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ON q-ANALGUE OF THE TWISTED L-FUNCTIONS AND q-TWISTED BERNOULLI NUMBERS

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Abstract. The aim of this work is to construct twisted q-L-series which interpolate twisted q-generalized Bernoulli numbers. By using generating function of q-Bernoulli numbers, twisted q-Bernoulli numbers and polynomials are defined. Some properties of this polynomials and numbers are described. The numbers $L_q(1-n,\chi,\xi)$ is also given explicitly.

1. Introduction

In this section, we aim at giving an elementary introduction to some functions which were found useful in number theory. The most famous are Dirichlet L-functions. We therefore give Drichlet L-functions and q-analogues of the Dirichlet series. We use the notation of Iwasawa [2], Koblitz [8] and Tsumura [12]. Let χ be a Dirichlet character of conductor f. The L-series attached to χ is defined as follows:

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s},$$

where $\mathrm{Re}\, s>1$. For $\chi=1$, this is the usual Riemann zeta function. It is well known that $L(s,\chi)$ may be continued analytically to the whole complex plane, except for a simple pole at $s=1$ when $\chi=1$. Hurwitz zeta function is defined as follows:

$$\zeta(s,b)=\sum_{n=0}^{\infty}(b+n)^{-s},$$

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where $\operatorname{Re} s > 1$ and $0 < b \leq 1$. For $b = 1$, this is the usual Riemann zeta function. It is well known that $\zeta(s, b)$ may be continued analytically to the whole complex plane, except for a simple pole at $s = 1$.

Iwasawa [2] gave fundamental properties of the generalized Bernoulli numbers and Dirichlet L-functions in more detail. The definition of ordinary Bernoulli numbers is well known: let t be an indeterminate and let

$$F(t) = \frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{1.1}$$

The coefficients $B_n, n \geq 0$, are called Bernoulli numbers. Let x be another indeterminate and let

$$F(t, x) = F(t)e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.2}$$

The coefficients $B_n(x), n \geq 0$, are called Bernoulli polynomials. The generalized Bernoulli numbers $B_{n,\chi}$ are defined by

$$F_\chi(t, x) = \sum_{a=0}^{f-1} \frac{\chi(a)te^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}. \tag{1.3}$$

If $\chi = \chi^0$, the principal character ($f = 1$), then (1.3) reduces to (1.1).

Note that when $\chi = \chi^0$, the principal character ($f = 1$), we have

$$\sum_{n \in \mathbb{N}} B_{n,1} \frac{t^n}{n!} = \frac{te^t}{e^t - 1} = \frac{t}{e^t - 1} + t,$$

so $B_{n,1} = B_n$ except for $n = 1$, when we have $B_{1,1} = \frac{1}{2}$, $B_1 = -\frac{1}{2}$. If $\chi \neq 1$ then $B_{0,\chi} = 0$, since $\sum_{a=1}^{d} \chi(a) = 0$. A relationship between $L(1 - n, \chi)$ and $B_{n,\chi}$ is given as follows [2]: for n be a positive integer

$$L(1 - n, \chi) = -\frac{B_{n,\chi}}{n}.$$

In [11], the author constructed an elementary introduction to twisted L-functions which are found useful in number theory and p-adic analysis. The author reviewed some of the basic facts about twisted L-series which interpolated twisted Bernoulli numbers. Their values at negative integers were given in terms of twisted Bernoulli numbers, $B_{n,\chi,\xi}$. Finally, the author discussed the value at 1 and analytic continuation of this function. Let r be a positive integer, and let $\varepsilon \neq 1$ be any nontrivial r-th root of 1. Let $\xi^f = \varepsilon$. Then twisted L-functions are defined as follows:

$$L(s,f,\xi)=\sum_{n=1}^{\infty}\frac{\chi(n)\xi^n}{n^s}.$$

Since the function $n \to \chi(n)\xi^n$ has period fr, this is a special case of the Dirichlet L-functions considered above. Such L-series (for $r = f$) are used classically to prove the formula for $L(1,\chi)$ by Fourier inversion. Koblitz ([7], [8]) gave a relation between $L(1-n,f,\xi)$ and $B_{n,\chi,\xi}$. He also defined p-adic twisted L-functions, $L_p(s,\chi,\xi)$, where s is p-adic number. Using these functions, he constructed p-adic measures and integration.

