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ON PSEUDO-FROBENIUS ELEMENTS OF SUBMONOIDS OF $\mathbb{N}^d$

J.I. GARCÍA-GARCÍA, I. OJEDA, J.C. ROSALES, AND A. VIGNERON-TENORIO

ABSTRACT. In this paper we study those submonoids of $\mathbb{N}^d$ which a non-trivial pseudo-Frobenius set. In the affine case, we prove that they are the affine semigroups whose associated algebra over a field has maximal projective dimension possible. We prove that these semigroups are a natural generalization of numerical semigroups and, consequently, most of their invariants can be generalized. In the last section we introduce a new family of submonoids of $\mathbb{N}^d$ and using its pseudo-Frobenius elements we prove that the elements in the family are direct limits of affine semigroups.

INTRODUCTION

Throughout this paper $\mathbb{N}$ will denote the set of nonnegative integers. Unless otherwise stated, all considered semigroups will be submonoids of $\mathbb{N}^d$. Finetelly generated submonoids of $\mathbb{N}^d$ will be called, affine semigroups, as usual. If $d = 1$, affine semigroups are called numerical semigroups. Numerical semigroup has been widely studied in the literature (see, for instance, [13] and the references therein). It is well-known that $S \subseteq \mathbb{N}$ is a numerical semigroup if and only if S is a submonoid of $\mathbb{N}$ such that $\mathbb{N} \setminus S$ is a finite set. Clearly, this does not hold for affine semigroups in general. In [8], the affine semigroups whose complementary in the cone that they generate is finite, in a suitable sense, are called $\mathcal{C}-$semigroups and the authors prove that a Wilf's conjecture can be generalized to this new situation.

In the numerical case, the finiteness of $\mathbb{N} \setminus S$ implies that there exists at least a positive integers $a \in \mathbb{N}\setminus S$ such that $a+S\setminus\{0\} \subseteq S$ (provided that $S \neq \mathbb{N}$). These integers are called pseudo-Frobenius numbers and the biggest one

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is the so-called Frobenius number. In this paper, we consider the semigroups such that there exists at least one element $\mathbf{a} \in \mathbb{N}^d \setminus S$ with $\mathbf{a} + S \setminus \{0\} \subseteq S$. By analogy, we call these elements the pseudo-Frobenius elements of S. We emphasize that the semigroups in the family of $\mathcal{C}-$semigroups have pseudo-Frobenius elements.

One the main results in this paper is Theorem 6 which states that an affine semigroup, S, has pseudo-Frobenius elements if and only if the length of the minimal free resolution of the semigroup algebra $\Bbbk[S]$, with $\Bbbk$ being a field, as a module over a polynomial ring is maximal, that is, if $\Bbbk[S]$ has the maximal projective dimension possible (see Section 2). For this reason we will say that affine semigroups with pseudo-Frobenius elements are maximal projective dimension semigroups, MPD-semigroups for short. Observe that, as minimal free resolution of semigroup algebras can be effectively computed, our results provides a computable necessary and sufficient condition for an affine semigroup to be a MPD-semigroup. In fact, using the procedure outlined [12], one could theoretically compute the minimal free resolution of $\Bbbk[S]$ from the pseudo-Frobenius elements. To this end, the effective computation of the pseudo-Frobenius is needed. We provide a bound of the pseudo-Frobenius elements in terms of d and the cardinal and size of the minimal generating set of S (see Corollary 9).

In order to emulate the maximal property of the Frobenius element of a numerical semigroup, we fix a term order on $\mathbb{N}^d$ and define Frobenius elements as the maximal elements for the fixed order of the elements in the integral points in the complementary of an affine semigroup S with respect to its cone. These Frobenius elements (if exist) are necesarily pseudo-Frobenius elements of S for a maximality matter (see Lemma 12). So, if Frobenius elements of S exist then $\Bbbk[S]$ has maximal projective dimension, unfortunately the converse it is not true in general. However, there are relevant families of affine semigroups with Frobenius elements as the family of $\mathcal{C}-$semigroups. In the section devoted to Frobenius elements (Section 3), we prove generalizations of well-known results for numerical semigroups such as Selmer's Theorem that relates the Frobenius numbers and Apéry sets (Theorem 16) or the characterization of the pseudo Frobenius elements in terms of the Apéry sets (Proposition 17). We close this section proving that affine semigroups having Frobenius elements are stable by gluing, by giving a formula for a Frobenius element in the gluing (Theorem 18).

In the Section 4, we deal with the problem of the irreducibility of the MPD-semigroups. Again, we prove the analogous results for MPD-semigroups than the known-ones for irreducible numerical semigroups. Of special interest is the characterization of the pseudo-Frobenius sets for $\mathcal{C}-$semigroups (Theorem 22 and Proposition 25). Pedro A. García-Sánchez communicated us that similar results were obtained by C. Cisto, G. Fiolla, C. Peterson and R. Utano in [4] for $\mathbb{N}^d-$semigroups, that is, those $\mathcal{C}-$semigroups whose associated cone is the whole $\mathbb{N}^d$.

Finally, in the last section of this paper, we introduce a new family of (non-necessarily finitely generated) submonoids of $\mathbb{N}^d$ which have pseudo-Frobenius elements, the elements of the family are called PI-monoids. These monoids are a natural generalization of the MED-semigroups (see [13, Chapter 3]). We conclude the paper by proving that any PI-monoid is direct limit of MPD-semigroups.

The study of $\mathbb{N}^d-$semigroups, which is a subfamily of the MPD-semigroups, is becoming to be an active research area in affine semigroups theory. For instance, in [3] algorithms for dealing with $\mathbb{N}^d-$semigroups are given. These algorithms are implemented in the development version site of the GAP ([11]) `NumericalSgps` package ([10]):

`https://github.com/gap-packages/numericalsgps`.

1. Pseudo-Frobenius elements of affine semigroups

Set $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$ and let S be the submonoid of $\mathbb{N}^d$ generated by $\mathcal{A}$. Consider the cone of S in $\mathbb{Q}^d_{\geq 0}$

$$\mathrm{pos}(S) := \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{a}_i \ \mid \ \lambda_i \in \mathbb{Q}_{\geq 0},\ i = 1, \ldots, n \right\}$$

and define $\mathcal{H}(S) := (\mathrm{pos}(S) \setminus S) \cap \mathbb{N}^d$.

Definition 1. An integer vector $\mathbf{a} \in \mathcal{H}(S)$ is called a **pseudo-Frobenius element** of S if $\mathbf{a} + S \setminus \{0\} \subseteq S$. The set of pseudo-Frobenius elements of S is denoted by $\mathrm{PF}(S)$.

Observe that the set $\mathrm{PF}(S)$ may be empty: indeed, let

$$\mathcal{A} = \{(2,0), (1,1), (0,2)\} \subset \mathbb{N}^2.$$

The semigroup S generated by $\mathcal{A}$ is the subset of points in $\mathbb{N}^2$ whose sum of coordinates is even. Thus, we has that $\mathcal{H}(S) + S = \mathcal{H}(S)$. Therefore $\mathrm{PF}(S) = \varnothing$.

On other hand, $\mathrm{PF}(S) \neq \mathcal{H}(S)$ in general as the following example shows.

