Kyushu J. Math. **76** (2022), 451–475
doi:10.2206/kyushujm.76.451

ON q-ANALOGUES OF ZETA FUNCTIONS OF ROOT SYSTEMS

Masaki KATO

(Received 21 July 2021 and revised 2 December 2021)

Abstract. Komori, Matsumoto and Tsumura introduced a zeta function $\zeta_r(\mathbf{s}, \Delta)$ associated with a root system Δ. In this paper, we introduce a q-analogue of this zeta function, denoted by $\zeta_r(\mathbf{s}, \mathbf{a}, \Delta; q)$, and investigate its properties. We show that a 'Weyl group symmetric' linear combination of $\zeta_r(\mathbf{s}, \mathbf{a}, \Delta; q)$ can be written as a multiple integral over a torus involving functions ψ_s. For positive integers k, functions ψ_k can be regarded as q-analogues of the periodic Bernoulli polynomials. When Δ is of type A_2 or A_3, the linear combinations can be expressed as the functions ψ_k, which are q-analogues of explicit expressions of Witten's volume formula. We also introduce a two-parameter deformation of the zeta function $\zeta_r(\mathbf{s}, \Delta)$ and study its properties.

1. Introduction

Let $\mathfrak{g}$ be a semisimple Lie algebra of rank r and s be a complex variable. We define the Witten zeta function by

$$\zeta_W(s, \mathfrak{g}) = \sum_{\varphi} (\dim \varphi)^{-s}, \tag{1.1}$$

where the summation on the right-hand side runs over all finite-dimensional irreducible representations φ of $\mathfrak{g}$. When $\mathfrak{g} = \mathfrak{sl}(2)$, the zeta function $\zeta_W(s, \mathfrak{sl}(2))$ becomes the Riemann zeta function $\zeta(s)$:

$$\zeta_W(s, \mathfrak{sl}(2)) = \zeta(s) := \sum_{n=1}^{\infty} n^{-s}.$$

The Witten zeta function was introduced by Zagier [**14**]. The reason the zeta function (1.1) was named 'Witten' comes from the fact that Witten [**13**] calculated volumes of certain moduli spaces in quantum gauge theory in terms of special values of (1.1) at positive even integers.

By using Weyl's dimension formula (for example, see [**11**, Section 3.8]), the Witten zeta function (1.1) can be written explicitly. Let $\langle \cdot\,, \cdot \rangle$ be the Killing form of $\mathfrak{g}$ and Δ_+ be the set of positive roots of $\mathfrak{g}$. For a root α of $\mathfrak{g}$, we denote the associated coroot of α by $\alpha^\vee$. Then Weyl's dimension formula states the following:

$$\dim \varphi = \prod_{\alpha \in \Delta_+} \frac{\langle \alpha^\vee, \lambda + \rho \rangle}{\langle \alpha^\vee, \rho \rangle}, \tag{1.2}$$

2010 Mathematics Subject Classification: Primary 33E20, 17B20.
Keywords: root system; Lie algebra; q-analogue; two-parameter deformation; Witten zeta function.

where λ is the dominant integral weight corresponding to an irreducible representation φ and ρ is the Weyl vector. Let P_+ be the set of all dominant integral weights. Then, by (1.2), the Witten zeta function can be written as follows:

$$\zeta_W(s, \mathfrak{g}) = \Bigg(\prod_{\alpha\in\Delta_+} \langle \alpha^\vee, \rho\rangle\Bigg)^s \sum_{\lambda\in P_+} \prod_{\alpha\in\Delta_+} \langle \alpha^\vee, \lambda+\rho\rangle^{-s}. \tag{1.3}$$

Komori, Matsumoto and Tsumura [**9, 10**] introduced the following zeta function associated with the root system Δ of $\mathfrak{g}$, as a multivariable generalization of (1.3). For a complex vector $\boldsymbol{s} = (s_\alpha)_{\alpha\in\Delta_+}$, we define the zeta function $\zeta_r(\boldsymbol{s}, \Delta)$ by

$$\zeta_r(\boldsymbol{s}, \Delta) := \sum_{\lambda\in P_+} \prod_{\alpha\in\Delta_+} \langle \alpha^\vee, \lambda+\rho\rangle^{-s_\alpha}. \tag{1.4}$$

The series on the right-hand side converges absolutely when $\operatorname{Re} s_\alpha > 1$ $(\alpha \in \Delta_+)$. When $s_\alpha = s$ for all $\alpha \in \Delta_+$, the function $\zeta_r((s, \ldots, s), \Delta)$ essentially coincides with the Witten zeta function (1.1). For details of the function $\zeta_r(\boldsymbol{s}, \Delta)$, see [**9, 10**].

