is constant, then

$$dr(X) = 0 \text{ for all } X. \tag{10}$$

In view of (10), (9) yields

$$A(QX) = -rA(X), \tag{11}$$

i.e.,

$$S(X, \rho) = -r\, g(X, \rho). \tag{12}$$

This leads to the following:

Theorem 2.1. In a concircularly symmetric $(PCRS)_n$ with constant scalar curvature, $-r$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ.

Since every concircularly flat manifold is concircularly symmetric. So by virtue of Theorem 2.1, we can state the following:

Corollary 2.1. In a concircularly flat $(PCRS)_n$ with constant scalar curvature, $-r$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ.

3. Concircularly Recurrent *(PCRS)$_n$*

Definition 3.1. A *(PCRS)$_n$* is said to be concircularly recurrent ([11, 12]) if its concircular curvature tensor $\tilde{C}$ satisfies the relation

$$(\nabla_X \tilde{C})(Y,Z,U,V) = A(X)\tilde{C}(Y,Z,U,V), \tag{13}$$

where A is a non-vanishing 1-form.

We now consider a concircularly recurrent *(PCRS)$_n$*. Then by virtue of (3), it follows from (13) that

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) - \frac{dr(X)}{n(n-1)}G(Y,Z,U,V)\\ = A(X)[R(Y,Z,U,V) - \frac{r}{n(n-1)}G(Y,Z,U,V)].\end{aligned} \tag{14}$$

Contracting (14) over Y and V, we get

$$(\nabla_X S)(Z,U) - \frac{dr(X)}{n}g(Z,U) = A(X)[S(Z,U) - \frac{r}{n}g(Z,U)]. \tag{15}$$

By virtue of (10), (15) yields

$$(\nabla_X S)(Z,U) = A(X)[S(Z,U) - \frac{r}{n}g(Z,U)]. \tag{16}$$

Using (16) in (2), we obtain

$$A(X)S(Y,Z) = -\frac{r}{n}[A(X)g(Y,Z) + A(Y)g(Z,X) + A(Z)g(X,Y)]. \tag{17}$$

Again taking contraction of (17) over Y and Z, we get

$$r[A(QX) + n\,A(X)] = 0 \qquad \text{for all } X. \tag{18}$$

Since the scalar curvature r of *(PCRS)$_n$* is always non-zero [4]. Therefore (3.6) yields

$$A(QX) = -\,n\,A(X), \tag{19}$$

i.e.,

$$S(X,\rho) = -\,n\,g(X,\rho). \tag{20}$$

Thus we can state the following:

Theorem 3.1. In a concircularly recurrent *(PCRS)$_n$* with constant scalar curvature, -n is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ.

4. Decomposable *(PCRS)$_n$*

A Riemannian manifold *(M^n,g)* is said to be decomposable manifold [13] if it can be expressed as $M_1^p \times M_2^{n-p}$ for $2 \leq p \leq n-2$, that is, in some coordinate neighbourhood of the Riemannian manifold *(M^n,g)*, the metric can be expressed as

$$ds^2 = g_{ij}dx^i dx^j = \overline{g}_{ab}dx^a dx^b + \overset{*}{g}_{\alpha\beta}\, dx^\alpha dx^\beta, \tag{21}$$

where $\overline{g}_{ab}$ are functions of x^1, x^2, … , x^p $(p<n)$ denoted by $\overline{x}$ and $\overset{*}{g}_{\alpha\beta}$ are functions of x^{p+1}, x^{p+2}, … , x^n denoted by $\overset{*}{x}$; a, b, c, … run from 1 to p and $\alpha,\beta,\gamma,...$ run from $p+1$ to n. The two parts of (21) are the metrics of $M_1^p\,(p \geq 2)$ and $M_2^{n-p}\,(n-p \geq 2)$ which are called the decompositions of the decomposable manifold

Let (M^n,g) be a decomposable Riemannian manifold such that for $2 \leq p \leq n-2$. Here throughout this section each object denoted by a 'bar' is assumed to be from M_1 and each object denoted by a 'star' is assumed to be from M_2.