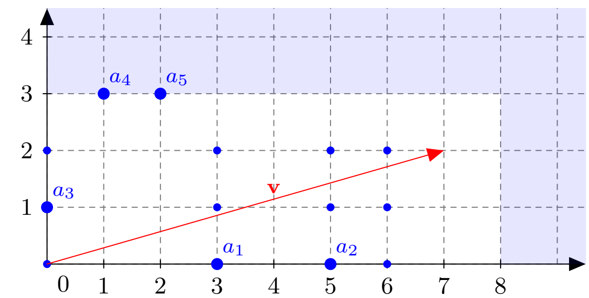
Example 2. Let S be the submonoid of $\mathbb{N}^2$ generated by the columns of the following matrix

$$A = \begin{pmatrix} 3 & 5 & 0 & 1 & 2 \\ 0 & 0 & 1 & 3 & 3 \end{pmatrix}$$

In this case,

$$\begin{aligned}\mathcal{H}(S) = \{&(1,0), (2,0), (4,0), (1,1), (2,1), (4,1), \\ &(1,2), (2,2), (4,2), (7,0), (7,1), (7,2)\},\end{aligned}$$

whereas $\mathrm{PF}(S) = \{(7,2)\}$:



The elements in S are the blue points and the points in the shadowed blue area. The red vector is the only pseudo-Frobenius element of S. The big blue points correspond to the minimal generators of S.

There are relevant families of numerical semigroups for which the pseudo-Frobenius element exist.

Definition 3. If $\mathcal{H}(S)$ is finite, then S is said to be a $\mathcal{C}-$**semigroup**, where $\mathcal{C}$ denotes the cone $\mathrm{pos}(S)$.

The $\mathcal{C}-$semigroups were introduced in [8].

Proposition 4. *If S is a $\mathcal{C}-$semigroup different from $\mathcal{C}\cap\mathbb{N}^d$, then $PF(S)\neq\varnothing$.*

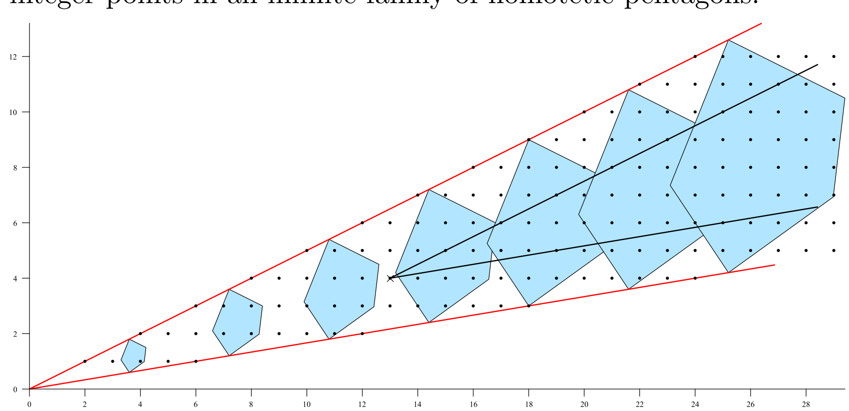
Proof. Let $\preceq$ be a term order on $\mathbb{N}^d$ and set $\mathbf{a} := max_{\preceq}(\mathcal{H}(S))$. If $\mathbf{b}\in S\setminus\{0\}$ is such that $\mathbf{a}+\mathbf{b}\notin S$, then $\mathbf{a}+\mathbf{b}\in\mathcal{H}(S)$ and $\mathbf{a}+\mathbf{b}\succ\mathbf{a}$ which contradicts the maximality of $\mathbf{a}$. $\square$

The converse of the above proposition is not true, as the following example shows.

Example 5. Let $\mathcal{A}\subset\mathbb{N}^2$ be the columns of the matrix

$$A=\begin{pmatrix} 18 & 18 & 4 & 20 & 23 & 8 & 11 & 11 & 10 & 14 & 7 & 7\\ 9 & 3 & 1 & 8 & 10 & 3 & 5 & 2 & 3 & 3 & 2 & 3\end{pmatrix}$$

and let S be the subsemigroup of $\mathbb{N}^2$ generated by $\mathcal{A}$. The elements in S are the integer points in an infinite family of homotetic pentagons.



This semigroup is a so-called multiple convex body semigroup (see [9] for further details). Clearly, S is not a $\mathcal{C}-$semigroup, but $(13,4) \in \mathrm{PF}(S)$.

2. Maximal projective dimension

As in the previous section, set $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$ and let S be the submonoid of $\mathbb{N}^d$ generated by $\mathcal{A}$. Let $\Bbbk$ be an arbitrary field.

The surjective $\Bbbk$-algebra morphism

$$\varphi_0 : R := \Bbbk[x_1, \ldots, x_n] \longrightarrow \Bbbk[S] := \bigoplus_{\mathbf{a}\in S} \Bbbk\, \chi^{\mathbf{a}};\ x_i \longmapsto \chi^{\mathbf{a}_i}$$

is S-graded, thus, the ideal $I_S := \ker(\varphi_0)$ is a S-homogeneous ideal called the ideal of S. Notice that I_S is a toric ideal generated by

$$\Big\{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \sum_{i=1}^{n} u_i \mathbf{a}_i = \sum_{i=1}^{n} v_i \mathbf{a}_i\Big\}.$$

Now, by using the S-graded Nakayama's lemma recursively, we may construct $S-$graded $\Bbbk-$algebra homomorphism

$$\varphi_{j+1} : R^{s_{j+1}} \longrightarrow R^{s_j},$$

corresponding to a choice of a minimal set of $S-$homogeneous generators for each module of syzygies $N_j := \ker(\varphi_j), j \geq 0$ (see [5] and the references therein). Notice that $N_0 = I_S$. Thus, we obtain a minimal free $S-$graded resolution for the $R-$module $\Bbbk[S]$ of the form

$$\ldots \longrightarrow R^{s_{j+1}} \xrightarrow{\varphi_{j+1}} R^{s_j} \longrightarrow \ldots \longrightarrow R^{s_2} \xrightarrow{\varphi_2} R^{s_1} \xrightarrow{\varphi_1} R \xrightarrow{\varphi_0} \Bbbk[S] \longrightarrow 0,$$

where $s_{j+1} := \sum_{\mathbf{b}\in S} \dim_{\Bbbk} V_j(\mathbf{b})$, with $V_j(\mathbf{b}) := (N_j)_{\mathbf{b}}/(\mathfrak{m}N_j)_{\mathbf{b}}$, is the so-called $(j+1)$th Betti number of $\Bbbk[S]$, where $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$ is the irrelevant maximal ideal.

Observe that the dimension of $V_j(\mathbf{b})$ is the number of generators of degree $\mathbf{b}$ in a minimal system of generators of the jth module of syzygies N_j (i.e. the multigraded Betti number $s_{j,\mathbf{b}}$). The $\mathbf{b} \in S$ such that $V_j(\mathbf{b}) \neq 0$ are called **$S-$degrees of the $j-$minimal syzygy** of $\Bbbk[S]$. So, by the Noetherian property of R, s_{j+1} is finite. Moreover, by the Hilbert's syzygy theorem and the Auslander-Buchsbaum's formula, it follows that $s_j = 0$ for $j > p = n - \mathrm{depth}_R \Bbbk[S]$ and $s_p \neq 0$. Such integer p is called the **projective dimension of S**.

Since $\mathrm{depth}_R \Bbbk[S] \geq 1$, the projective dimension of S is lesser than or equal to $n-1$. We will say that S is a **maximal projective dimension semigroup** (MPD-semigroup, for short) if its projective dimension is $n-1$, equivalently, if $\mathrm{depth}_R \Bbbk[S] = 1$.

Recall that S is said to be Cohen-Macaulay if $\mathrm{depth}_R \Bbbk[S] = \dim(\Bbbk[S])$. So, if S is a MPD-semigroup, then S is Cohen-Macaulay if and only if $\Bbbk[S]$ is the coordinate ring of a monomial curve; equivalently, S is a numerical semigroup.

