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ON PU'S THEOREM FOR ODD DIMENSIONAL REAL PROJECTIVE SPACES

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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1. Introduction. Let (P^n, g) be any Riemannian structure on the n dimensional real projective space. Let $\mathrm{vol}(P^n, g)$ denote the volume of P^n with respect to the canonical measure derived from g, and $L_g(c)$ denote the length of a curve c relative to g. Following M. Berger ([1]), we define

$$\mathrm{carc}_1(P^n, g) \equiv \mathrm{Inf}\{L_g(c) \mid c;\ \text{homologically non-trivial piecewise smooth closed curve on } P^n\}$$

$$\mathrm{quot}_1(P^n, g) \equiv \mathrm{vol}(P^n, g)/\{\mathrm{carc}_1(P^n, g)\}^n .$$

Then Pu ([3]) has showen the following: $\mathrm{quot}_1(P^2, g) \geqq \mathrm{quot}_1(P^2, g_0) = 1/(2\pi)$ holds, where g_0 denotes the canonical Riemannian structure of constant curvature. Moreover, the equality holds if and only if g is of constant curvature.

Now it is natural to ask whether the higher dimensional analogue of Pu's theorem is valid; that is, $\mathrm{quot}_1(P^n, g) \geqq \mathrm{quot}_1(p^n, g_0)$ holds, where the equality holds if and only if g is of constant curvature. Pu's proof depends on the fact that any Riemannian structure g over P^2 is conformally related to the canonical structure, and his method is valid for a class of Riemannian structures on P^n which are conformally related to the canonical structure of constant curvature. Berger treats various generalizations of Pu's theorem (see ([1]).

The purpose of the present note is to show that, on odd dimensional real projective spaces, for some classes of Riemannian structures which are not conformally related to the canonical structure, the version of Pu's theorem holds good.

Now we shall explain these special classes of Riemannian structures on P^{2n+1}. First we shall review the notion of contact Riemannian structure. Let ξ be a unit Killing vector field on a Riemannian manifold (M^{2n+1}, g) which satisfies

$$R(X, \xi)Y = k\{g(X, Y)\xi - g(Y, \xi)X\}$$

for some positive constant k. Then (M, g, ξ) is called a Sasakian structure on M. Note that sectional curvatures of plane sections containing ξ are equal to a constant k. (M, g, ξ) is called to be of constant ϕ-holomorphic sectional curvature H, if sectional curvature $K(X, \phi X) \equiv H$ for every $X \perp \xi$, where we put $\phi(X) = -\nabla_X \xi$. Now it is well known that every odd dimensional space form of positive curvature has a natural contact Riemannian structure (see [8]).

Let $(S^{2n+1}, \tilde{g}_0, \tilde{\xi}_0)$ be the canonical regular Sasakian structure of constant curvature k on S^{2n+1}, and Φ_t be the one parameter group of transformations generated by $\tilde{\xi}_0$. Then $\{\text{id.}, \Phi_{\pi/\sqrt{k}}\}$ acts on S^{2n+1} as a deck transformation group. Dividing $(S^{2n+1}, \tilde{g}_0, \tilde{\xi}_0)$ by $\{\text{id.}, \Phi_{\pi/\sqrt{k}}\}$, we have the canonical Sasakian structure (P^{2n+1}, g_0, ξ_0) of constant curvature k over P^{2n+1}. We have the Boothby-Wang's fiberings $\tilde{p}$: $(S^{2n+1}, \tilde{g}_0, \tilde{\xi}_0) \to (P^n(\boldsymbol{C}), h_0, J_0)$ and p: $(P^{2n+1}, g_0, \xi_0) \to (P^n(\boldsymbol{C}), h_0, J_0)$, where $(P^n(\boldsymbol{C}), h_0, J_0)$ denotes the complex projective space of complex dimension n with constant holomorphic sectional curvature $4k$. Then we have the following canonical representations of $\tilde{g}_0$ and g_0,

$$\tilde{g}_0 = \tilde{p}^* h_0 + \tilde{\eta}_0 \otimes \tilde{\eta}_0 \,, \qquad g_0 = p^* h_0 + \eta_0 \otimes \eta_0 \,,$$

where $\tilde{\eta}_0$ (resp. η_0) denotes the contact form corresponding to $\tilde{\xi}_0$ (resp. ξ_0); i.e. $\tilde{\eta}_0(\tilde{X}) = \tilde{g}_0(\tilde{\xi}_0, \tilde{X})$ for any vector field $\tilde{X}$ on S^{2n+1} (resp. $\eta_0(X) = g_0(\xi_0, X)$ for any vector field X on P^{2n+1}).