Let $\overline{X},\overline{Y},\overline{Z},\overline{U},\overline{V} \in \chi(M_1)$ and $\overset{*}{X},\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V} \in \chi(M_2)$, $\chi(M_i)$ being the Lie algebra of smooth vector fields on M_i, i =1, 2. Then we have the following relations [13]:

$$R(\overset{*}{X},\overline{Y},\overline{Z},\overline{U}) = R(\overline{X},\overset{*}{Y},\overline{Z},\overset{*}{U}) = R(\overline{X},\overset{*}{Y},\overset{*}{Z},\overset{*}{U}) = 0,$$

$$(\nabla_{\overset{*}{X}}R)(\overline{Y},\overline{Z},\overline{U},\overline{V}) = (\nabla_{\overline{X}}R)(\overline{Y},\overset{*}{Z},\overline{U},\overset{*}{V}) = (\nabla_{\overset{*}{X}}R)(\overline{Y},\overset{*}{Z},\overline{U},\overset{*}{V}) = 0,$$

$$R(\overline{X},\overline{Y},\overline{Z},\overline{U}) = \overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{U}),$$

$$R(\overset{*}{X},\overset{*}{Y},\overset{*}{Z},\overset{*}{U}) = \overset{*}{R}(\overset{*}{X},\overset{*}{Y},\overset{*}{Z},\overset{*}{U}),$$

$$S(\overline{X},\overline{Y}) = \overline{S}(\overline{X},\overline{Y}),$$

$$S(\overset{*}{X},\overset{*}{Y}) = \overset{*}{S}(\overset{*}{X},\overset{*}{Y}),$$

$$(\nabla_{\overline{X}}S)(\overline{Y},\overline{Z}) = (\overline{\nabla}_{\overline{X}}S)(\overline{Y},\overline{Z}),$$

$$(\nabla_{\overset{*}{X}}S)(\overset{*}{Y},\overset{*}{Z}) = (\overset{*}{\nabla}_{\overset{*}{X}}S)(\overset{*}{Y},\overset{*}{Z}),$$

$$r = \overline{r} + \overset{*}{r},$$

where r, $\overline{r}$ and $\overset{*}{r}$ are the scalar curvature of M, M_1, M_2 respectively.

Let us consider a Riemannian manifold (M^n,g) which is decomposable $(PCRS)_n$. Then $M^n = M_1^p \times M_2^{n-p}, (2 \leq p \leq n-2)$.

Now from (2), we have

$$\begin{aligned}
&(\nabla_{\overline{X}}S)(\overline{Y},\overline{Z}) + (\nabla_{\overline{Y}}S)(\overline{Z},\overline{X}) + (\nabla_{\overline{Z}}S)(\overline{X},\overline{Y}) \\
&\quad = 2A(\overline{X})S(\overline{Y},\overline{Z}) + A(\overline{Y})S(\overline{X},\overline{Z}) + A(\overline{Z})S(\overline{Y},\overline{X}) \\
&\text{or,} \\
&(\nabla_{\overline{X}}S)(\overline{Y},\overline{Z}) + (\nabla_{\overline{Y}}S)(\overline{Z},\overline{X}) + (\nabla_{\overline{Z}}S)(\overline{X},\overline{Y}) \\
&\quad = 2A(\overline{Y})S(\overline{Z},\overline{X}) + A(\overline{Z})S(\overline{X},\overline{Y}) + A(\overline{X})S(\overline{Y},\overline{Z}) \\
&\text{or,} \\
&(\nabla_{\overline{X}}S)(\overline{Y},\overline{Z}) + (\nabla_{\overline{Y}}S)(\overline{Z},\overline{X}) + (\nabla_{\overline{Z}}S)(\overline{X},\overline{Y}) \\
&\quad = 2A(\overline{Z})S(\overline{X},\overline{Y}) + A(\overline{X})S(\overline{Y},\overline{Z}) + A(\overline{Y})S(\overline{Z},\overline{X}),
\end{aligned} \tag{22}$$

and

$$\left(\nabla_{\overset{*}{X}} S\right)(\overset{*}{Y},\overset{*}{Z}) + \left(\nabla_{\overset{*}{Y}} S\right)(\overset{*}{Z},\overset{*}{X}) + \left(\nabla_{\overset{*}{Z}} S\right)(\overset{*}{X},\overset{*}{Y})$$
$$= 2A(\overset{*}{X})S(\overset{*}{Y},\overset{*}{Z}) + A(\overset{*}{Y})S(\overset{*}{X},\overset{*}{Z}) + A(\overset{*}{Z})S(\overset{*}{Y},\overset{*}{X})$$

or,

$$\left(\nabla_{\overset{*}{X}} S\right)(\overset{*}{Y},\overset{*}{Z}) + \left(\nabla_{\overset{*}{Y}} S\right)(\overset{*}{Z},\overset{*}{X}) + \left(\nabla_{\overset{*}{Z}} S\right)(\overset{*}{X},\overset{*}{Y}) \tag{23}$$
$$= 2A(\overset{*}{Y})S(\overset{*}{Z},\overset{*}{X}) + A(\overset{*}{Z})S(\overset{*}{X},\overset{*}{Y}) + A(\overset{*}{X})S(\overset{*}{Y},\overset{*}{Z})$$