Theorem 6. *The necessary and sufficient condition for S to be a MPD-semigroup is that* $\mathrm{PF}(S) \neq \varnothing$. *In this case,* $\mathrm{PF}(S)$ *has finite cardinality.*

Proof. By definition, S is a MPD-semigroup if and only if $V_{n-2}(\mathbf{b}) \neq \varnothing$ for some $\mathbf{b} \in S$. By [5, Theorem 2.1], given $\mathbf{b} \in S$, $V_{n-2}(\mathbf{b}) \neq \varnothing$ if and only if $\mathbf{b} - \sum_{i=1}^n \mathbf{a}_i \notin S$ and $\mathbf{b} - \sum_{i \in F} \mathbf{a}_i \in S$, for every $F \subsetneq \{1, \ldots, n\}$. Clearly, if $\mathrm{PF}(S) \neq \varnothing$, there exists $\mathbf{a} \in \mathcal{H}(S)$ such that $\mathbf{a} + S \setminus \{0\} \subseteq S$. So, by taking $\mathbf{b} = \mathbf{a} + \sum_{i=1}^n \mathbf{a}_i$, one has that $\mathbf{b} - \sum_{i=1}^n \mathbf{a}_i \notin S$ and $\mathbf{b} - \sum_{i \in F} \mathbf{a}_i \in S$, for every $F \subsetneq \{1, \ldots, n\}$, and we conclude S is a MPD-semigroup. Conversely, if there exists $\mathbf{b} \in S$ such that $V_{n-2}(\mathbf{b}) \neq \varnothing$, then we have that $\mathbf{a} := \mathbf{b} - \sum_{i=1}^n \mathbf{a}_i \in \mathbb{Z}^d \setminus S$. Clearly, $\mathbf{a} \in \mathbb{Z}^d \setminus S$ and $\mathbf{a} + S \setminus \{0\} \subseteq S$, because $\mathbf{a} + \mathbf{a}_j \in S$, for every $j \in \{1, \ldots, n\}$. Thus, in order to see that $\mathbf{a} \in \mathcal{H}(S)$ it suffices to prove that $\mathbf{a} \in \mathrm{pos}(S)$. Without loss of generality, we assume that $\{a_1, \ldots, a_\ell\}$ is a minimal set of generators of $\mathrm{pos}(S)$. By Farkas' Lemma, $\mathrm{pos}(S)$ is a rational convex polyhedral cone. Then, for each $j \in \{1, \ldots, \ell\}$ there exists $\mathbf{c}_j \in \mathbb{R}^d$ such that $\mathbf{a}_j \cdot \mathbf{c}_j = 0$ and $\mathbf{a}_i \cdot \mathbf{c}_j \geq 0,\ i \neq j$, where $\cdot$ denotes the usual inner product on $\mathbb{R}^n$. Now, since $\mathbf{a} + \mathbf{a}_j \in S$, one has that $\mathbf{a} + \mathbf{a}_j = \sum_{i=1}^n u_{ij} \mathbf{a}_i$, for some $u_{ij} \in \mathbb{N}$. Therefore

$$\mathbf{a} \cdot \mathbf{c}_j = (\mathbf{a} + \mathbf{a}_j) \cdot \mathbf{c}_j = (\sum_{i=1}^n u_{ij} \mathbf{a}_i) \cdot \mathbf{c}_j = \sum_{i=1}^n u_{ij} (\mathbf{a}_i \cdot \mathbf{c}_j) \geq 0,$$

for every $j \in \{1, \ldots, \ell\}$. That is to say $\mathbf{a} \in \mathrm{pos}(S)$.

Finally, if $\mathrm{PF}(S) \neq \varnothing$, the finiteness of $\mathrm{PF}(S)$ follows from the finiteness of the Betti numbers. □

Observe that from the arguments in the proof of Theorem 6 it follows that s_p is the cardinality of $\mathrm{PF}(S)$. In fact, we have proved the following fact:

Corollary 7. *If S is a MPD-semigroup, then* $\mathbf{b} \in S$ *is the $S-$degree of the $(n-2)$th minimal syzygy of* $\Bbbk[S]$ *if and only if* $\mathbf{b} \in \{\mathbf{a} + \sum_{i=1}^n \mathbf{a}_i,\ \mathbf{a} \in \mathrm{PF}(S)\}$.

In [12], it is outlined a procedure for a partial computation of the minimal free resolution of S starting from a set of $S-$degrees of the jth minimal syzygy of $\Bbbk[S]$, for some j. This procedure gives a whole free resolution if one knows all the $S-$degrees of the pth minimal syzygies of $\Bbbk[S]$, where p is the projective dimension of S. Therefore, by Corollary 7, if S is a MPD-semigroup, we can use the proposed method in [12] to compute the minimal free resolution of $\Bbbk[S]$, provided that we were able to compute $\mathrm{PF}(S)$. However, this is not easy at all, for this reason it is highly interesting to given bounds for the elements in $\mathrm{PF}(S)$.

Given $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$, let $\ell(u)$ be the length of $\mathbf{u}$ that is, $\ell(\mathbf{u}) = \sum_{i=1}^n u_i$, and for an $d \times n-$integer matrix $B = (\mathbf{b}_1 | \cdots | \mathbf{b}_n)$, we will write $||B||_\infty$ for $\max_i \sum_{j=1}^n |b_{ij}|$. In [6], the authors provide an explicit bound for the $S-$degrees of the minimal generators of N_j, for every $j \in \{1, \ldots, p\}$.

Let $A \in \mathbb{N}^{d \times n}$ be the matrix whose ith column is $\mathbf{a}_i,\ i = 1, \ldots, n$.

Theorem 8. [6, Theorem 3.2] *If* $\mathbf{b} \in S$ *is an* $S-$*degree of a minimal* $j-$*syzygy of* $\Bbbk[S]$*, then* $\mathbf{b} = A\mathbf{u}$ *with* $\mathbf{u} \in \mathbb{N}^n$ *such that*

$$\ell(\mathbf{u}) \leq (1 + 4\,||A||_\infty)^{d(d_j-1)} + (j+1)d_j - 1,$$

where $d_j = \left(\begin{array}{c} n \\ j+1 \end{array}\right)$.

By Corollary 7, this bound can be particularized for $j = n-2$ as follows.

Corollary 9. *Let* S *be a MPD-semigroup. If* $\mathbf{a} \in \mathrm{PF}(S)$*, then* $\mathbf{a} = A\mathbf{v}$ *for some* $\mathbf{v} \in \mathbb{N}^n$ *satisfying*

$$\ell(\mathbf{v}) \leq (1 + 4\,||A||_\infty)^{d(n-1)} + n^2 - 1.$$

Proof. Assuming $\mathbf{a}$ is a pseudo-Frobenius element of S, by Corollary 7, there exists $\mathbf{b} = A\mathbf{u} \in S$ for some $\mathbf{u} \in \mathbb{N}^n$, with $V_{n-2}(\mathbf{b}) \neq \varnothing$ and such that $\mathbf{a} = \mathbf{b} - \sum_{i=1}^n \mathbf{a}_i$; in particular $\mathbf{a} = A\mathbf{u} - A\,\mathbf{1} = A(\mathbf{u} - \mathbf{1})$. Consider $\mathbf{v} = \mathbf{u} - \mathbf{1}$. Note that $||\mathbf{v}||_1 \leq ||\mathbf{u}||_1 + ||\mathbf{1}||_1 = ||\mathbf{u}||_1 + n$. By Theorem 8, $||\mathbf{u}||_1 \leq (1 + 4||A||_\infty)^{d(d_j-1)} + (j+1)d_j - 1$ where $j = n-2$ and $d_{n-2} = n$. So,

