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ON PSEUDO-PRIMALITY OF THE n-TH POWER OF PRIME ENTIRE FUNCTIONS

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I. Introduction and main results.

Let $g_0(z)$ be a transcendental entire function which is prime or pseudo-prime. We pose the following question: are the functions $g_0(z)^n$ always pseudo-prime for $n=2, 3, \cdots$? The answer is affirmative if and only if n is an odd number. That is to say that $g_0(z)^n$ is pseudo-prime if n is odd; while for even number n, there exists a prime entire function $g_0(z)$ such that $g_0(z)^n$ is not pseudo-prime. This assertion is contained in the following two theorems.

THEOREM 1. *Let $g_0(z)$ be a pseudo-prime entire function, and n ($\geqq 3$) be an odd number. Then $F(z)=g_0(z)^n$ is also pseudo-prime.*

THEOREM 2. *The function*

$$F(z)=(\sin z)e^{\cos z} \tag{1}$$

is prime.

Remark 1. If $F(z)$ is the function of the form (1), and n is an even number, then $F(z)^n$ is not pseudo-prime, as is shown by the following factorization

$$F(z)^n=(\sin^n z)e^{n \cos z}=((1-w^2)^{n/2}e^{nw})\circ\cos z .$$

Remark 2. The function $F(z)$ of the form (1) is also an example of prime periodic entire functions. In 1971, Gross [4] asked if there exist such functions. Later on, Ozawa [8, 9], Baker & Yang [2], Gross & Yang [6] constructed various examples of such kind of entire functions. Our example here is a much simpler one.

From Theorem 2 and Remark 1, it is easy to derive the following

COROLLARY. *For any polynomial $P(z)$ of degree 2, there exists a prime entire function $g^*(z)$ such that $F(z)=P(g^*(z))$ is not pseudo-prime.*

The basic notions in the factorization theory of entire and meromorphic

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functions, such as prime, E-prime, pseudo-prime, etc., shall not be stated here. One may find the definitions of these notions in the references.

2. Preliminary lemmas.

In proving our theorems we shall need several known results.

LEMMA 1 (Hayman [7]). *Let $f(z)$ be an entire function. Then*

$$\sum_{a\neq\infty}\left(1-\frac{1}{v(a)}\right)\leqq 1,$$

where $v(a)$ stands for the least order of almost all a-points of $f(z)$. Especially, there is at most one complex number a such that $v(a)\geqq 3$.

LEMMA 2 (Edrei [3]). *Let $g(z)$ be an entire function. If there exists an unbounded sequence $\{a_n\}$ such that almost all the roots of $g(z)=a_n$ $(n=1, 2, \cdots)$ lie on one straight line, then $g(z)$ is a polynomial of degree at most two.*

LEMMA 3 (Baker & Gross [1]). *Let $h(z)$ be a periodic entire function of finite lower order, and c be a non-zero constant. Then*

$$H(z)=h(z)+cz$$

is prime.

LEMMA 4 (Gross [5]). *All meromorphic solutions of the functional equation*

$$f(z)^2+g(z)^2=1 \tag{2}$$

are of the form

$$f(z)=\frac{2s(z)}{1+s(z)^2}, \qquad g(z)=\frac{1-s(z)^2}{1+s(z)^2},$$

where $s(z)$ is any meromorphic function. In particular, there is no non-constant $f(z)=z^{-n}f_0(z)$ where n is a non-negative integer and $f_0(z)$ is a polynomial satisfying equation (2).

3. Proof of theorem 1.

Let $n=p_1\cdot p_2\cdots p_k$ with prime numbers $p_j\geqq 3$, $j=1, \cdots, k$. If $g_1(z)=g_0(z)^{p_1}$ is proved to be pseudo-prime, so is $g_2(z)=g_1(z)^{p_2}=g_0(z)^{p_1p_2}$, and so on. Therefore, we may assume that $n=p$ is a prime number. Also, $g_0(z)$ may be assumed transcendental.

Suppose $F(z)=g_0(z)^p=f(g(z))$ with transcendental entire functions f and g. By Lemma 1, it is easily seen that among zeros of $f(w)$ there is at most one zero with order q such that $(q, p)=1$. Hence 2 cases may occur.