Now, let φ, β be arbitrary positive C^∞-functions on $P^n(C)$. We shall consider a class of Riemannian structures g on P^{2n+1} which have the form

$$g = p^*(\varphi^2 h_0) + p^*(\beta^2) \eta_0 \otimes \eta_0 \,.$$

If β is a constant, $(P^{2n+1}, g, (1/\beta)\xi_0)$ defines an almost contact Riemannian structure on P^{2n+1}. But this almost contact Riemannian structure is not a contact Riemannian structure, unless φ is a constant function. If $\varphi = \beta = 1$, then $p^* h_0 + \eta_0 \otimes \eta_0$ reduces to the canonical metric of constant curvature k. More generally, if φ, β are constants and $\varphi^2 = \beta$ holds, then we have a Sasakian structure $(P^{2n+1}, g = \beta p^*(h_0) + \beta^2 \eta_0 \otimes \eta_0, (1/\beta)\xi_0)$ of constant ϕ-holomorphic sectional curvature $((4 - 3\beta)/\beta)k$ ([5], [7]).

Now we shall state our results.

THEOREM A. *Let (P^{2n+1}, g, ξ) be any regular Sasakian structure of constant ϕ-holomorphic sectional curvature H. Then we have*

$$\mathrm{quot}_1(P^{2n+1}, g, \xi) \geqq \mathrm{quot}_1(P^{2n+1}, g_0, \xi_0) = 1/n!\, \pi^n \,,$$

where the equality holds if and only if g is of constant curvature.

REMARK. I. Chavel ([2]) proved Pu's theorem for a class of normal homogeneous Riemannian structures on odd dimensional real projective spaces. On the other hand, the author has proved that these spaces have natural Sasakian structures of constant ϕ-holomorphic sectional curvature ([4]). Thus Theorem A covers the result of Chavel.

THEOREM B. *Let* (P^{2n+1}, g) *denote the Riemannian structure over* P^{2n+1} *with* $g = p^*(\varphi^2 h_0) + (p^*\beta^2)\eta_0 \otimes \eta_0$, *where* φ, β *are arbitrary positive* C^∞-*functions on* $P^n(C)$. *Then we have*

$$\mathrm{quot}_1(P^{2n+1}, g) \geqq \mathrm{quot}_1(P^{2n+1}, g_0) = 1/n!\,\pi^n ,$$

where the equality holds if and only if g *is of constant curvature.*

REMARK. Theorem A is a corollary to Theorem B. But we shall give an independent proof, because we can give a proof which uses only elementary geometry of Sasakian manifolds of constant ϕ-holomorphic sectional curvature.

All facts about contact Riemannian structure used in this note may be found in the Lecture Note by S. Sasaki ([5]) and papers of S. Tanno ([6], [7]).

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2. **Proof of Theorem A.** It suffices to show the theorem in the case $k = 1$ (k is a constant appeared in the definition of Sasakian structure). Let (P^{2n+1}, g, ξ) be a regular Sasakian structure of constant ϕ-holomorphic sectional curvature H over the real projective space P^{2n+1}. Then it is known that $H + 3 > 0$ holds ([7]). Let $(S^{2n+1}, \tilde{g}, \tilde{\xi})$ be a regular Sasakian structure on S^{2n+1} which is derived from (P^{2n+1}, g, ξ) via the covering projection $q: S^{2n+1} \to P^{2n+1}$ and is of constant ϕ-holomorphic sectional curvature H. Then we have the Boothby-Wang's fibering p (resp. $\tilde{p}$) of (P^{2n+1}, g, ξ) (resp. $(S^{2n+1}, \tilde{g}, \tilde{\xi})$) over the complex projective space $(P^n(C), h, J)$ of constant holomorphic sectional curvature $H + 3$, and $\tilde{p} = pq$ holds. Then we get