In this paper, we introduce a q-analogue of the zeta function (1.4) and investigate its basic properties. The q-analogue of (1.4) is defined by the following:

$$\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta; q) := \sum_{\lambda\in P_+} \prod_{\alpha\in\Delta_+} \frac{a_\alpha^{\langle\alpha^\vee,\lambda+\rho\rangle}}{(1-q^{\langle\alpha^\vee,\lambda+\rho\rangle})^{s_\alpha}} \quad (\boldsymbol{a} = (a_\alpha)_{\alpha\in\Delta_+}).$$

When $\Delta = \Delta(A_1)$, the function $\zeta_1(s, q^{s-1}, \Delta(A_1); q)$ is essentially the same as a q-analogue of the Riemann zeta function, introduced by Kaneko, Kurokawa and Wakayama [**7**]. In Section 2, we establish basic properties of the function $\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta; q)$, including its analytic continuation. In Section 3, we show that a 'Weyl group symmetric' linear combination of functions $\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta; q)$ can be written as a multiple integral over a torus involving functions ψ_s. In Section 4, we investigate basic properties of functions ψ_k for positive integers k. In particular, we show in Proposition 4.2 that the functions ψ_k can be regarded as q-analogues of the periodic Bernoulli polynomials. In Section 5, we show that, when $\Delta = \Delta(A_1),\ \Delta(A_2),\ \Delta(A_3)$ and all components of the vector $\boldsymbol{s}$ are positive integers, the linear combination introduced in Section 4 can be written in terms of the functions ψ_k. When $\Delta = \Delta(A_2),\ \Delta(A_3)$, these expressions can be considered to be q-analogues of explicit expressions of Witten's volume formula, discovered independently by Zagier, Garoufalidis and Weinstein for the A_2 case (see [**14**]) and by Gunnells and Sczech [**5**] for the A_3 case. In Section 6, we introduce a p-deformation $\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\beta}, \Delta; p, q)$ of $\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta; q)$ and establish its basic properties. When $\Delta = \Delta(A_1)$, the function $\zeta_1(1, qe^{2\pi\sqrt{-1}x}, 1, \Delta(A_1); p, q)$ is considered to be a generating function of the elliptic zeta values, introduced by Felder and Varchenko [**4**].

2. q-Analogues of zeta functions of root systems

In this section, we introduce a q-analogue of the zeta function of a root system (1.4) and investigate its basic properties.

To do this, we prepare some notation of a root system. For details of the theory of root systems, we refer to [**1, 6**]. Let V be an r-dimensional real vector space with an inner product $\langle\cdot\,,\cdot\rangle$. We identify the dual space V^* with V via this inner product of V. Let Δ be a root system of V and

$$\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$$

be the coroot of $\alpha \in \Delta$. Let $\alpha_1, \ldots, \alpha_r$ be simple roots of Δ and put $\Psi := \{\alpha_1, \ldots, \alpha_r\}$. We denote the sets of positive and negative roots of Δ by Δ_+ and Δ_-, respectively:

$$\Delta_+ := \{c_1\alpha_1 + \cdots + c_r\alpha_r \in \Delta \mid c_i \geq 0 \ (i = 1, \ldots, r)\},$$
$$\Delta_- := \{c_1\alpha_1 + \cdots + c_r\alpha_r \in \Delta \mid c_i \leq 0 \ (i = 1, \ldots, r)\}.$$

Let $\lambda_1, \ldots, \lambda_r$ be the fundamental weights of Δ and P, P_+ and ρ be the weight lattice, the set of all dominant integral weights and the Weyl vector, respectively:

$$P := \bigoplus_{i=1}^{r} \mathbb{Z}\lambda_i, \quad P_+ := \bigoplus_{i=1}^{r} \mathbb{Z}_{\geq 0}\lambda_i, \quad \rho := \lambda_1 + \cdots + \lambda_r.$$

We are now in a position to define a q-analogue of the zeta function associated with the root system Δ. Let q be a real number satisfying $0 < q < 1$. For complex vectors $\boldsymbol{s} = (s_\alpha)_{\alpha \in \Delta_+}$ and $\boldsymbol{a} = (a_\alpha)_{\alpha \in \Delta_+}$, we define the function $\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta; q)$ by

$$\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta; q) := \sum_{\lambda \in P_+} \prod_{\alpha \in \Delta_+} \frac{a_\alpha^{\langle \alpha^\vee, \lambda + \rho \rangle}}{(1 - q^{\langle \alpha^\vee, \lambda + \rho \rangle})^{s_\alpha}}. \tag{2.1}$$

The series on the right-hand side of (2.1) converges absolutely for $|a_\alpha| < 1$ ($\alpha \in \Delta_+$).