In [11], the author defined generalized of the functions $F(t)$ and $F(t,x)$, which are mentioned in the above. Let χ be a Dirichlet character with conductor f and let ξ be rth root of 1 and let t be an indeterminate. $F_{\chi,\xi}(t)$ is defined as follows:

$$F_{\chi,\xi}(t)=\sum_{a=0}^{f-1}\frac{\chi(a)\xi^a te^{at}}{\xi^f e^{ft}-1}=\sum_{n=0}^{\infty}B_{n,\chi,\xi}\frac{t^n}{n!}. \tag{1.4}$$

If $r = 1$, then (1.4) reduces to (1.3). The coefficients $B_{n,\chi,\xi}$, $n \geq 0$, are called twisted Bernoulli numbers. Let x be another indeterminate and $F_{\chi,\xi}(t,x)$ is defined as follows:

$$F_{\chi,\xi}(t,x)=F_{\chi,\xi}(t)e^{xt}=\sum_{a=0}^{f-1}\frac{\chi(a)\xi^a te^{(a+x)t}}{\xi^f e^{ft}-1}=\sum_{n=0}^{\infty}B_{n,\chi,\xi}(x)\frac{t^n}{n!}. \tag{1.5}$$

The coefficients $B_{n,\chi,\xi}(x)$, $n \geq 0$, are called twisted Bernoulli polynomials.

Theorem 1. ([11]) *Let χ be a Dirichlet character with conductor f and let ξ be rth root of* 1. *Then we have*

$$F_{\chi,\xi}(t,x)=\frac{1}{rf}\sum_{a=0}^{f-1}\chi(a)\xi^a\sum_{b=0}^{r-1}\xi^{bf}F(trf,\frac{a+bf+x-rf}{rf}). \tag{1.6}$$

Definition 1. ([11]) Let χ be a Dirichlet character with conductor f and let ξ be rth root of 1. Then we have

$$B_{n,\chi,\xi}(x)=(rf)^{n-1}\sum_{a=0}^{f-1}\sum_{b=0}^{r-1}\chi(a)\xi^{a+bf}B_n(\frac{a+bf+x-rf}{rf}), \tag{1.7}$$

and

$$B_{n,\chi,\xi}=(rf)^{n-1}\sum_{a=0}^{f-1}\sum_{b=0}^{r-1}\chi(a)\xi^{a+bf}B_n(\frac{a+bf-rf}{rf}). \tag{1.8}$$

where $B_{n,\chi,\xi}(x)$ and $B_{n,\chi,\xi}$ are twisted generalized Bernoulli polynomials and numbers, respectively.

In [1], Carlitz defined q-extensions of Bernoulli numbers and polynomials and proved properties generalizing those satisfied by B_k and $B_k(x)$. He defined a set of numbers $\eta_k = \eta_k(q)$ inductively by

q-Bernoulli numbers: $\eta_0 = 1,\ (q\eta+1)^k - \eta_k = \begin{cases} 1 \text{ if } k=1, \\ 0 \text{ if } k>1, \end{cases}$ with the usual convention about replacing η^k by η_k. These numbers are q-analogues of the ordinary Bernoulli numbers B_k, but they do not remain finite when $q = 1$. So he modified the definition as

$$\beta_0 = 1, (q\beta+1)^k - \beta_k = \begin{cases} 1 \text{ if } k=1, \\ 0 \text{ if } k>1. \end{cases}$$