(a) $f(w)=(w-w_0)^q h(w)^p$ and $g(z)=w_0+s(z)^p$ with transcendental entire functions h and s. Then

$$g_0(z)^p=s(z)^{qp}(h(w_0+s(z)^p))^p$$

or

$$g_0(z)=us(z)^q h(w_0+s(z)^p) \qquad (u^p=1),$$

which gives a contradiction as $g_0(z)$ is assumed to be pseudo-prime.

(b) $f(w)=h(w)^p$ with a transcendental entire function h. Then $g_0(z)=uh(g(z))$. Again a contradiction.

Now let $F=f(g)$ with f being meromorphic (not entire) and g entire, both transcendental. Then f must have exactly one pole, w_0 say, which g doesn't take. And we may write

$$f(w)=(w-w_0)^{-k}f^*(w), \qquad g(z)=w_0+e^{M(z)},$$

where k is a positive integer, f^* is transcendental entire with $f^*(w_0)\neq 0$, and $M(z)$ is non-constant entire.

If $f^*(w)$ has no zeros, or each zero of f^* is of order mp with a positive integer m, then $f^*(w)=h(w)^p$ with a transcendental entire function h, which implies

$$g_0(z)=ue^{-kM(z)/p}h(w_0+e^{M(z)}) \quad (u^p=1)$$

or

$$g_0(z)=(uw^{-k}h(w_0+w^p))\circ e^{M(z)/p}.$$

But this violates the pseudo-primality of $g_0(z)$.

If $f^*(w)$ has a zero, w_1 say $(w_1\neq w_0)$, of order q with $(q, p)=1$ (By the same reasoning as in case(a), f^* has at most one such zero). Then $g(z)$ must be of the form

$$g(z)=w_1+s(z)^p$$

with an entire function s. But this is impossible, since the entire function $g(z)$, which has a Picard exceptional value w_0, can not have any completely ramified values.

The proof of theorem 1 is completed.

4. Proof of theorem 2.

Let $F(z)=f(g(z))$ with non-linear entire functions f and g. We discuss two cases.

Case (a). f has infinitely many zeros. Then by lemma 2, $g(z)$ must be a polynomial of degree 2. Hence, $F(z)$ may be expressed by

$$F(z)=f_1((z-c)^2)$$

with an entire function f_1 and a constant c. This yields

$$\sin(z+c)e^{\cos(z+c)}=\sin(-z+c)e^{\cos(-z+c)}.$$

But the above equality can not hold, as is shown by substituting some special values of z.

Case (b). f has only finitely many zeros. Then we may write

$$f(w)=Q(w)e^{L(w)}$$

with a non-constant polynomial Q and entire function L. Thus

$$Q(g(z))e^{L(g(z))}=(\sin z)e^{\cos z}. \tag{3}$$

Since $F(z)=(\sin z)e^{\cos z}$ is of infinite order and its convergent exponent of zeros, denoted by $\rho^*(F)$, is one, if L is a constant, then Q must have exactly one (simple) zero, *i.e.* Q is linear, which is out of our consideration. Therefore, L must be non-constant.

By considering the growth of functions in both sides of (3), we see that the order of $g(z)$

$$\rho(g)\leqq\rho(L(g))=\rho(\cos z)=1$$

and

$$\rho(g)=\rho(Q(g))\geqq\rho^*(Q(g))=\rho^*(\sin z)=1.$$

So that $\rho(g)=1$.

Putting

$$s(z)=\frac{Q(g(z))}{\sin z}\exp(\cos z-L(g(z))),$$

we have $\rho(s)\leqq 1$, which implies

$$\cos z-L(g(z))=az+b, \qquad a \text{ and } b \text{ are constants.}$$

If $a\neq 0$, then by lemma 3, $L(g(z))=\cos z-az-b$ is prime, so that L is linear. And we may write

$$g(z)=c_1\cos z+c_2 z+b_1,$$

where c_1, c_2 and b_1 are constants with $c_1c_2\neq 0$. On the other hand

$$Q(g(z))=(\sin z)e^{az+b}. \tag{4}$$

Therefore, substituting $z=2n\pi$ into both sides of (4), we see that the right side of (4) is 0, while the left side tends to ∞, which is a contradiction.