When $\Delta = \Delta(A_1)$, the function $\zeta_1(s, q^{s-1}, \Delta(A_1); q)$ is a q-analogue of the Riemann zeta function introduced in [**7**], multiplied by $(1 - q)^s$. For general root systems Δ, the functions $\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta; q)$ can be regarded as q-analogues of zeta functions (1.4). In fact, when $a_\alpha = q^{t_\alpha}$ ($\mathrm{Re}\, t_\alpha > 0$) and $\mathrm{Re}\, s_\alpha > 1$ for $\alpha \in \Delta_+$, we have

$$\lim_{q \to 1} (1 - q)^{|\boldsymbol{s}|} \zeta_r(\boldsymbol{s}, (q^{t_\alpha})_{\alpha \in \Delta_+}, \Delta; q) = \zeta_r(\boldsymbol{s}, \Delta),$$

where we put

$$|\boldsymbol{s}| = \sum_{\alpha \in \Delta_+} s_\alpha.$$

For $\Delta = \Delta(A_r), \Delta(B_r), \Delta(C_r), \Delta(D_r)$, the functions $\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta : q)$ can be expressed explicitly, as follows.

Example 2.1. Let $\Delta = \Delta(A_r)$. Let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{r+1}\}$ be the standard basis of $(r+1)$-dimensional real vector space $\mathbb{R}^{r+1}$. Then we have the following:

$$V = \left\{ \sum_{i=1}^{r+1} x_i \boldsymbol{e}_i \,\middle|\, \sum_{i=1}^{r+1} x_i = 0 \right\},$$
$$\Delta(A_r) = \{\boldsymbol{e}_i - \boldsymbol{e}_j \mid 1 \leq i, j \leq r+1, \ i \neq j\},$$
$$\Delta_+(A_r) = \{\boldsymbol{e}_i - \boldsymbol{e}_j \mid 1 \leq i < j \leq r+1\},$$
$$\Psi(A_r) = \{\boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{e}_2 - \boldsymbol{e}_3, \ldots, \boldsymbol{e}_r - \boldsymbol{e}_{r+1}\}.$$

By putting $\alpha_k = \boldsymbol{e}_k - \boldsymbol{e}_{k+1}$, the positive coroots can be written as

$$(\boldsymbol{e}_i - \boldsymbol{e}_j)^\vee = \sum_{i \leq k < j} \alpha_k^\vee.$$

Thus we have

$$\zeta_r(\boldsymbol{s},\boldsymbol{a},\Delta(A_r);q)=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{1\le i<j\le r+1}\frac{a_{ij}^{\langle\sum_{i\le k<j}\alpha_k^\vee,m_1\lambda_1+\cdots+m_r\lambda_r\rangle}}{(1-q^{\langle\sum_{i\le k<j}\alpha_k^\vee,m_1\lambda_1+\cdots+m_r\lambda_r\rangle})^{s_{ij}}}$$
$$=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{1\le i<j\le r+1}\frac{a_{ij}^{m_i+\cdots+m_{j-1}}}{(1-q^{m_i+\cdots+m_{j-1}})^{s_{ij}}},$$

where we put $a_\alpha=a_{ij}$ and $s_\alpha=s_{ij}$ for $\alpha=\boldsymbol{e}_i-\boldsymbol{e}_j$. In particular, by putting

$$s_{ij}=0\quad((i,j)\neq(1,2),(1,3),\ldots,(1,r+1)),$$
$$a_{ij}=\begin{cases}q^{s_{ij}-1} & (i,j)=(1,2),(1,3),\ldots,(1,r+1),\\ 0 & \text{otherwise},\end{cases}$$

we obtain a q-analogue of the multiple zeta function (see [**15**])

$$\zeta_q(s_{12},\ldots,s_{1,r+1}):=(1-q)^{s_{12}+\cdots+s_{1,r+1}}\sum_{k_1>\cdots>k_r>0}\frac{q^{k_1(s_{12}-1)+\cdots+k_r(s_{1,r+1}-1)}}{(1-q^{k_1})^{s_{12}}\cdots(1-q^{k_r})^{s_{1,r+1}}},\tag{2.2}$$

multiplied by $(1-q)^{-s_{12}-\cdots-s_{1,r+1}}$.