These numbers $\beta_k = \beta_k(q)$ were called the q-Bernoulli numbers, which reduce to B_k when $q = 1$. Some properties of $\beta_k(q)$ were investigated by a lot of authors. Koblitz [9] constructed a q-analogue of p-adic Dirichlet L-series which interpolated Carlitz's q-Bernoulli numbers at non-positive integers and he raised two questions. In [9], Koblitz gave properties of q-extension of Bernoulli numbers and polynomials and he constructed p-adic measure and Dirichlet L function. In [10], Satoh constructed a complex analytic q-L-series which is a q-analogue of Dirichlet's L-functions and interpolates q-Bernoulli numbers, which is an answer to Koblitz's question 1. He induced this q-L-series from the generating function of q-Bernoulli numbers. Tsumura [12] defined q-L-series which is slightly different from the one in [10]. He also gave q-analogues of the Dirichlet L-series and Dedekind ζ-function. In [3], Kim showed that Carlitz's q-Bernoulli number can be represented as an integral by the q-analogue μ_q of ordinary p-adic invariant measure and he gave an answer to a part of a question of Koplitz. In [4], Kim gave a proof of the distribution relation for q-Bernoulli polynomials by using q-integral and evaluated the values of p-adic q-L-function.

Tsumura [12] modified the definition of the q-Bernoulli numbers $B_k = B_k(q)$ as follows: $B_0(q) = \frac{q-1}{\log q}, (qB+1)^k - B_k(q) = \begin{cases} 1 \text{ if } k=1, \\ 0 \text{ if } k>1. \end{cases}$

The usual convention about replacing B_k by B^k. We can see that $B_k(q) \to B_k$ when $q \to 1$.

EXAMPLE 1. We give some Tsumura's q-Bernoulli numbers:

$$B_1(q) = \frac{\log q + 1 - q}{(q-1)\log q}, B_2(q) = \frac{[2](1-q) - 2\log q}{[2](q-1)^2 \log q}, \ldots$$

We now summarize our present paper in detail as follows.

In Section 2, q-analogues of Dirichlet series are given due to Tsumura [12]. The relation between $\zeta_q(s,a)$ and $L_q(s,\chi)$ are proved.

In Section 3, we construct a twisted q-L-series which interpolates twisted q-generalized Bernoulli numbers.

In Section 4, we define the generating functions of twisted q-Bernoulli polynomials and numbers. We prove the numbers $L_q(1-n,\chi,\xi)$ which is related to "twisted" q- generalized Bernoulli numbers.

2. q -analogue of the Dirichlet L-series

Let $\mathbb{R}$ and $\mathbb{C}$ be the field of real and complex numbers as usual. Let q be a real number with $0<q<1$. We denote $[x]=[x;q]=\frac{1-q^x}{1-q}$. Note that $[x;q]\to x$ if $q\to 1$. q-analogues of the Dirichlet sires is defined as follows [12]: for a set of complex numbers $\{c_n\}$,

$$f(s)=\sum_{n=1}^{\infty}\frac{c_nq^{-n}}{(q^{-n}[n])^s},$$

for $s\in\mathbb{C}$. Tsumura [12] investigated these series by using a method similar to the method used to treat the ordinary Dirichlet series. For example the q-Riemann ζ-function can be defined by

$$\zeta_q(s)=\sum_{n=1}^{\infty}\frac{q^{-n}}{(q^{-n}[n])^s}.$$

We can see that the right-hand side of this series converges when $\mathrm{Re}(s)>1$. And $\zeta_q(s)$ may be analytically continued to the whole complex plane, except for a simple pole at $s=1$ with residue $\frac{q-1}{\log q}$(for detail see [12]).

Definition 2. (q-analogue of the Hurwitz ζ-functions [12]).

$$\zeta_q(s,b)=q^{s-1}\sum_{n=0}^{\infty}\frac{q^{-n}}{(q^{-n}[n]+b)^s}, \tag{2.1}$$

for $0<b\leq 1$, and $s\in\mathbb{C}$. $\zeta_q(s,b)\to\zeta(s,b)$ if $q\to 1$, where $\zeta(s,b)$ is the ordinary Hurwitz ζ-function.