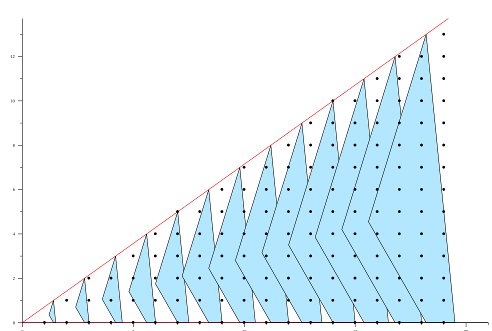
$$\begin{aligned}\ell(\mathbf{v}) &\leq (1 + 4\,||A||_\infty)^{d(n-1)} + (n-1)n - 1 + n \\ &= (1 + 4||A||_\infty)^{d(n-1)} + n^2 - 1,\end{aligned}$$

as claimed. □

Note that, given any affine semigroup and the graded minimal free resolution of its associated algebra over a field, Theorem 6 and Corollary 7 allow us to check if the semigroup has pseudo-Frobenius elements and, in affirmative case, to compute them. Thus, the combination of both results provide an algorithm for the computation of the pseudo-Frobenius elements of a affine semigroup, provided that they exist (i.e. if the depth of the algebra is one). The following example illustrates this fact:

Example 10. Let S be the multiple convex body semigroup associated to the convex hull $\mathcal{P}$ of the set $\{(1.2, .35), (1.4, 0), (1.5, 0), (1.4, 1)\}$, that is to say,

$$S = \bigcup_{k \in \mathbb{N}} k\,\mathcal{P} \cap \mathbb{N}^2.$$



Using the `Mathematica` package `PolySGTools` introduced in [9], we obtain that the minimal generating system of S is the set of columns of the following matrix

$$A = \begin{pmatrix} 3 & 4 & 4 & 5 & 7 & 7 & 7 & 7 & 8 & 9 \\ 0 & 1 & 2 & 2 & 0 & 3 & 4 & 5 & 1 & 2 \end{pmatrix}.$$

Now, we can easily check that S is not a $\mathcal{C}-$semigroup, because $(3,2) + \lambda(7,5) \in \mathcal{H}(S)$, for every $\lambda \in \mathbb{N}$. Moreover, we can compute the $S-$graded minimal free resolution of $\Bbbk[S]$ using Singular ([7]) as follows:

```
LIB "toric.lib";
LIB "multigrading.lib";
ring r = 0, (x(1..10)), dp;
intmat A[2][10] = 3, 4, 4, 5, 7, 7, 7, 7, 8, 9,
                  0, 1, 2, 2, 0, 3, 4, 5, 1, 2;
setBaseMultigrading(A);
ideal i = toric_ideal(A,"ect");
def L = multiDegResolution(i,9,0);
```

Finally, using the command `multiDeg(L[9])`, we obtain that the degrees of the minimal generators of the $9-$th syzygy module are $(72,20)$ and $(73,21)$. So, S has two pseudo-Frobenius elements: $(11,0) = (72,20) - (61,20)$ and $(12,1) = (73,21) - (61,20)$.

3. On Frobenius elements of MPD-semigroups

Throughout this section, S will be a MPD-semigroup generated by $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$.

Definition 11. We say that $\mathbf{f} \in \mathcal{H}(S)$ is a **Frobenius element** of S if $\mathbf{f} = \max_{\prec} \mathcal{H}(S)$ for some term order $\prec$ on $\mathbb{N}^d$. Let us write $\mathrm{F}(S)$ for the set of Frobenius elements of S.

Frobenius elements of S may not exist. However, if S is a $\mathcal{C}-$semigroup, then it has Frobenius elements because $\mathcal{H}(S)$ is finite.

Lemma 12. *Every Frobenius element of S is a pseudo-Frobenius element of S, in symbols:* $\mathrm{F}(S) \subseteq \mathrm{PF}(S)$.

Proof. If $\mathbf{f} \in \mathrm{F}(S)$, then there is a term order $\prec$ on $\mathbb{N}^d$ such that $\mathbf{f} = \max_{\prec} \mathcal{H}(S)$. If there exists $\mathbf{a} \in S \setminus \{0\}$ such that $\mathbf{f} + \mathbf{a} \notin S$, then $\mathbf{f} \prec \mathbf{f} + \mathbf{a} \in \mathcal{H}(S)$, in contradiction to the maximality of $\mathbf{f}$. Therefore $\mathbf{f} \in \mathrm{PF}(S)$. □

The following notion of Frobenius vectors was introduced in [2]: we say that S has a **Frobenius vector** if there exists $\mathbf{f} \in G(\mathcal{A}) \setminus S$ such that

$$\mathbf{f} + \mathrm{relint}(\mathrm{pos}(S)) \cap G(\mathcal{A}) \subseteq S \setminus \{0\} \subseteq S,$$

where $G(\mathcal{A})$ denotes the group generated by $\mathcal{A}$ in $\mathbb{Z}^d$ and $\mathrm{relint}(\mathrm{pos}(S))$ the relative interior of the cone $\mathrm{pos}(S)$.

Proposition 13. *Every Frobenius element of S is a Frobenius vector of S.*

Proof. Let $\mathbf{f} \in \mathrm{F}(S)$. Since $\mathbf{a} := \mathbf{f} + \mathbf{a}_1 \in S$, then $\mathbf{f} = \mathbf{a} - \mathbf{a}_1 \in G(\mathcal{A}) \setminus S$. Let $\mathbf{b} \in \mathrm{relint}(\mathrm{pos}(S)) \cap G(\mathcal{A})$. If $\mathbf{b} \in S$, then $\mathbf{f} + \mathbf{b} \in S$ by Lemma 12. If $\mathbf{b} \notin S$, then $\mathbf{b} \in \mathcal{H}(S) = (\mathrm{pos}(S) \setminus S) \cap \mathbb{N}^d$ and therefore either $\mathbf{f} + \mathbf{b} \in S$ or $\mathbf{f} + \mathbf{b} \in \mathcal{H}(S)$. However, since $\mathbf{f} + \mathbf{b}$ is greater than $\mathbf{f}$ for every term order on $\mathbb{N}^d$, we are done. □

As a consequence of the above result, we have that the set of $\mathcal{C}$-semigroups is a new family of affine semigroups for which Frobenius vectors exist.

Although Frobenius vectors may not exist in general, there are families of submonoids of $\mathbb{N}^d$ with Frobenius vectors that are not MPD-semigroups (see [2]). However, even for MPD-semigroups, the converse of the above proposition does not hold in general, for instance, the MPD-semigroup in Example 5 has a Frobenius vector which is not a Frobenius element.

Now, similarly to the numerical case ($n = 1$), we can give a Selmer's formula type (see [13, Proposition 2.12(a)]) for $\mathcal{C}-$semigroups. To do this, we need to recall the notion of Apéry set.

Definition 14. The **Apéry set** of a submonoid S of $\mathbb{N}^d$ relative to $\mathbf{b} \in S \setminus \{0\}$ is defined as $\mathrm{Ap}(S, \mathbf{b}) = \{\mathbf{a} \in S \mid \mathbf{a} - \mathbf{b} \in \mathrm{pos}(S) \setminus S\}$.

Clearly $\mathrm{Ap}(S, \mathbf{b}) - \mathbf{b} \subseteq \mathcal{H}(S)$; in particular, if S is $\mathcal{C}-$semigroup, we have that $\mathrm{Ap}(S, \mathbf{b})$ is finite for every $\mathbf{b} \in S \setminus \{0\}$.

Proposition 15. *Let S be a submonoid of $\mathbb{N}^d$ and $\mathbf{b} \in S \setminus \{0\}$. For each $\mathbf{a} \in S$ there exists an unique $(k, \mathbf{c}) \in \mathbb{N} \times \mathrm{Ap}(S, \mathbf{b})$ such that $\mathbf{a} = k\,\mathbf{b} + \mathbf{c}$. In particular, $\big(\mathrm{Ap}(S, \mathbf{b}) \setminus \{0\}\big) \cup \{\mathbf{b}\}$ is a system of generators of S.*

Proof. If suffices to take k as the highest non-negative integer such that $\mathbf{b} - k\mathbf{a} \in S$. □

The following result is the generalization of [13, Proposition 2.12(a)].