Example 2.2. When $\Delta=\Delta(B_r)$, we have the following:

$$V=\mathbb{R}^r,$$
$$\Delta(B_r)=\{\pm\boldsymbol{e}_i\pm\boldsymbol{e}_j\mid 1\le i<j\le r,\}\cup\{\pm\boldsymbol{e}_i\mid 1\le i\le r\},$$
$$\Delta_+(B_r)=\{\boldsymbol{e}_i\pm\boldsymbol{e}_j\mid 1\le i<j\le r\}\cup\{\boldsymbol{e}_i\mid 1\le i\le r\},$$
$$\Psi(B_r)=\{\alpha_j=\boldsymbol{e}_j-\boldsymbol{e}_{j+1}\mid 1\le j\le r-1\}\cup\{\alpha_r=\boldsymbol{e}_r\}.$$

The simple coroots are given by

$$\alpha_j^\vee=\boldsymbol{e}_j-\boldsymbol{e}_{j+1}\quad(1\le j\le r-1),$$
$$\alpha_r^\vee=2\boldsymbol{e}_r,$$

and the positive coroots can be written as

$$\begin{cases}(\boldsymbol{e}_i+\boldsymbol{e}_j)^\vee=\sum_{i\le k<j}\alpha_k^\vee+2\sum_{j\le k<r}\alpha_k^\vee+\alpha_r^\vee & (1\le i<j\le r),\\ (\boldsymbol{e}_i-\boldsymbol{e}_j)^\vee=\sum_{i\le k<j}\alpha_k^\vee & (1\le i<j\le r),\\ (\boldsymbol{e}_i)^\vee=2\sum_{i\le k<r}\alpha_k^\vee+\alpha_r^\vee & (1\le i\le r).\end{cases}$$

Thus we have

$$\zeta_r(\boldsymbol{s},\boldsymbol{a},\Delta(B_r);q)=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{1\le i\le r}\frac{a_i^{2(m_i+\cdots+m_{r-1})+m_r}}{(1-q^{2(m_i+\cdots+m_{r-1})+m_r})^{s_i}}$$
$$\times\prod_{1\le i<j\le r}\frac{a_{ij,-}^{m_i+\cdots+m_{j-1}}}{(1-q^{m_i+\cdots+m_{j-1}})^{s_{ij,-}}}$$
$$\times\prod_{1\le i<j\le r}\frac{a_{ij,+}^{m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-1})+m_r}}{(1-q^{m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-1})+m_r})^{s_{ij,+}}}.$$

Example 2.3. When $\Delta = \Delta(C_r)$, we have the following:

$$\begin{aligned} V &= \mathbb{R}^r, \\ \Delta(C_r) &= \{\pm \boldsymbol{e}_i \pm \boldsymbol{e}_j \mid 1 \le i < j \le r, \} \cup \{\pm 2\boldsymbol{e}_i \mid 1 \le i \le r\}, \\ \Delta_+(C_r) &= \{\boldsymbol{e}_i \pm \boldsymbol{e}_j \mid 1 \le i < j \le r\} \cup \{2\boldsymbol{e}_i \mid 1 \le i \le r\}, \\ \Psi(C_r) &= \{\alpha_j = \boldsymbol{e}_j - \boldsymbol{e}_{j+1} \mid 1 \le j \le r-1\} \cup \{\alpha_r = 2\boldsymbol{e}_r\}. \end{aligned}$$

The simple coroots are given by

$$\begin{aligned} \alpha_j^\vee &= \boldsymbol{e}_j - \boldsymbol{e}_{j+1} \quad (1 \le j \le r-1), \\ \alpha_r^\vee &= \boldsymbol{e}_r, \end{aligned}$$

and the positive coroots can be written as

$$\begin{cases} (\boldsymbol{e}_i + \boldsymbol{e}_j)^\vee = \sum_{i \le k < j} \alpha_k^\vee + 2 \sum_{j \le k \le r} \alpha_k^\vee & (1 \le i < j \le r), \\ (\boldsymbol{e}_i - \boldsymbol{e}_j)^\vee = \sum_{i \le k < j} \alpha_k^\vee & (1 \le i < j \le r), \\ (\boldsymbol{e}_i)^\vee = \sum_{i \le k < r} \alpha_k^\vee & (1 \le i \le r). \end{cases}$$

Thus we have

$$\begin{aligned} \zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta(C_r); q) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{1 \le i \le r} & \frac{a_i^{m_i + \cdots + m_r}}{(1 - q^{m_i + \cdots + m_r})^{s_i}} \\ \times \prod_{1 \le i < j \le r} & \frac{a_{ij,-}^{m_i + \cdots + m_{j-1}}}{(1 - q^{m_i + \cdots + m_{j-1}})^{s_{ij,-}}} \\ \times \prod_{1 \le i < j \le r} & \frac{a_{ij,+}^{m_i + \cdots + m_{j-1} + 2(m_j + \cdots + m_{r-1}) + m_r}}{(1 - q^{m_i + \cdots + m_{j-1} + 2(m_j + \cdots + m_r)})^{s_{ij,+}}}. \end{aligned}$$