Proposition 1. ([12]) *If* $k\geq 1$ *and* $0<b\leq 1$, *then*

$$\zeta_q(1-k,b)=\frac{(-1)^{k+1}B_k(b,q)}{kq^k}. \tag{2.2}$$

DEFINITION 3. (q-analogue of the Dirichlet L-series [12]). Let χ be a Dirichlet character of conductor f.

$$L_q(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)q^{-n}}{(q^{-n}[n])^s},$$

for $s\in\mathbb{C}$. $L_q(s,\chi)\to L(s,\chi)$ if $q\to 1$.

For χ as above, the generalized q-Bernoulli numbers are defined as follows [12]:

$$B_{k,\chi}(q)=[f]^{k-1}\sum_{a=1}^{f}\chi(a)q^{ak}B_k(\frac{[a]}{[f]},q^f),$$

for $k\geq 1$. In the case when $\chi=1$, $B_{k,1}(q)=q^{-k}B_k(1,q)=B_k(q)$.

Tsumura [12] gave a connection between $L_q(s,\chi)$ and $\zeta_q(s,a)$ as follows:

THEOREM 2. *Let χ be a Dirichlet character of conductor f.*

$$L_q(s,\chi)=[f]^{-s}\sum_{a=1}^{f}\chi(a)q^{(1-s)(f-a)}\zeta_{q^f}(s,\frac{[a]}{[f]}),$$

for $s\in\mathbb{C}$ and $s>1$.

Proof. Substituting $n=mf+a$, where $m=0,1,2,\dots,\infty$, and $a=1,2,\dots,f$ into definition of $L_q(s,\chi)$ as above, we obtain

$$\begin{aligned}
L_q(s,\chi)&=\sum_{a=1}^{f}\chi(a)\sum_{n=0}^{\infty}\frac{q^{-a-mf}}{(q^{-a-mf}[a+mf])^s}\\
&=\sum_{a=1}^{f}\chi(a)q^{as-a}\sum_{n=0}^{\infty}\frac{q^{-mf}}{(\frac{q^{-mf}-q^a}{1-q})^s}\\
&=\sum_{a=1}^{f}\chi(a)q^{as-a}\sum_{n=0}^{\infty}\frac{q^{-mf}}{\left\{q^{-mf}(\frac{1-q^{mf}}{1-q^f})(\frac{1-q^f}{1-q})+(\frac{1-q^a}{1-q})\right\}^s}\\
&=\sum_{a=1}^{f}\chi(a)q^{as-a}\sum_{n=0}^{\infty}\frac{q^{-mf}}{\{q^{-mf}[m;q^f][f]+[a]\}^s}\\
&=[f]^{-s}\sum_{a=1}^{f}\chi(a)q^{as-a-fs+f}\zeta_{q^f}(s,\frac{[a]}{[f]}).
\end{aligned}$$

where

$$q^a\left(\frac{1-q^{mf}}{1-q^f}\right)\left(\frac{1-q^f}{1-q}\right)+\left(\frac{1-q^a}{1-q}\right)=[m;q^f][f]q^a+[a]=[a+mf]. \tag{2.3}$$

Therefore we obtain the desired result. □

We wish to give the numbers $L_q(1-n,\chi,\xi)$ explicitly in the later section. For this we need the q-twisted Bernoulli numbers, which are defined below.

3. q-twisted L-functions

Our primary goal in this section is to construct q-twisted L-functions which interpolate q-twisted generalized Bernoulli numbers $B_{n,\chi,\xi}(q)$. We discuss some of the fundamental properties of these numbers which are needed in the later section.

By using the definition of $L_q(s,\chi)$ and $L(s,\chi,\xi)$, we can define a q-analogue of twisted L-function.