Theorem 16. *If $\mathbf{f} \in \mathrm{F}(S)$, there exists a term order $\prec$ on $\mathbb{N}^d$ such that*

$$\mathbf{f} = \max_{\prec} \mathrm{Ap}(S, \mathbf{b}) - \mathbf{b},$$

for every $\mathbf{b} \in S \setminus \{0\}$.

Proof. By definition, there exists a term order $\prec$ on $\mathbb{N}^d$ such that $\mathbf{f} = \max_{\prec} \mathcal{H}(S)$. By Lemma 12, $\mathbf{f}+\mathbf{b} \in S$ and clearly $(\mathbf{f}+\mathbf{b})-\mathbf{b} = \mathbf{f} \notin \mathrm{pos}(S)\setminus S$. Thus, $\mathbf{f} + \mathbf{b} \in \mathrm{Ap}(S, \mathbf{b})$. Now, suppose that there exist $\mathbf{a} \in \mathrm{Ap}(S, \mathbf{b})$ such that $\mathbf{f} + \mathbf{b} \prec \mathbf{a}$. In this case, $\mathbf{a} - \mathbf{b} \in \mathcal{H}(S)$, so $\mathbf{a} - \mathbf{b} \preceq \mathbf{f}$ and therefore $\mathbf{a} \prec \mathbf{f} + \mathbf{b}$ which contradicts the anti-symmetry property of $\prec$. □

Next result generalizes [13, Proposition 2.20].

Proposition 17. *Let S be a submonoid of $\mathbb{N}^d$ and $\mathbf{b} \in S \setminus \{0\}$. Then $\mathrm{PF}(S) \neq \varnothing$ if and only if $\mathrm{maximals}_{\preceq_S} \mathrm{Ap}(S, \mathbf{b}) \neq \varnothing$. In this case,*

$$\mathrm{PF}(S) = \{\mathbf{a} - \mathbf{b} \mid \mathbf{a} \in \mathrm{maximals}_{\preceq_S} \mathrm{Ap}(S, \mathbf{b})\}. \tag{1}$$

Proof. Suppose that there exists $\mathbf{b}' \in \mathrm{PF}(S)$ and let $\mathbf{a} = \mathbf{b}' + \mathbf{b}$. Clearly $\mathbf{a} \in \mathrm{Ap}(S, \mathbf{b})$, and we claim that $\mathbf{a} \in \mathrm{maximals}_{\preceq_S} \mathrm{Ap}(S, \mathbf{b})$. Otherwise, there exists $\mathbf{a}' \in \mathrm{Ap}(S, \mathbf{b})$ such that $\mathbf{a}' - \mathbf{a} \in S$, then

$$\mathbf{a}' - \mathbf{b} = (\mathbf{a}' - \mathbf{b}) + \mathbf{b}' - \mathbf{b}' = \mathbf{b}' + (\mathbf{a}' - (\mathbf{b}' + \mathbf{b})) = \mathbf{b}' + (\mathbf{a}' - \mathbf{a}) \in S,$$

which contradicts the definition of Apéry set of S relative to $\mathbf{b}$. Therefore, $\mathbf{b}' = \mathbf{a} - \mathbf{b}$ with $\mathbf{a} \in \mathrm{maximals}_{\preceq_S} \mathrm{Ap}(S, \mathbf{b})$. Consider now $\mathbf{a}'' \in \mathrm{maximals}_{\preceq_S} \mathrm{Ap}(S, \mathbf{b})$ and let $\mathbf{b}'' = \mathbf{a}'' - \mathbf{b}$. If $\mathbf{b}'' \notin \mathrm{PF}(S)$, then $\mathbf{b}'' + \mathbf{a}_i \notin S$, for some $i \in \{1, \ldots, n\}$, that is to say, $\mathbf{a}'' + \mathbf{a}_i \in \mathrm{Ap}(S, \mathbf{b})$ which is not possible by the maximality of $\mathbf{a}''$ in $\mathrm{Ap}(S, \mathbf{b})$ with respect to $\preceq_S$. $\square$

Observe that (1) holds for every MPD-semigroup.

We end this section by proving that MPD-semigroups are stable by gluing. First of all, let us recall the notion of gluing of affine semigroups.

Given an affine semigroup $S \subseteq \mathbb{N}^d$, denote by $G(S)$ the group spanned by S, that is,

$$G(S) = \big\{\mathbf{z} \in \mathbb{Z}^m \mid \mathbf{z} = \mathbf{a} - \mathbf{b}, \mathbf{a}, \mathbf{b} \in S\big\}.$$

Assume that S is finitely generated. Let $\mathcal{A}$ be the minimal generating system of S and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ be a nontrivial partition of $\mathcal{A}$. Let S_i be the submonoid of $\mathbb{N}^d$ generated by $A_i,\ i \in \{1, 2\}$. Then $S = S_1 + S_2$. We say that S is the **gluing** of S_1 and S_2 by $\mathbf{d}$ if

- $\mathbf{d} \in S_1 \cap S_2$ and,
- $G(S_1) \cap G(S_2) = \mathbf{d}\mathbb{Z}$.

We will denote this fact by $S = S_1 +_{\mathbf{d}} S_2$.

Theorem 18. *Let S be an finitely generated submonoid of $\mathbb{N}^d$. Assume that $S = S_1 +_{\mathbf{d}} S_2$. If S_1 and S_2 are MPD-semigroups, and $\mathbf{b}_i \in \mathrm{PF}(S_i),\ i = 1, 2$, then $\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{d} \in \mathrm{PF}(S)$. In particular, S is a MPD-semigroup.*

Proof. Let $\mathbf{b} := \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{d}$. Since $\mathbf{b}_i \in \mathrm{pos}(S_i),\ i = 1, 2$, and $\mathbf{d} \in S_1 \cap S_2$, we conclude that $\mathbf{b} \in \mathrm{pos}(S)$. If $\mathbf{b} \in S$, then there exist $\mathbf{b}'_i \in S_i,\ i = 1, 2$, such that $\mathbf{b} = \mathbf{b}'_1 + \mathbf{b}'_2$. Then $\mathbf{b}_1 + \mathbf{d} - \mathbf{b}'_1 = \mathbf{b}'_2 - \mathbf{b}_2 \in G(S_1) \cap G(S_2) = \mathbf{d}\mathbb{Z}$. So, there exist $k \in \mathbb{Z}$ such that $\mathbf{b}_1 + \mathbf{d} - \mathbf{b}'_1 = \mathbf{b}'_2 - \mathbf{b}_2 = k\mathbf{d}$. If $k \leq 0$, then $\mathbf{b}_2 = \mathbf{b}'_2 - k\mathbf{d} \in S_2$, which is impossible. If $k > 0$, then $\mathbf{b}_1 = \mathbf{b}'_1 + (k-1)\mathbf{d} \in S_1$, which is also impossible. All this prove that $\mathbf{b} \in \mathcal{H}(S)$.

Now, let $\mathbf{a} \in S \setminus \{0\}$. Again there exist $\mathbf{b}'_i \in S_i,\ i = 1, 2$, such that $\mathbf{a} = \mathbf{b}'_1 + \mathbf{b}'_2$. Since $\mathbf{d} \in S_1 \cap S_2 \subset S$. We have that $\mathbf{b}_1 + \mathbf{b}'_1 \in S_1$ and $\mathbf{b}_2 + \mathbf{b}'_2 + \mathbf{d} \in S_2$. Thus $\mathbf{b} + \mathbf{a} \in S$, and we are done. $\square$

Example 19. Let $S_1 = \{(x, y, z) \in \mathbb{N}^3 \mid z = 0\} \setminus \{(1, 0, 0)\}$ and $S_2 = \{(x, y, z) \in \mathbb{N}^3 \mid x = y\} \setminus \{(0, 0, 1)\}$. Clearly, $(1, 0, 0) \in \mathrm{PF}(S_1)$ and $(0, 0, 1) \in \mathrm{PF}(S_2)$. They are minimally generated by $\{(2, 0, 0), (3, 0, 0), (0, 1, 0), (1, 1, 0)\}$ and $\{(1, 1, 0), (1, 1, 1), (0, 0, 2), (0, 0, 3)\}$, respectively. The set $G(S_1) \cap G(S_2)$ is equal to $(1, 1, 0)\mathbb{Z}$ and $S_1 + S_2$ is generated by

$$\{(2, 0, 0), (3, 0, 0), (0, 1, 0), (1, 1, 0), (1, 1, 1), (0, 0, 2), (0, 0, 3)\}$$

By Theorem 18, $(1, 0, 0) + (0, 0, 1) + (1, 1, 0) = (2, 1, 1)$ belongs to $\mathrm{PF}(S_1 + S_2)$.