$$g = p^*h + \eta \otimes \eta, \quad \tilde{g} = \tilde{p}^*h + \tilde{\eta} \otimes \tilde{\eta}, \quad q_*\tilde{\xi} = \xi ,$$

where η (resp. $\tilde{\eta}$) denotes the contact form corresponding to ξ (resp. $\tilde{\xi}$). Note that $H = 1$ holds if and only if (P^{2n+1}, g, ξ) is the canonical Sasakian structure of constant curvature 1 ([7]). Now it is known that every Sasakian structure of constant ϕ-holomorphic sectional curvature H may be obtained from the canonical structure by the so-called D-homothetic deformation. That is, (P^{2n+1}, g, ξ) (resp. $(S^{2n+1}, \tilde{g}, \tilde{\xi})$) has the form

$$g = \alpha g_0 + (\alpha^2 - \alpha)\eta_0 \otimes \eta_0 , \qquad \xi = (1/\alpha)\xi_0$$
$$(\text{resp. } \tilde{g} = \alpha \tilde{g}_0 + (\alpha^2 - \alpha)\tilde{\eta}_0 \otimes \tilde{\eta}_0 , \qquad \tilde{\xi} = (1/\alpha)\tilde{\xi}_0)$$

with $\alpha = 4/(H + 3)$. So the fibres of p are closed geodesics of length

$\delta = 4\pi/(H+3)$. Since $p\colon (P^{2n+1}, g, \xi) \to (P^n(C), h, J)$ is a Riemannian submersion and $(P^n(C), h, J)$ is of constant holomorphic sectional curvature $H+3$, we have by ([1] p. 8)

$$\mathrm{vol}(P^{2n+1}, g, \xi) = \delta\, \mathrm{vol}(P^n(C), h, J) = \delta\{(H+3)/4\}^{-n}\pi^n/n!\ .$$

Case 1. $\delta \leqq 2\pi/\sqrt{H+3}$ (that is, $H \geqq 1$). Since fibres of p define a non-trivial one dimensional homology cycle of length δ, we get

$$\begin{aligned}\mathrm{quot}_1(P^{2n+1}, g, \xi) &= \mathrm{vol}(P^{2n+1}, g, \xi)/\{\mathrm{carc}_1(P^{2n+1}, g, \xi)\}^{2n+1}\\ &\geqq \pi^n\{(H+3)/4\}^{-n}/\{\delta^{2n}n!\} \geqq 1/(\pi^n n!) = \mathrm{quot}_1(P^{2n+1}, g_0, \xi_0)\ ,\end{aligned}$$

where the equality holds if and only if $H = 1$ holds.

Case 2. $\delta \geqq 2\pi/\sqrt{H+3}$ (that is, $H \leqq 1$). First take a half great circle $\tilde{c}$ on $(S^{2n+1}, \tilde{g}_0, \tilde{\xi}_0)$ connecting $x \in S^{2n+1}$ and its antipodal point which is horizontal with respect to the Riemannian submersion $\tilde{p}\colon (S^{2n+1}, \tilde{g}_0, \tilde{\xi}_0) \to (P^n(C), h_0, J_0)$. Since through D-homothetic deformation, horizontal geodesics are taken into horizontal geodesics, closed geodesic $q\tilde{c}$ defines a non-trivial one dimensional homology cycle of length $2\pi/\sqrt{H+3}$ on (P^{2n+1}, g, ξ). So we have

$$\begin{aligned}\mathrm{quot}_1(P^{2n+1}, g, \xi) &\geqq \delta\frac{\pi^n}{n!}\Big(\frac{H+3}{4}\Big)^{-n}\Big/\Big(\frac{2\pi}{\sqrt{H+3}}\Big)^{2n+1} = \frac{1}{\pi^n n!}\Big(\frac{\delta}{\pi}\frac{\sqrt{H+3}}{2}\Big)\\ &\geqq 1/(\pi^n n!) = \mathrm{quot}_1(P^{2n+1}, g_0, \xi_0)\ ,\end{aligned}$$

where the equality holds if and only if $H = 1$.