DEFINITION 4. (q-analogue of the twisted L-functions). Let χ be a Dirichlet character with conductor f and let ξ be rth root of 1.

$$L_q(s,\chi,\xi)=\sum_{n=1}^{\infty}\frac{\chi(n)\xi^n q^{-n}}{(q^{-n}[n])^s}, \tag{3.1}$$

for $s\in\mathbb{C}$. $L_q(s,\chi,\xi)\to L(s,\chi,\xi)$ if $q\to 1$. Since the function $n\to\chi(n)\xi$ has period fr, this is a special case of the Dirichlet L_q-functions considered above.

REMARK 1. Koblitz ([7], [8]) defined p-adic twisted L-functions, $L_p(s,\chi,\xi)$, where s is p-adic number. Using these functions, he constructed p-adic measures and integration, neither of which we have include here. In [9], Koblitz constructed a q-analogue of the p-adic L-function $L_{p,q}(s,\chi)$ which interpolated Carlitz's q-Bernoulli numbers. q-analogue of the p-adic twisted L-function $L_{p,q}(s,\chi,\xi)$ may be defined. We have also omitted a discussion of p-adic case.

Now, we give a relation between $L_q(s,\chi,\xi)$ and $\zeta_q(s,a)$ as follows.

THEOREM 3. *Let χ be a Dirichlet character with conductor f and let ξ be rth root of 1.*

$$L_q(s,\chi,\xi) = [rf]^{-s}\sum_{a=0}^{f-1}\sum_{b=0}^{r-1}\chi(a)\xi^{a+bf}q^{(1-s)(2rf-a-bf)} \cdot \zeta_{q^{rf}}(s,\frac{[a+bf-rf]}{[rf]}), \tag{3.2}$$

for $s \in \mathbb{C}$.

Proof. Substituting $n = a+bf-rf+rfm$ with $m = 0,1,\dots,\infty$, $a = 1,2,\dots,f-1$, and $b = 1,2,\dots,r-1$ into (3.1), we obtain

$$L_q(s,\chi,\xi) = \sum_{a=0}^{f-1}\sum_{b=0}^{r-1}\sum_{m=0}^{\infty}\chi(a)\xi^{a+bf} \cdot \frac{q^{-(a+bf-rf+rfm)}}{(q^{-(a+bf-rf)}q^{-rfm}[a+bf-rf+rfm])^s}.$$

After some calculations we get

$$L_q(s,\chi,\xi) = \sum_{a=0}^{f-1}\sum_{b=0}^{r-1}\chi(a)\xi^{a+bf}q^{as-a+bfs-bf+rf-rfs} \cdot \sum_{m=0}^{\infty}\frac{q^{-rfm}}{(q^{-rfm}\frac{1-q^{a+bf-rf+rfm}}{1-q})^s}.$$

By using (2.3) and (2.1) in the above, we obtain

$$L_q(s,\chi,\xi) = [rf]^{-s}\sum_{a=0}^{f-1}\sum_{b=0}^{r-1}\chi(a)\xi^{a+bf}q^{(1-s)(2rf-bf-a)} \cdot \zeta_{q^{rf}}(s,\frac{[a+bf-rf]}{[rf]}).$$

We obtain the desired result. □

4. q-twisted Bernoulli numbers and polynomials

The main purpose of this section is to give the numbers $L_q(1-n,\chi,\xi)$ which is related to twisted q- generalized Bernoulli numbers. Some basic facts about $F_q(t)$ and L_q-series are reviewed. Then their values at negative integers are given in terms of twisted q- generalized Bernoulli numbers.