or,

$$\left(\nabla_{\overset{*}{X}} S\right)(\overset{*}{Y},\overset{*}{Z}) + \left(\nabla_{\overset{*}{Y}} S\right)(\overset{*}{Z},\overset{*}{X}) + \left(\nabla_{\overset{*}{Z}} S\right)(\overset{*}{X},\overset{*}{Y})$$
$$= 2A(\overset{*}{Z})S(\overset{*}{X},\overset{*}{Y}) + A(\overset{*}{X})S(\overset{*}{Y},\overset{*}{Z}) + A(\overset{*}{Y})S(\overset{*}{Z},\overset{*}{X}).$$

From (22), we find

$$A(\overset{*}{X})S(\overline{Y},\overline{Z}) = 0, \tag{24}$$

$$A(\overline{X})S(\overset{*}{Y},\overset{*}{Z}) = 0. \tag{25}$$

Now from (24) it follows that either $A(\overset{*}{X}) = 0$ for any vector field $\overset{*}{X} \in$ *χ(M₂)* or $S(\overline{Y},\overline{Z}) = 0$ for all vector fields $\overline{Y},\overline{Z} \in$ *χ(M₁)*, i.e., the decomposition *M₁* is Ricci flat.

Again if $A(\overset{*}{X}) = 0$ then from (23), we get

$$\left(\nabla_{\overset{*}{X}} S\right)(\overset{*}{Y},\overset{*}{Z}) + \left(\nabla_{\overset{*}{Y}} S\right)(\overset{*}{Z},\overset{*}{X}) + \left(\nabla_{\overset{*}{Z}} S\right)(\overset{*}{X},\overset{*}{Y}) = 0,$$

that is, the Ricci tensor of the decomposition M_2 is cyclic parallel.

Similarly from (25), we obtain either the Ricci tensor of the decomposition M_1 is cyclic parallel or the decomposition M_2 is Ricci flat. Thus, we can state the following:

Theorem 4.1. In a decomposable $(PCRS)_n$, one of the decompositions is Ricci flat and the Ricci tensor of the other decomposition is cyclic parallel.

5. Totally Umbilical Hypersurfaces of $(PCRS)_n$

Let $(\overline{V},\overline{g})$ be an *(n+1)*-dimensional Riemannian manifold covered by a system of coordinate neighbourhoods $\{U, y^\alpha\}$. Let *(V,g)* be a hypersurface of $(\overline{V},\overline{g})$ defined in a locally coordinate system by means of a system of parametric equation $y^\alpha = y^\alpha(x^i)$, where Greek indices take values 1, 2, … , *n* and Latin indices take values 1,2,…,*(n+1)*. Let N^α be the components of a local unit normal to *(V,g)*. Then we have

$$g_{ij} = \bar{g}_{\alpha\beta} y_i^\alpha y_j^\beta, \tag{26}$$

$$\bar{g}_{\alpha\beta} N^\alpha y_j^\beta = 0, \ \bar{g}_{\alpha\beta} N^\alpha N^\beta = e = 1, \tag{27}$$

$$y_i^\alpha y_j^\beta g^{ij} = \bar{g}^{\alpha\beta} - N^\alpha N^\beta, y_i^\alpha = \frac{\partial y^\alpha}{\partial x^i}. \tag{28}$$

The hypersurface *(V,g)* is called a totally umbilical hypersurface ([14, 15]) of $(\bar{V}, \bar{g})$ if its second fundamental form Ω_{ij} satisfies

$$\Omega_{ij} = H g_{ij}, y_{i,j}^\alpha = g_{ij} H N^\alpha, \tag{29}$$

where the scalar function H is called the mean curvature of *(V,g)* given by $H = \frac{1}{n} \Sigma g^{ij} \Omega_{ij}$. If, in particular, $H=0$, i.e.,

$$\Omega_{ij} = 0, \tag{30}$$

then the totally umbilical hypersurface is called a totally geodesic hypersurface of $(\bar{V}, \bar{g})$.