Example 2.4. When $\Delta = \Delta(D_r)$, we have the following:

$$\begin{aligned} V &= \mathbb{R}^r, \\ \Delta(D_r) &= \{\pm \boldsymbol{e}_i \pm \boldsymbol{e}_j \mid 1 \le i < j \le r\}, \\ \Delta_+(D_r) &= \{\boldsymbol{e}_i \pm \boldsymbol{e}_j \mid 1 \le i < j \le r\}, \\ \Psi(D_r) &= \{\alpha_j = \boldsymbol{e}_j - \boldsymbol{e}_{j+1} \mid 1 \le j \le r-1\} \cup \{\alpha_r = \boldsymbol{e}_{r-1} + \boldsymbol{e}_r\}. \end{aligned}$$

The simple coroots are give by

$$\begin{aligned} \alpha_j^\vee &= \boldsymbol{e}_j - \boldsymbol{e}_{j+1} \quad (1 \le j \le r-1), \\ \alpha_r^\vee &= \boldsymbol{e}_{r-1} + \boldsymbol{e}_r, \end{aligned}$$

and the positive coroots can be written as

$$\begin{cases} (\boldsymbol{e}_i + \boldsymbol{e}_r)^\vee = \sum_{i \le k \le r-2} \alpha_k^\vee + \alpha_r^\vee & (1 \le i < r), \\ (\boldsymbol{e}_i - \boldsymbol{e}_j)^\vee = \sum_{i \le k < j} \alpha_k^\vee & (1 \le i < j \le r), \\ (\boldsymbol{e}_i + \boldsymbol{e}_j)^\vee = \sum_{i \le k < j} \alpha_k^\vee + 2 \sum_{j \le k \le r-2} \alpha_k^\vee + \alpha_{r-1}^\vee + \alpha_r^\vee & (1 \le i < j < r). \end{cases}$$

Thus we have

$$\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta(D_r); q) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{1\le i<r} \frac{a_i^{m_i+\cdots+m_{r-2}+m_r}}{(1-q^{m_i+\cdots+m_{r-2}+m_r})^{s_{ir}^+}}$$
$$\times \prod_{1\le i<j\le r} \frac{a_{ij,-}^{m_i+\cdots+m_{j-1}}}{(1-q^{m_i+\cdots+m_{j-1}})^{s_{ij,-}}}$$
$$\times \prod_{1\le i<j<r} \frac{a_{ij,+}^{m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-2})+m_{r-1}+m_r}}{(1-q^{m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-2})+m_{r-1}+m_r})^{s_{ij,+}}}.$$

The following proposition implies that the function $\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta; q)$ is meromorphically continued to the whole space as a function of $\boldsymbol{s}$ and $\boldsymbol{a}$.

PROPOSITION 2.5. *We have the following expression:*

$$\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta; q) = \sum_{\substack{r_\alpha=0\\ a\in\Delta_+}}^{\infty} \left(\prod_{\alpha\in\Delta_+} \binom{s_\alpha + r_\alpha - 1}{r_\alpha}\right) \prod_{i=1}^{r} \frac{\prod_{\alpha\in\Delta_+}(aq^{r_\alpha})^{\langle\alpha^\vee,\lambda_i\rangle}}{1-\prod_{\alpha\in\Delta_+}(aq^{r_\alpha})^{\langle\alpha^\vee,\lambda_i\rangle}}.$$

Proof. By the binomial expansion, we obtain

$$(1-q^{\langle\alpha^\vee,\lambda+\rho\rangle})^{-s_\alpha} = \sum_{r_\alpha=0}^{\infty} \binom{s_\alpha + r_\alpha - 1}{r_\alpha} q^{\langle\alpha^\vee,\lambda+\rho\rangle r_\alpha}.$$