If $a=0$, then $L(g(z))=\cos z-b$ and $Q(g(z))=b_2\sin z$, and we obtain an identity

$$Q^*(g(z))^2+L^*(g(z))^2\equiv 1 \qquad \text{or} \qquad Q^*(w)^2+L^*(w)^2\equiv 1$$

with a polynomial Q^* and an entire function L^* for every $w \in \boldsymbol{C}$, which violates lemma 4.

Up to now we have proved that $F(z)$ is E-prime.

Now, let $F=f(g)$ with meromorphic functions f and g (f is not entire), we discuss three cases.

Case (i). f is transcendental. Then g must be entire, and we have

$$f(w)=(w-w_0)^{-n}f_1(w), \qquad g(z)=w_0+e^{M(z)}, \tag{5}$$

where $f_1(w)$ is a transcendental entire function with $f_1(w_0)\neq 0$, n is a positive integer, and $M(z)$ is a non-constant entire function. We derive

$$F(z)=(\sin z)e^{\cos z}=e^{-nM(z)}f_1(w_0+e^{M(z)})$$

$$=[e^{-nw}f_1(w_0+e^{w})]\circ M(z)$$

Since $F(z)$ is E-prime, $M(z)$ must be linear, and we may write

$$F(z)=(\sin z)e^{\cos z}=e^{-naz}f^*(e^{az}), \tag{6}$$

where f^* is trancendental entire.

By the same argument as in case (a), we conclude that f^* has only finitely many zeros. Then we may write

$$f^*(w)=P(w)e^{N(w)}$$

with a non-constant polynomial P and entire function N. We obtain

$$(\sin z)e^{\cos z}=e^{-anz}P(e^{az})\exp(N(e^{az}))$$

Putting

$$T(z)=\frac{P(e^{az})}{\sin z}=\exp(\cos z+naz-N(e^{az})).$$

Obviously, $\rho(T)\leqq 1$. Hence

$$\cos z+naz-N(e^{az})=Az+B,\ A \text{ and } B \text{ are constants}.$$

If $A\neq na$, then N is linear (by using lemma 3), and we would get

$$e^{az}=A_1\cos z+A_2 z+B_1,$$

which is apparently impossible.

If $A=na$, then $N(e^{az})=\cos z-B$ and $P(e^{az})=B_2(\sin z)e^{naz}$, and we would derive an identity

$$P^*(e^{az})^2e^{-2naz}+N^*(e^{az})^2\equiv 1 \quad \text{or} \quad \frac{P^*(w)^2}{w^{2n}}+N^*(w)^2\equiv 1$$

with a polynomial P^* and a positive integer n. This again violates lemma 4.

Case (ii). f is rational and g entire. Then we obtain (5) and (6) with f_1 (and f^*) being a polynomial. And we may deduce that $M(z)$ is linear. But in this case the function in the right side of (6) would be of finite order, which is also a contradiction.

Case (iii). f is rational and g meromorphic (not entire). Let x_0 be a pole of f, then $g(z)$ doesn't assume x_0, so that

$$g_1(z)=\frac{1}{g(z)-x_0}$$

is entire. Denoting

$$R(w)=f\left(\frac{1}{w}+x_0\right),$$

we get a factorization $F=R\circ g_1$ which is equivalent to $F=f\circ g$. Then this case reduces to case (ii).

The proof is thus completed.

5. Final remark.

We propose the following questions:

(1) Does there exist an entire function $g_0(z)$ which is prime and of *finite order* such that $g_0(z)^2$ is not pseudo-prime?

(2) Let $P(z)$ be a polynomial of degree$\geqq 3$ which has no right factor of the form $(z-a)^2+b$ with constants a, b and let $g_0(z)$ be a pseudo-prime entire function. Can we conclude that the function $F(z)=P(g_0(z))$ is also pseudo-prime?

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