The equation of Weingarten for *(V,g)* can be written as $N_{,j}^\alpha = -\frac{H}{n} y_j^\alpha$. The structure equations of Gauss and Codazzi ([14, 15]) for *(V,g)* and $(\bar{V}, \bar{g})$ are respectively given by

$$R_{ijkl} = \bar{R}_{\alpha\beta\gamma\delta} B_{ijkl}^{\alpha\beta\gamma\delta} + H^2 G_{ijkl}, \tag{31}$$

$$\bar{R}_{\alpha\beta\gamma\delta} B_{ijk}^{\alpha\beta\gamma} N^\delta = H_{,i} g_{jk} - H_{,j} g_{ik}, \tag{32}$$

where R_{ijkl} and $\bar{R}_{\alpha\beta\gamma\delta}$ are curvature tensors of *(V,g)* and $(\bar{V}, \bar{g})$ respectively, and

$$B_{ijkl}^{\alpha\beta\gamma\delta} = B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta, B_i^\alpha = y_i^\alpha, G_{ijkl} = g_{il} g_{jk} - g_{ik} g_{jl}.$$

Also we have ([14, 15])

$$\bar{S}_{\alpha\delta} B_i^\alpha B_j^\delta = S_{ij} - (n-1) H^2 g_{ij}, \tag{33}$$

$$\bar{S}_{\alpha\delta} N^\alpha B_i^\delta = (n-1) H_{,i}, \tag{34}$$

$$\bar{r} = r - n(n-1) H^2, \tag{35}$$

where S_{ij} and $\bar{S}_{\alpha\delta}$ are the Ricci tensors of *(V,g)* and $(\bar{V}, \bar{g})$ respectively and r and $\bar{r}$ are the scalar curvatures of *(V,g)* and $(\bar{V}, \bar{g})$ respectively.

In terms of local coordinates the relation (2) can be written as

$$\begin{aligned} &S_{ij,k} + S_{jk,i} + S_{ki,j} = 2A_k S_{ij} + A_i S_{jk} + A_j S_{ki} \\ &\text{or,} \\ &S_{ij,k} + S_{jk,i} + S_{ki,j} = 2A_i S_{jk} + A_j S_{ki} + A_k S_{ij} \\ &\text{or,} \\ &S_{ij,k} + S_{jk,i} + S_{ki,j} = 2A_j S_{ki} + A_k S_{ij} + A_i S_{jk}. \end{aligned} \tag{36}$$

Let $(\overline{V},\overline{g})$ be a *(PCRS)$_n$*. Then we get

$$\overline{S}_{\alpha\beta,\gamma}+\overline{S}_{\beta\gamma,\alpha}+\overline{S}_{\gamma\alpha,\beta}=2A_\gamma\overline{S}_{\alpha\beta}+A_\alpha\overline{S}_{\beta\gamma}+A_\beta\overline{S}_{\gamma\alpha}$$
or,
$$\overline{S}_{\alpha\beta,\gamma}+\overline{S}_{\beta\gamma,\alpha}+\overline{S}_{\gamma\alpha,\beta}=2A_\alpha\overline{S}_{\beta\gamma}+A_\beta\overline{S}_{\gamma\alpha}+A_\gamma\overline{S}_{\alpha\beta} \tag{37}$$
or,
$$\overline{S}_{\alpha\beta,\gamma}+\overline{S}_{\beta\gamma,\alpha}+\overline{S}_{\gamma\alpha,\beta}=2A_\beta\overline{S}_{\gamma\alpha}+A_\gamma\overline{S}_{\alpha\beta}+A_\alpha\overline{S}_{\beta\gamma}.$$

where *A*, *B* are nowhere vanishing 1-forms.

Multiplying both sides of (37) by $B^{\alpha\beta\gamma}_{ijk}$ and then using (33) and (36), we obtain either

$$H = 0$$

or

$$2[H_{,k}g_{ij}+H_{,i}g_{kj}+H_{,j}g_{ik}]=H[2A_kg_{ij}+A_ig_{kj}+A_jg_{ik}]. \tag{38}$$

Transvecting (38) by g^{ij}, we obtain

$$H_{,k}=\frac{n+1}{n+2}A_k \tag{39}$$

for all *k*. This leads to the following:

Theorem 5.1. If the totally umbilical hypersurface of a *(PCRS)$_n$* is a *(PCRS)$_n$* then either the manifold is a totally geodesic hypersurface or the associated 1-form *A* satisfies the relation (39).

We now consider that the space *(V,g)* is totally geodesic hypersurface, i.e.,

$$H = 0. \tag{40}$$

In view of (40), (33) yields

$$\overline{S}_{\alpha\delta}B^{\alpha}_{i}B^{\delta}_{j}=S_{ij}. \tag{41}$$

Using (41) in (37), we have the relation (36). Thus we can state the following:

Theorem 5.2. The totally geodesic hypersurface of a *(PCRS)$_n$* is also *(PCRS)$_n$*.

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