3. **Proof of Theorem B.** Before the proof of Theorem B, we note that $SU(n+1) \times R$ acts on $(S^{2n+1}, \tilde{g}_0, \tilde{\xi}_0)$ as an automorphism group. We explain this fact as follows. (See S. Tanno ([6]) also). Let $(SU(n+1)/SU(n), \tilde{g})$ be a normal homogeneous Riemannian manifold of Berger. This space is diffeomorphic to a sphere and on this space there exists a unit Killing vector field $\tilde{\xi}$ which defines a Sasakian structure of constant ϕ-holomorphic sectional curvature $4 - (3/4)\gamma^2$ ($\gamma = \sqrt{2(n+1)/n}$) and $SU(n+1) \times R$ acts on $(SU(n+1)/SU(n), \tilde{g}, \tilde{\xi})$ as an automorphism group ([4].) Since this structure may be obtained from the canonical structure of constant curvature via D-homothetic deformation, $\tilde{g}$ takes a form $\tilde{g} = \alpha\tilde{g}_0 + \beta\tilde{\eta}_0 \otimes \tilde{\eta}_0$ where α, β are constants (See the proof of Theorem A). Since $SU(n+1) \times R$ leaves $\tilde{g}, \tilde{\xi}_0, \tilde{\eta}_0$ invariant, $SU(n+1) \times R$ leaves also $\tilde{g}_0$ invariant and acts on $(S^{2n+1}, \tilde{g}_0, \tilde{\xi}_0)$ as an automorphism group. (But $\tilde{g}_0$ is not a normal homogeneous Riemannian metric with respect to $SU(n+1)$ if $n > 1$ holds.) Moreover, let $\tilde{p}\colon (S^{2n+1}, \tilde{g}_0, \tilde{\xi}_0) \to (P^n(C), h_0, J_0)$ be the Boothby-Wang's fibering. Then $SU(n+1)$ acts also on $P(C) = SU(n+1)/U(n)$ as an automorphism group and elements of $SU(n+1)$ acting on $(S^{2n+1}, \tilde{g}_0, \tilde{\xi}_0)$

commute with $\tilde{p}$.

Now we shall return to the proof of Theorem B. We may assume that $k=1$ holds, where k is a constant appeared in the definition of Sasakian structure. We choose a Haar measure ν on $SU(n+1)$ with total measure 1, and define the function ψ on $P^n(C)$ by

$$\psi \equiv \int_{\sigma \in SU(n+1)} \sigma^* \varphi \cdot \nu$$

which reduces to a constant because $SU(n+1)$ acts transitively on $P^n(C)$. Then as Pu showed ([3]), by Hölder inequality, we have

$$\mathrm{vol}(P^n(C), \psi^2 h_0) \leqq \mathrm{vol}(P^n(C), \varphi^2 h_0) ,$$

where the equality holds if and only if φ is a constant.

Next recall that $p\colon (P^{2n+1}, g) \longrightarrow (P^n(C), \varphi^2 h_0)$ is a Riemannian submersion and fiber $p^{-1}(m)$ over $m \in P^n(C)$ is a closed curve of length $\pi\beta(m)$. Since $(P^n(C), h_0)$ is of constant holomorphic sectional curvature 4, we have

$$\mathrm{vol}(P^{2n+1}, g) = \pi \int_{P^n(C)} \beta(m) dM \geqq b\pi\, \mathrm{vol}(P^n(C), \psi^2 h_0) = b\pi\psi^{2n}\pi^n/n! ,$$

where dM denotes the volume element of $(P^n(C), \varphi^2 h_0)$ and we put $b = \mathrm{Min}_{m \in P^n(C)} \beta(m)$.