Thus we have

$$\begin{aligned}
\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta; q) &= \sum_{\lambda\in P_+} \prod_{\alpha\in\Delta_+} a^{\langle\alpha^\vee,\lambda+\rho\rangle} \left(\sum_{r_\alpha=0}^{\infty} \binom{s_\alpha + r_\alpha - 1}{r_\alpha} q^{\langle\alpha^\vee,\lambda+\rho\rangle r_\alpha}\right)\\
&= \sum_{\substack{r_\alpha=0\\ a\in\Delta_+}}^{\infty} \left(\prod_{\alpha\in\Delta_+} \binom{s_\alpha + r_\alpha - 1}{r_\alpha}\right) \sum_{\lambda\in P_+} \prod_{\alpha\in\Delta_+} (aq^{r_\alpha})^{\langle\alpha^\vee,\lambda+\rho\rangle}\\
&= \sum_{\substack{r_\alpha=0\\ a\in\Delta_+}}^{\infty} \left(\prod_{\alpha\in\Delta_+} \binom{s_\alpha + r_\alpha - 1}{r_\alpha}\right) \sum_{m_1,\ldots,m_r=1}^{\infty} \prod_{\alpha\in\Delta_+} (aq^{r_\alpha})^{\langle\alpha^\vee,m_1\lambda_1+\cdots+m_r\lambda_r\rangle}\\
&= \sum_{\substack{r_\alpha=0\\ a\in\Delta_+}}^{\infty} \left(\prod_{\alpha\in\Delta_+} \binom{s_\alpha + r_\alpha - 1}{r_\alpha}\right) \prod_{i=1}^{r} \frac{\prod_{\alpha\in\Delta_+}(aq^{r_\alpha})^{\langle\alpha^\vee,\lambda_i\rangle}}{1-\prod_{\alpha\in\Delta_+}(aq^{r_\alpha})^{\langle\alpha^\vee,\lambda_i\rangle}}.
\end{aligned}$$

□

Remark 2.6. It is obscure that it holds for a generic complex vector $\boldsymbol{s}$ that

$$\lim_{q\to 1}(1-q)^{|\boldsymbol{s}|}\zeta_r(\boldsymbol{s}, (q^{t_\alpha})_{\alpha\in\Delta_+}, \Delta; q) = \zeta_r(\boldsymbol{s}, \Delta).$$

We note that, when $r = 1$, Kaneko, Kurokawa and Wakayama [**7**] showed that

$$\lim_{q\to 1}(1-q)^s\zeta_1(s, q^{s-1}, \Delta; q) = \zeta(s)$$

for all $s \in \mathbb{C}$, $s \neq 1$. This Kaneko–Kurokawa–Wakayama result was generalized to the q-multiple zeta function (2.2) by Zhao [**15**].

3. Weyl group symmetry

Let W be the Weyl group of a root system Δ. That is, W is a group generated by reflections σ_α with respect to the hyperplane orthogonal to $\alpha \in \Delta$: $W = \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$. Let $B_k(\cdot)$ ($k = 0, 1, 2, \ldots$) be Bernoulli polynomials defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!},$$

and for $\boldsymbol{k} := (k_\alpha)_{\alpha \in \Delta}$ with $k_\alpha \in \mathbb{Z}_{\geq 0}$, we put

$$B_{\boldsymbol{k}}(\Delta) := \int_0^1 \cdots \int_0^1 \Bigg(\prod_{\alpha \in \Delta_+ \setminus \Psi} B_{k_\alpha}(\{x_\alpha\}) \Bigg) \prod_{j=1}^{r} B_{k_{\alpha_j}} \Bigg(\Bigg\{ - \sum_{\alpha \in \Delta_+ \setminus \Psi} x_\alpha \langle \alpha^\vee, \lambda_j \rangle \Bigg\} \Bigg)$$
$$\times \prod_{\alpha \in \Delta_+ \setminus \Psi} dx_\alpha,$$

where, for a real number x, $\{x\}$ denotes the fractional part of x. Komori, Matsumoto and Tsumura [**9**] obtained the following result.

THEOREM 3.1. [**9**, III, Theorem 8] *Assume that Δ is an irreducible root system. For $v \in V$, we denote the norm of v by $||v|| := \langle v, v \rangle^{1/2}$ and put $\boldsymbol{k} = (k_{||\alpha||})_{\alpha \in \Delta_+} \in \mathbb{Z}_{>0}^{|\Delta_+|}$. Then we have*

$$\zeta_r(2\boldsymbol{k}, \Delta) = \frac{(-1)^{|\Delta_+|}}{|W|} \Bigg(\prod_{\alpha \in \Delta_+} \frac{(2\pi\sqrt{-1})^{2k_\alpha}}{(2k_\alpha)!} \Bigg) B_{2\boldsymbol{k}}(\Delta).$$

Theorem 3.1 implies that

$$\zeta_r((2k, \ldots, 2k), \Delta) \in \mathbb{Q}\pi^{2k|\Delta_+|} \tag{3.1}$$

for $k \in \mathbb{Z}_{>0}$. This result is called Witten's volume formula.

Komori, Matsumoto and Tsumura [**9**] deduced Theorem 3.1 from an integral representation of a sum of zeta functions (1.4) which has the Weyl group symmetry. We define the action of the Weyl group W to the complex vector $\boldsymbol{s} = (s_\alpha)_{\alpha \in \Delta_+}$ by

$$w\boldsymbol{s} = (s_{w^{-1}\alpha})_{\alpha \in \Delta}$$

for $w \in W$, where we put $s_\alpha = s_{-\alpha}$ for $\alpha \in \Delta_-$.