4. On the irreducibility of MPD-semigroups

Now, let us study the irreducibility of MPD-semigroups with special emphasis in the $\mathcal{C}-$semigroups case. Recall that a submonoid of $\mathbb{N}^d$ is **irreducible** if cannot be expressed as an intersection of two submonoids of $\mathbb{N}^n$ containing it properly.

Lemma 20. *Let $\mathbf{a} \in \mathrm{PF}(S)$. If $2\,\mathbf{a} \in S$, then $S \cup \{\mathbf{a}\}$ is the submonoid of $\mathbb{N}^d$ generated by $\mathcal{A} \cup \{\mathbf{a}\}$. Moreover,*

(a) if $\mathbf{a} \in \mathrm{F}(S)$ and $\mathrm{PF}(S) \neq \{\mathbf{a}\}$ then $S \cup \{\mathbf{a}\}$ is a MPD-semigroup;
(b) if S is a $\mathcal{C}-$semigroup, then $S \cup \{\mathbf{a}\}$ is a $\mathcal{C}-$semigroup.

Proof. By definition, $\mathbf{a}+\mathbf{b} \in S \subset S\cup\{\mathbf{a}\}$, for every $\mathbf{b} \in S$, and by hypothesis $2\,\mathbf{a} \in S$, so $k\,\mathbf{a} \in S$ for every $k \in \mathbb{N}$; thus, $S \cup \{\mathbf{a}\}$ is the submonoid of $\mathbb{N}^d$ generated by $\mathcal{A} \cup \{\mathbf{a}\}$.

Suppose now that $\mathbf{a}$ is a Frobenius element of S and that $\mathrm{PF}(S) \neq \{\mathbf{a}\}$. Let $\mathbf{b} \in \mathrm{PF}(S) \setminus \{\mathbf{a}\}$. Clearly, $\mathbf{a} + \mathbf{b} \in S$, because $\mathbf{a} + \mathbf{b}$ is greater than $\mathbf{a}$ for every term order on $\mathbb{N}^d$, therefore $\mathbf{b} + (S \cup \{\mathbf{a}\}) \setminus \{0\} \subset S \subset S \cup \{\mathbf{a}\}$ and we conclude that $\mathrm{PF}(S \cup \{\mathbf{a}\}) \neq \varnothing$ which proves (a).

Finally, since $\mathbf{a} \in \mathcal{H}(S)$, we have that $\mathrm{pos}(S \cup \{\mathbf{a}\}) = \mathrm{pos}(S)$. Therefore $\mathcal{H}(S \cup \{\mathbf{a}\}) \subset \mathcal{H}(S)$, that is, $S \cup \{\mathbf{a}\}$ is a $\mathcal{C}-$semigroup if S it so as claimed in (b). □

Proposition 21. *If S has a Frobenius element, $\mathbf{f}$, and is irreducible then*

(a) S is maximal among all the submonoids of $\mathbb{N}^d$ having $\mathbf{f}$ as a Frobenius element.
(b) S has an unique Frobenius element.

Proof. Suppose that S is irreducible and let S' be a submonoid of $\mathbb{N}^d$ having $\mathbf{f}$ as Frobenius element. Since, by Lemma 20, $S \cup \{\mathbf{f}\}$ is a MPD-semigroup and $S = (S \cup \{\mathbf{f}\}) \cap S'$, we conclude that $S = S'$. Finally, if S has two Frobenius elements, say $\mathbf{f}_1$ and $\mathbf{f}_2$, then $S = (S \cup \{\mathbf{f}_1\}) \cap (S \cup \{\mathbf{f}_2\})$ which contradicts the irreducibility of S. □

Notice that the submonoids in condition (a) are necessarily MPD-semigroups; indeed the existence of a Frobenius elements in a submonoid of $\mathbb{N}^d$ implies that the submonoid is a MPD-semigroup by Lemma 12.

Theorem 22. *If S has a Frobenius element, $\mathbf{f}$, and is irreducible, then either $\mathrm{PF}(S) = \{\mathbf{f}\}$ or $\mathrm{PF}(S) = \{\mathbf{f}, \mathbf{f}/2\}$.*

Proof. Suppose that $\mathrm{PF}(S) \neq \{\mathbf{f}\}$. Now, since $\mathrm{PF}(S)$ has cardinality greater than or equal to two, there exists $\mathbf{a} \in \mathrm{PF}(S)$ different from $\mathbf{f}$. If $2\mathbf{a} \in S$, then, by Lemma 20, $S \cup \{\mathbf{a}\}$ are $S \cup \{\mathbf{f}\}$ are a submonoid of $\mathbb{N}^d$ whose intersection is S, in contradiction with irreducibility of S. Therefore, we may assume that $2\mathbf{a} \notin S$ which implies $2\mathbf{a} \in \mathrm{PF}(S)$. Then $\mathbf{f} + \mathbf{u} = 2\mathbf{a}$ for some $\mathbf{u} \in \mathbb{N}^d$, because $\mathbf{f}$ is greater than or equal to $\mathbf{b}$ for every term order on

$\mathbb{N}^d$. Notice that $4\mathbf{a} = \mathbf{f} + (\mathbf{f} + 2\mathbf{u}) \in S$; so, by Lemma 20, $\mathbf{S} \cup \{2\mathbf{a}\}$ is a MPD-semigroup. Now, if $\mathbf{u} \neq 0$, then $S = (\mathbf{S} \cup \{2\mathbf{a}\}) \cap (\mathbf{S} \cup \{\mathbf{f}\})$, in contradiction with irreducibility of S. Therefore $2\mathbf{a} = \mathbf{f}$ and we are done. □

Example 23. Let S be the MPD-semigroup of Example 2. Let us see that S is irreducible. If S is not irreducible, there exist two submonoids, S_1 and S_2, of $\mathbb{N}^2$ such that $S = S_1 \cap S_2$. Since $\mathrm{pos}(S) = \mathbb{N}^2$, we have that $\mathrm{pos}(S_1) = \mathrm{pos}(S_2) = \mathbb{N}^2$ and it follows that $\mathcal{H}(S_i) \subseteq \mathcal{H}(S),\ i = 1, 2$. On other hand, since $(7, 2) \notin S$, then $(7, 2) \notin S_1$ or $(7, 2) \notin S_2$. Therefore, S_1 or S_2 is a submonoid of $\mathbb{N}^2$ such that $\mathbf{f} \in \mathrm{F}(S_1)$ or $\mathbf{f} \in \mathrm{F}(S_2)$, respectively. Now, by Proposition 21, we conclude that $S = S_1$ or $S = S_2$, that is, S is irreducible.

If $n = 1$, the converse Theorem 22 is also true (see [13, Section 4.1]). Let us see that this is also happen if we fix the cone.

Definition 24. If S is $\mathcal{C}-$semigroup, we say that S is $\mathcal{C}-$**irreducible** if cannot be expressed as an intersection of two finitely generated submonoids S_1 and S_2 of $\mathbb{N}^d$ with $\mathrm{pos}(S_1) = \mathrm{pos}(S_2) = \mathrm{pos}(S)$ containing it properly.