Case 1. $\psi \geqq b$. In this case we have

$$\mathrm{quot}_1(P^{2n+1}, g) \geqq \mathrm{vol}(P^{2n+1}, g)/\{\text{minimum length of closed curves which are fibers of } p\}^{2n+1} \geqq \frac{b\pi\psi^{2n}\pi^n}{(b\pi)^{2n+1}n!} \geqq \frac{1}{\pi^n n!} .$$

If the equality holds then $\varphi = \beta \ (=\psi)$ is a constant and g takes the form $g = \psi(p^* h_0 + \eta_0 \otimes \eta_0) = \psi g_0$.

Case 2. $\psi \leqq b$. Let $\tilde{c} = \{\tilde{c}(t);\ 0 \leqq t \leqq \pi\}$ be a geodesic connecting $x \in (S^{2n+1}, \tilde{g}_0, \tilde{\xi}_0)$ and its antipodal point which is horizontal with respect to the Riemannain submersion $\tilde{p}\colon (S^{2n+1}, \tilde{g}_0, \tilde{\xi}_0) \longrightarrow (P^n(C), h_0, J_0)$. This curve is again horizontal with respect to the Riemannian submersion $p\colon (S^{2n+1}, \tilde{g}) \longrightarrow (P^n(C), \varphi^2 h_0)$, because $\tilde{g} = \tilde{p}^*(\varphi^2 h_0) + (\tilde{p}^* \beta^2)\eta_0 \otimes \eta_0$ holds. By projecting this curve by q we have a non-trivial one dimensional cycle of P^{2n+1}.

Now let C be a set of closed geodesics of length π on $(P^n(C), h_0, J_0)$. Then C has a natural topology of compact $(4n-2)$-dimensional manifold over which the function $\underline{c} \in C \longrightarrow \int_0^\pi \varphi(\underline{c}(t))dt$ is continuous. So this function takes a minimum value at some closed geodesic $\underline{c}_0$. If we lift $\underline{c}_0$ to a horizontal geodesic $\tilde{c}_0$ of $(S^{2n+1}, \tilde{g}_0, \tilde{\xi}_0)$ connecting x and its antipodal point, we have

$$L_{\tilde{g}}(\tilde{c}_0) = \int_0^{\pi} (\tilde{p}^*\varphi)(\tilde{c}_0(t))dt = \int_0^{\pi} \varphi(\underline{c}_0(t))dt \leqq \int_0^{\pi} \varphi(\sigma\underline{c}_0(t))dt$$
$$= \int_0^{\pi} (\tilde{p}^*\varphi)(\sigma\tilde{c}_0(t))dt = L_{\tilde{g}}(\sigma\tilde{c}_0)$$

for any $\sigma \in SU(n+1)$, where we have used the fact that σ and $\tilde{p}$ are commutative. Now we get,

$$L_g(q\tilde{c}_0) = \int_0^{\pi} (\tilde{p}^*\varphi)(\tilde{c}_0(t))dt \leqq \int_{\sigma \in SU(n+1)} \left\{ \int_0^{\pi} (\tilde{p}^*\varphi)(\sigma\tilde{c}_0(t))dt \right\} \nu$$
$$= \int_0^{\pi} \left\{ \int_{\sigma \in SU(n+1)} \sigma^*\varphi \cdot \nu \right\} (\tilde{p}\tilde{c}_0(t))dt = \psi\pi \, .$$

Since $q\tilde{c}_0$ defines a non-trivial one dimensional homology cycle of (P^{2n+1}, g) which is horizontal relative to the Riemannian submersion $p: (P^{2n+1}, g) \to (P^n(C), \varphi^2 h_0)$, we have finally

$$\mathrm{quot}_1(P^{2n+1}, g) \geqq \mathrm{vol}(P^{2n+1}, g)/\{L_g(q\tilde{c}_0)\}^{2n+1} \geqq \frac{b\pi \cdot \varphi^{2n}\pi^n}{(\psi\pi)^{2n+1} n!} = \frac{1}{n!\,\pi^n}\frac{b}{\psi} \geqq \frac{1}{n!\,\pi^n} \, .$$

If the equality holds then $\varphi = \beta \; (=\psi)$ is a constant and g takes the form $g = \psi(p^*h_0 + \eta_0 \otimes \eta_0) = \psi g_0$.

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