THEOREM 3.2. [**9**, III, Theorem 6] *We put*

$$S(\boldsymbol{s}, \Delta) := \sum_{w \in W} \Bigg(\prod_{\alpha \in \Delta_+ \cap w\Delta_-} (-1)^{-s_\alpha} \Bigg) \zeta_r(w^{-1}\boldsymbol{s}),$$

and assume that $\mathrm{Re}\, s_\alpha > 1$ *for $\alpha \in \Delta_+$. Then we have*

$$S(\boldsymbol{s}, \Delta) = (-1)^{|\Delta_+|} \Bigg(\prod_{\alpha \in \Delta_+} \frac{(2\pi\sqrt{-1})^{s_\alpha}}{\Gamma(s_\alpha + 1)} \Bigg) \int_0^1 \cdots \int_0^1 \Bigg(\prod_{\alpha \in \Delta_+ \setminus \Psi} L(s_\alpha, x_\alpha) \Bigg)$$
$$\times \prod_{j=1}^{r} L\Bigg(s_{\alpha_j}, - \sum_{\alpha \in \Delta_+ \setminus \Psi} x_\alpha \langle \alpha^\vee, \lambda_j \rangle \Bigg) \prod_{\alpha \in \Delta_+ \setminus \Psi} dx_\alpha,$$

where $\Gamma(s)$ *denotes the gamma function and we put*

$$L(s, x) := -\frac{\Gamma(s+1)}{(2\pi\sqrt{-1})^s} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{e^{2\pi\sqrt{-1}nx}}{n^s}.$$

In this section, we consider a q-analogue of Theorem 3.2. We define the action of the Weyl group W to the complex vector $\boldsymbol{a} = (a_\alpha)_{\alpha\in\Delta_+}$ by

$$w^{-1}\boldsymbol{a} = (a_{w\alpha})_{\alpha\in\Delta_+} \quad (w \in W),$$

where we put $a_\alpha = q^{s_{-\alpha}} a_{-\alpha}^{-1}$ for $\alpha \in \Delta_-$. We introduce the Weyl group symmetric sum $S(\boldsymbol{s}, \boldsymbol{a}, \Delta; q)$ defined by

$$S(\boldsymbol{s}, \boldsymbol{a}, \Delta; q) := \sum_{w\in W} \left(\prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{-s_\alpha} \right) \zeta_r(w^{-1}\boldsymbol{s}, w^{-1}\boldsymbol{a}, \Delta; q).$$

THEOREM 3.3. *Assume that* $\Re s_\alpha > 0$, $q^{\Re s_\alpha} < |a_\alpha| < 1$ *for all* $\alpha \in \Delta_+$. *Then we have the following:*

$$S(\boldsymbol{s}, \boldsymbol{a}, \Delta; q) = \frac{1}{(2\pi\sqrt{-1})^{|\Delta_+\setminus\Psi|}} \int_{\mathbb{T}^{|\Delta_+\setminus\Psi|}} \left(\prod_{\alpha\in\Delta_+\setminus\Psi} \psi_{s_\alpha}(a_\alpha z_\alpha; q) \right)$$
$$\times \prod_{j=1}^{r} \psi_{s_{\alpha_j}} \left(a_{\alpha_j} \prod_{\alpha\in\Delta_+\setminus\Psi} z_\alpha^{-\langle\alpha^\vee, \lambda_j\rangle} \right) \prod_{\alpha\in\Delta_+\setminus\Psi} \frac{dz_\alpha}{z_\alpha},$$

where $\mathbb{T}$ *is the unit circle* $\{z \in \mathbb{C} \mid |z| = 1\}$ *and we put*

$$\psi_s(a; q) := \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{a^n}{(1-q^n)^s}.$$

Proof. By definition, we have

$$S(\boldsymbol{s}, \boldsymbol{a}, \Delta; q) = \sum_{w\in W} \left(\prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{-s_\alpha} \right) \sum_{\lambda\in P_+} \prod_{\alpha\in\Delta_+} \frac{a_{w\alpha}^{\langle\alpha^\vee, \lambda+\rho\rangle}}{(1-q^{\langle\alpha^\vee, \lambda+\rho\rangle})^{s_{w\alpha}}}.$$

The product $\prod_{\alpha\in\Delta_+}$ can be decomposed as follows:

$$\prod_{\alpha\in\Delta_+} = \prod_{\alpha\in\Delta_+\cap w^{-1}\Delta_+} \quad \prod_{\alpha\in\Delta_+\cap w^{-1}\Delta_-}.$$