We define the following $F_q(t)$ function which is similar to the one in [12]. The generating function of q-Bernoulli numbers $F_q(t)$ is given by

$$F_q(t) = \sum_{k=0}^{\infty} B_k(q)\frac{t^k}{k!} = \sum_{n=0}^{\infty} tq^{-n}e^{-q^{-n}[n]t}. \tag{4.1}$$

The remarkable point is that the series on the right-hand side of (4.1) is uniformly convergent in the wider sense. Hence we have

$$B_k(q) = \frac{d^k}{dt^k}F_q(t).$$

This is used to construct a q-Dirichlet series which are given above. By using this idea, Satoh [10] constructed the complex q-L-series which interpolated Carlitz's q-Bernoulli numbers $\beta_n(q)$. Higer order of the q-Bernoulli numbers and polynomials, $\beta_n^{(-m,k)}(q)$, for $m, k \in \mathbb{N}$, are defined by Kim [5], Kim and Rim [6]. They gave relations between these numbers and $L_{q,p}$-series (see for detail [5], [6]). Tsumura [12] studied a q-analogue of the Dirichlet L-series which interpolated Tsumura's q-Bernoulli numbers $B_n(q)$.

We shall explicitly determine the generating function $F_q(t)$ of $B_k(q)$:

$$F_q(t) = \sum_{k=0}^{\infty} B_k(q)\frac{t^k}{k!}.$$

This is the unique solution of the following q-difference equation:

$$F_q(t) = e^t F_q(qt) - qte^t. \tag{4.2}$$

LEMMA 1.

$$F_q(t) = \sum_{k=0}^{\infty} tq^{-n}e^{-q^{-n}[n]t}. \tag{4.3}$$

Proof. The right hand side is uniformly convergent in the wider sense, and satisfies (4.2). □

REMARK 2. i) By using (2.1) and (4.3), then we arrive at proof of (2.2). ii) As $q \to 1$ in (4.3), we have $F_q(t) \to F(t)$ in (1.1).

THEOREM 4. *Let* $k > 0$, $\zeta_q(1-k) = -\frac{(-1)^k B_k(q)}{k}$.

Proof. By using definition of $\zeta_q(s)$ and Lemma1, we obtain

$$B_k(q) = \frac{d^k}{dt^k}F_q(t) = (-1)^{k-1}k\zeta_q(1-k),$$

for $k > 0$. So we obtain the desired result. □

The generating function of q-Bernoulli polynomials $F_q(t,x)$ is defined by

$$F_q(t,x)=\sum_{k=0}^{\infty} tq^{-n}e^{(-q^{-n}[n]+[x])t}.$$

As $q\to 1$ in, we have $F_q(t)\to F(t)$ in (1.2).

Let χ be a Dirichlet character of conductor f. Then we define the following $F_{q,\chi}(t,x)$ function which is generating q-generalized Bernoulli polynomials $B_{q,\chi}(q,x)$.

DEFINITION 5.

$$F_{q,\chi}(t,x)=\frac{1}{[f]}\sum_{a=0}^{f}\chi(a)F_q(t[f],\frac{[a-f+x]}{[f]}). \tag{4.4}$$

LEMMA 2. *Let χ be a Dirichlet character of conductor f.*

$$F_{q,\chi}(t,x)=t\sum_{a=0}^{f-1}\chi(a)e^{([a]+[x]q^a-[f]q^{a+x-f})t}\sum_{k=0}^{\infty}q^{-n}e^{-q^{-n}[n][f])t}.$$

Proof. By using (4.4) and (2.3) ($[a+x]=[a]+[x]q^a$ and $[fa]=[f][a;q^f]$), we obtain

$$F_{q,\chi}(t,x)=t\sum_{a=0}^{f-1}\chi(a)\sum_{k=0}^{\infty}q^{-n}e^{(-q^{-n}[n][f]+[a]+[x]q^a-[f]q^{a+x-f})t}.$$

After some elementary calculations, we get the desired result. □

REMARK 3. As $q\to 1$, we have $F_{q,\chi}(t,x)\to F_\chi(t,x)$ in (1.3).

By using the definition of $F_{q,\chi}(t,x)$, we can define a twisted generating function of twisted q-Bernoulli polynomials.

DEFINITION 6. Let χ be a Dirichlet character with conductor f and let ξ be rth root of 1.

$$F_{q,\chi,\xi}(t,x)=\frac{1}{[rf]}\sum_{a=0}^{f-1}\chi(a)\xi^a\sum_{b=0}^{r-1}\xi^{bf}F_q(t[rf],\frac{[a-rf+bf+x]}{[rf]}). \tag{4.5}$$

REMARK 4. As $q\to 1$, we have $F_{q,\chi,\xi}(t,x)\to F_{\chi,\xi}(t,x)$ in (1.5).