Proposition 25. *If S is a $\mathcal{C}-$semigroup such that* $\mathrm{PF}(S) = \{\mathbf{f}\}$ *or* $\mathrm{PF}(S) = \{\mathbf{f}, \mathbf{f}/2\}$*, then S is $\mathcal{C}-$irreducible.*

Proof. Suppose there exist two finitely generated submonoids S_1 and S_2 of $\mathbb{N}^d$ with $\mathrm{pos}(S_1) = \mathrm{pos}(S_2) = \mathrm{pos}(S)$ such that $S = S_1 \cap S_2$; in particular, S_1 and S_2 are $\mathcal{C}-$semigroups. For $i = 1, 2$, we take $\mathbf{b}_i \in \mathrm{maximals}_{\preceq_S} S_i \setminus S$. Since $S_i \setminus S$ is finite, $\mathbf{b}_i$ is well-defined for $i = 1, 2$. By maximality, $\mathbf{b}_i + \mathbf{a} \in S$ for every $\mathbf{a} \in S \setminus \{0\},\ i = 1, 2$, that is to say, $\mathbf{b}_i \in \mathrm{PF}(S)$. Therefore, $\mathbf{b}_i = \mathbf{f},\ i = 1, 2$ or $\mathbf{b}_i = \mathbf{f}$ and $\mathbf{f}_j = \mathbf{f}/2,\ \{i, j\} = \{1, 2\}$. In the first case, we obtaim $\mathbf{b}_1 = \mathbf{b}_2$ which is not possible because $\mathbf{b}_i \notin S,\ i = 1, 2$. In the second case, we obtain that $\mathbf{f} \in S_1 \cap S_2 = S$ which obviously is impossible. Therefore, $\mathbf{b}_1$ or $\mathbf{b}_2$ does not exist and we conclude that $S = S_1$ or $S = S_2$. □

5. PI-monoids

Let $\preceq_{\mathbb{N}^d}$ be the usual partial order in $\mathbb{N}^d$, that is, $\mathbf{a} = (a_1, \ldots, a_d) \preceq_{\mathbb{N}^d} \mathbf{b} = (b_1, \ldots, b_d)$ if and only if $a_i \leq b_i,\ i \in \{1, \ldots, d\}$.

Definition 26. If S is a submonoid of $\mathbb{N}^d$, we define the **multiplicity** of S as $m(S) := inf_{\preceq_{\mathbb{N}^d}}(S \setminus \{0\})$.

If $d = 1$, the notion of multiplicity defined above agrees with the notion of multipliciy of a numerical semigroup (see [13, Section 2.2]) Let us introduce a new family of submonoids of $\mathbb{N}^d$, that we have called **principal ideal monoids**, or PI$-$monoids for short. This family generalizes the notion of MED-semigroups (see [13, Chapter 3] for $d > 1$.

Definition 27. A submonoid S of $\mathbb{N}^d$ is said to be a **PI-monoid** if there exist a submonoid T of $\mathbb{N}^d$ and $\mathbf{a} \in T \setminus \{0\}$ such that $S = \big(\mathbf{a}\ + T\big) \cup \{0\}$.

Clearly, PI-monoids are not always affine semigroups, since they are not necessarily finitely generated. We will explicitly provide a minimal generating system of any PI-monoid later on, first let us explore some its properties.

Example 28. In $\mathbf{N}^2$, an example of finitely generated PI-monoid is $S_1 = (2,2) + \langle(1,1)\rangle = \langle(2,2),(3,3)\rangle$.

To obtain a non-finitely generated PI-monoid of $\mathbf{N}^2$, consider $T = \mathbf{N}^2$ and $a = (1,1)$. The PI-monoid $S_2 = (1,1) + \mathbf{N}^2$ is equal to $\{(x,y) \mid x \geq 1,\ y \geq 1\} \cup \{(0,0)\}$ which is not a finitely generated submonoid of $\mathbf{N}^2$.

Lemma 29. *If $S \subseteq \mathbb{N}^d$ is a PI-monoid, then $m(S) \in S \setminus \{0\}$. In particular, $m(S) = \min_{\preceq_{\mathbb{N}^d}}(S \setminus \{0\})$.*

Proof. Since S is a PI-monoid, there exist a submonoid T of $\mathbb{N}^d$ and $\mathbf{a} \in T \setminus \{0\}$ such that $S = \big(\mathbf{a} + T\big) \cup \{0\}$. Clearly, $\mathbf{a} = \min_{\preceq_{\mathbb{N}^d}}(S \setminus \{0\})$. □

The following result is the generalization of [13, Proposition 3.12].

Proposition 30. *Let S be a submonoid of $\mathbb{N}^d$. Then, S is a PI-monoid if and only if $m(S) \in S \setminus \{0\}$ and $(S \setminus \{0\}) - m(S)$ is a submonoid of $\mathbb{N}^d$.*

Proof. If S is a PI-monoid, by Lemma 29, $m(S) = \min_{\preceq_{\mathbb{N}^d}}(S \setminus \{0\})$; moreover, there is a submonoid T of $\mathbb{N}^d$ such that $S = (m(S) + T) \cup \{0\}$. So, $(S \setminus \{0\}) - m(S) = T$ is a submonoid of $\mathbb{N}^d$. For the converse implication, it suffices to note that $S = (m(S)+T) \cup \{0\}$ and that $T = (S \setminus \{0\}) - m(S)$. □

Corollary 31. *If $S \subseteq \mathbb{N}^d$ is a PI-monoid, then there exist an unique submonoid T of $\mathbb{N}^d$ and an unique $\mathbf{a} \in T \setminus \{0\}$ such that $S = (\mathbf{a} + T) \setminus \{0\}$*

Proof. It is clear that $\mathbf{a}$ must be equal to $m(S)$ and that T must be equal to $(S \setminus \{0\}) - m(S)$. □

Remark 32. Given a submonoid S of $\mathbb{N}^d$, we will write $\mathrm{PI}(S)$ for the set

$$\big\{(\mathbf{a} + S) \cup \{0\} \ \mid \mathbf{a} \in S \setminus \{0\}\big\}.$$

Observe that, as an immediate consequence of Corollary 31, we have that the set $\{\mathrm{PI}(S) \mid S \text{ is a submonoid of } \mathbb{N}^d\}$ is a partition of the set of all PI-monoids of $\mathbb{N}^d$. Moreover, if $\mathscr{A}$ denotes the set of all submonoids of $\mathbb{N}^d$, for some d, and $\mathscr{P}i$ denotes the set of all PI-monoids of $\mathbb{N}^d$, for some d, we have an injective map

$$\mathscr{A} \longrightarrow \mathscr{P}i;\ S \mapsto (\min_{lex}(S \setminus \{0\}) + S) \cup \{0\},$$

where lex means the lexicographic term order on $\mathbb{N}^d$.

Recall that a system of generators $\mathcal{A}$ of a submonoid A of $\mathbb{N}^d$ is said to be minimal if no proper subset of $\mathcal{A}$ generates A. The following result identifies a minimal system of generators of an PI-monoid.

Proposition 33. *Let S be a submonoid of $\mathbb{N}^d$. Then S is a PI-monoid if and only if*

$$\big(\mathrm{Ap}(S, m(S)) \setminus \{0\}\big) \cup \{m(S)\}$$

is a minimal system of generators of S.