By using (1.6), (1.7), (1.8) and (4.5), we can define a twisted q-Bernoulli numbers $B_{k,\chi,\xi}(q)$.

DEFINITION 7. Let χ be a Dirichlet character with conductor f and let ξ be rth root of 1.

$$B_{k,\chi,\xi}(q) = \frac{1}{[rf]^{1-k}} \sum_{a=0}^{f-1} \chi(a)\xi^a q^{-ak} \sum_{b=0}^{r-1} \xi^{bf} q^{-bfk} B_k(\frac{[a-rf+bf]}{[rf]}, q^{rf}).$$

We shall next describe some properties of $B_{n,\chi,\xi}(x,q)$ and $B_{n,\chi,\xi}(q)$ as follows:

i) if $\chi = \chi^0$, the principal character ($f = 1$), and $r = 1$, then

$$F_{q,\chi,\xi}(t,x) = F_q(t,x)$$

and

$$F_{\chi,\xi}(t,x) = F_q(t,[x]),$$

so that

$$B_{n,\chi^0,1}(x,q) = B_n(x,q),$$
$$B_{n,\chi^0,1} = B_n(q), n \geq 0.$$

ii)

$$B_{n,\chi,\xi}(0,q) = B_{n,\chi,\xi}(q), n \geq 0.$$

iii)

$$\begin{aligned} B_{0,\chi,\xi}(q) &= [rf]^{n-1} \sum_{a=0}^{f-1} \sum_{b=0}^{r-1} \chi(a)\xi^{a+bf} B_0 \\ &= \frac{q-1}{\log q}[rf]^{n-1} \sum_{a=0}^{f-1} \chi(a)\xi^a \sum_{b=0}^{r-1} \xi^{bf} \\ &= \frac{q-1}{\log q}[rf]^{n-1} \sum_{a=0}^{f-1} \chi(a)\xi^a \frac{\xi^{rf}-1}{\xi^f-1} \\ &= 0. \end{aligned}$$

Thus we have

$$B_{0,\chi,1}(q) = \frac{q-1}{\log q}[f]^{n-1} \sum_{0 \leq a < f} \chi(a) = 0.$$

Hence,

$$\deg(B_{n,\chi,\xi}(x,q)) < n$$

if $\chi = \chi^0$, the principal character ($f = 1$) and $r = 1$.

iv) If $r = 1$, then

$$B_{n,\chi,1}(x,q) = B_{n,\chi}(x,q),$$

and

$$B_{n,\chi,\xi}(q) = B_{n,\chi}(q), n \geq 0.$$

We now give a relation between twisted q-Bernoulli numbers and q-twisted L-functions as follows.

THEOREM 5. *Let χ be a Dirichlet character with conductor f and let ξ be rth root of 1. Let $n \geq 1$. Then*

$$L_q(1-n,\chi,\xi) = (-1)^{n+1} q^{rfk} \frac{B_{n,\chi,\xi}(q)}{n}.$$

Proof. Setting $s = 1-n$ in (3.2), we have

$$L_q(1-n,\chi,\xi) = [rf]^{n-1} \sum_{a=0}^{f-1} \sum_{b=0}^{r-1} \chi(a) \xi^{a+bf} q^{n(2rf-a-bf)} \cdot \zeta_{q^{rf}}\left(1-n, \frac{[a+bf-rf]}{[rf]}\right).$$

Writing $q \to q^{rf}$ and $b \to \frac{[a+bf-rf]}{[rf]}$ in (2.2) and substituting this result into the above equation, we arrive at the desired result. □

REMARK 5. $L_q(s,\chi,\xi)$ values at $s = 1$ may be calculate and relations with class numbers may be found. We do not discuss these properties here.

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