Proof. By Lemma 29, if S is a PI-monoid, then $m(S) \in S \setminus \{0\}$ Moreover, by Proposition 15, we have that $\mathcal{A} := \big(\mathrm{Ap}(S, m(S)) \setminus \{0\}\big) \cup \{m(S)\}$ is a system of generators of S. So, it suffices to prove that $\mathcal{A}$ is minimal. Let us assume the contrary, that is, there exists $\mathbf{a} \in \mathcal{A}$ such that $\mathcal{A} \setminus \{\mathbf{a}\}$ generates S. By the minimality of $m(S)$, $\mathbf{a} \neq m(S)$. Thus, $\mathbf{a} \in \mathrm{Ap}(S, m(S)) \setminus \{0\}$ and there exists $\mathbf{b}$ and $\mathbf{c} \in S$ with $\mathbf{a} = \mathbf{b} + \mathbf{c}$. By Proposition 30, we know that $\mathbf{b} - m(S) + \mathbf{c} - m(S) = \mathbf{d} - m(S)$ for some $\mathbf{d} \in S \setminus \{0\}$. Therefore, $\mathbf{a} = \mathbf{d} + m(S) \notin \mathrm{Ap}(S, m(S))$, which is impossible. Conversely, if S is not a PI-monoid, by Proposition 30, we have that $(S \setminus \{0\}) - m(S)$ is not a submonoid of $\mathbb{N}^d$. So, there exists $\mathbf{a}$ and $\mathbf{b} \in \mathrm{Ap}(S, m(S)) \setminus \{0\}$ such that $\mathbf{a} - m(S) + \mathbf{b} - m(S) \notin (S \setminus \{0\}) - m(S)$. In particular, $\mathbf{a} + \mathbf{b} - m(S) \notin S$ and consequently $\mathbf{a} + \mathbf{b} \in \mathrm{Ap}(S, m(S))$. So, $\mathrm{Ap}(S, m(S))$ is not a minimal system of generators of S. □

Now, we will show that PI-monoids have non-trivial infinite pseudo-Frobenius set. Recall that every submonoid S of $\mathbb{N}^d$ defines a natural partial order on $\mathbb{N}^d$ as follows: $\mathbf{x} \preceq \mathbf{y}$ if and only if $\mathbf{y} - \mathbf{x} \in S$. As in the previous section this partial order will be denoted as $\preceq_S$.

Corollary 34. *A submonoid S of $\mathbb{N}^d$ is a PI-monoid if and only if $m(S) \in S \setminus \{0\}$ and $\mathrm{Ap}(S, m(S)) \setminus \{0\} = m(S) + \mathrm{PF}(S)$.*

Proof. If S is a PI-monoid, then $m(S) \in S$ by Lemma 29 and, by Proposition 33, $\big(\mathrm{Ap}(S, m(S)) \setminus \{0\}\big) \cup \{m(S)\}$ is a minimal system of generators of S. Therefore, $\mathrm{Ap}(S, m(S)) \setminus \{0\} = \mathrm{maximals}_{\preceq_S}(\mathrm{Ap}(S, m(S))$. Now, by Proposition 17, we are done. Conversely, let us suppose that $m(S) \in S$ and that $\mathrm{Ap}(S, m(S)) \setminus \{0\} = m(S) + \mathrm{PF}(S)$. By Proposition 17, we have that $\mathrm{PF}(S) = \mathrm{maximals}_{\preceq_S}(\mathrm{Ap}(S, m(S)) - m(S)$. Therefore, $\mathrm{Ap}(S, m(S)) \setminus \{0\} = \mathrm{maximals}_{\preceq_S}(\mathrm{Ap}(S, m(S))$, that is, $\big(\mathrm{Ap}(S, m(S)) \setminus \{0\}\big) \cup \{m(S)\}$ is a minimal system of generators of S. Now, by Proposition 33, we are done. □

Putting all this together, we have the following characterization of the PI-monoids.

Theorem 35. *Let S be a submonoid of $\mathbb{N}^d$. The following conditions are equivalent:*

(1) *S is a PI-monoid.*
(2) *$m(S) \in S \setminus \{0\}$ and $(S \setminus \{0\}) - m(S)$ is closed under addition.*
(3) *$\big(\mathrm{Ap}(S, m(S)) \setminus \{0\}\big) \cup \{m(S)\}$ is a minimal system of generators of S.*
(4) *$\{m(S) + \mathrm{PF}(S)\} \cup \{m(S)\}$ is a minimal system of generators of S.*

Example 36. Let S_1 and S_2 be the PI-monoids of Example 28. For S_1 we have $m(S_1) = \{(2,2)\}$ and $\mathrm{Ap}(S_1,(2,2)) = \{(0,0),(3,3)\}$ obtaining that $\{(2,2),(3,3)\}$ is a system of generators of S_1 and that $\mathrm{PF}(S_1) = \{(1,1)\}$. For S_2, $m(S_2) = (1,1)$ and $\mathrm{Ap}(S_2,(1,1)) = \{(0,0)\} \cup \{(x,1) \mid x \in \mathbf{N} \setminus \{0,1\}\} \cup \{(1,y) \mid y \in \mathbf{N} \setminus \{0,1\}\}$. So $\{(1,1)\} \cup \{(x,1) \mid x \in \mathbf{N} \setminus \{0,1\}\} \cup \{(1,y) \mid y \in \mathbf{N} \setminus \{0,1\}\}$ is a non-finite system of generators of S_2 and $\mathrm{PF}(S_2) = \{(x,0) \mid x \in \mathbf{N} \setminus \{0\}\} \cup \{(0,y) \mid y \in \mathbf{N} \setminus \{0\}\}$.

Finally, our last results state the relationship between PI-monoids and MPD-semigroups.

Corollary 37. *Let S be a PI-monoid. Then S is an affine semigroup if and only if S is an MPD-semigroup. In this case,* $\mathrm{Ap}(S, m(S))$ *is finite.*

Proof. Let S be a finitely generated PI-monoid. By Corollary 34, $\mathrm{PF}(S) \neq \varnothing$. So, by Theorem 6, S is a MPD-semigroup. The converse is trivial as MPD-semigroups are affine semigroups by definition. The last part is a direct consequence of Theorem 6 and Corollary 34. □

The following result was inspired by [1, Lemma 2.2]:

Corollary 38. *Let S be a PI-monoid. Then there exist a direct system $(S_\lambda, i_{\lambda\mu})$ of MPD-semigroups contained in S such that $S = \varinjlim_{\lambda\in\Lambda} S_\lambda$, where $i_{\lambda\mu} : S_\lambda \to S_\mu$ is the inclusion map.*

Proof. Let $\Lambda = \{\lambda \subset \{m(S) + \mathrm{PF}(S)\} \mid \lambda \text{ is finite}\}$, partially ordered by inclusion, and define S_λ to be the affine semigroup generated by $\lambda \cup \{m(S)\}$. Clearly, we have that $S_\lambda \subseteq S_\mu$ if $\lambda \subseteq \mu$; in this case, let $i_{\lambda\mu} : S_\lambda \to S_\mu$ is the inclusion map. Now, since $S_\lambda \subseteq S$ for every $\lambda \in \Lambda$, we conclude that $S = \varinjlim_{\lambda\in\Lambda} S_\lambda$ by Theorem 35, because $\{m(S) + \mathrm{PF}(S)\} \cup \{m(S)\}$ is a minimal system of generators of S.

Finally, let us see that S_λ is a MPD-semigroup for every $\lambda \in \Lambda$. To do that, we first note that $m(S_\lambda) = m(S) \in S_\lambda$, for every $\lambda \in \Lambda$. Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subseteq \mathrm{PF}(S)$ and let $\lambda = \{m(S) + \mathcal{A}\} \in \Lambda$. Then, $\mathrm{Ap}(S_\lambda, m(S)) = \{0, \mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is finite, in particular, $\mathrm{maximals}_{\preceq_S}(\mathrm{Ap}(S, m(S)) - m(S)$ is a non-empty finite set. Therefore, by Proposition 17, $\mathrm{PF}(S) \neq \varnothing$, that is to say, S_λ is a MPD-semigroup. □

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