Furthermore it holds that

$$\prod_{\alpha\in\Delta_+\cap w^{-1}\Delta_-} \frac{a_{w\alpha}^{\langle\alpha^\vee, \lambda+\rho\rangle}}{(1-q^{\langle\alpha^\vee, \lambda+\rho\rangle})^{s_{w\alpha}}} = \prod_{\alpha\in\Delta_+\cap w^{-1}\Delta_-} \frac{(q^{s_{-w\alpha}} a_{-w\alpha}^{-1})^{\langle\alpha^\vee, \lambda+\rho\rangle}}{(1-q^{\langle\alpha^\vee, \lambda+\rho\rangle})^{s_{-w\alpha}}}$$
$$= \prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{s_\alpha} \prod_{\alpha\in\Delta_-\cap w^{-1}\Delta_+} \frac{a_{w\alpha}^{\langle\alpha^\vee, \lambda+\rho\rangle}}{(1-q^{\langle\alpha^\vee, \lambda+\rho\rangle})^{s_{w\alpha}}}.$$

Thus we have

$$\begin{aligned}
S(\boldsymbol{s}, \boldsymbol{a}, \Delta; q) &= \sum_{w\in W}\sum_{\lambda\in P_+}\prod_{\alpha\in w^{-1}\Delta_+}\frac{a_{w\alpha}^{\langle\alpha^\vee,\lambda+\rho\rangle}}{(1-q^{\langle\alpha^\vee,\lambda+\rho\rangle})^{s_{w\alpha}}}\\
&= \sum_{w\in W}\sum_{\lambda\in P_+}\prod_{\alpha\in\Delta_+}\frac{a_{\alpha}^{\langle w^{-1}\alpha,\lambda+\rho\rangle}}{(1-q^{\langle w^{-1}\alpha^\vee,\lambda+\rho\rangle})^{s_{\alpha}}}\\
&= \sum_{w\in W}\sum_{\lambda\in P_+}\prod_{\alpha\in\Delta_+}\frac{a_{\alpha}^{\langle \alpha,w(\lambda+\rho)\rangle}}{(1-q^{\langle \alpha^\vee,w(\lambda+\rho)\rangle})^{s_{\alpha}}}.
\end{aligned}$$

Let H_Δ be the union of boundaries of all Weyl chambers. Then, for $\lambda\in P\setminus H_\Delta$, there exist unique $w\in W$ and $\lambda'\in P_+$ satisfying $\lambda=w(\lambda'+\rho)$. Thus we have

$$S(\boldsymbol{s}, \boldsymbol{a}, \Delta; q) = \sum_{\lambda\in P\setminus H_\Delta}\prod_{\alpha\in\Delta_+}\frac{a_{\alpha}^{\langle \alpha,\lambda\rangle}}{(1-q^{\langle \alpha^\vee,\lambda\rangle})^{s_{\alpha}}}.$$

Here, by observing

$$\frac{a_{\alpha}^{\langle \alpha^\vee,\lambda\rangle}}{(1-q^{\langle \alpha^\vee,\lambda\rangle})^{s_{\alpha}}} = \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}} z_\alpha^{-\langle\alpha^\vee,\lambda\rangle}\psi_{s_\alpha}(a_\alpha z_\alpha; q)\,\frac{dz_\alpha}{z_\alpha},$$

we find that

$$\begin{aligned}
S(\boldsymbol{s}, \boldsymbol{a}, \Delta; q) &= \sum_{\lambda\in P\setminus H_\Delta}\prod_{\alpha\in\Psi}\frac{a_{\alpha}^{\langle \alpha^\vee,\lambda\rangle}}{(1-q^{\langle \alpha^\vee,\lambda\rangle})^{s_{\alpha}}}\prod_{\alpha\in\Delta_+\setminus\Psi}\frac{a_{\alpha}^{\langle \alpha^\vee,\lambda\rangle}}{(1-q^{\langle \alpha^\vee,\lambda\rangle})^{s_{\alpha}}}\\
&= \sum_{\lambda\in P\setminus H_\Delta}\prod_{\alpha\in\Psi}\frac{a_{\alpha}^{\langle \alpha^\vee,\lambda\rangle}}{(1-q^{\langle \alpha^\vee,\lambda\rangle})^{s_{\alpha}}}\\
&\quad\times\prod_{\alpha\in\Delta_+\setminus\Psi}\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}} z_\alpha^{-\langle\alpha^\vee,\lambda\rangle}\psi_{s_\alpha}(a_\alpha z_\alpha; q)\,\frac{dz_\alpha}{z_\alpha}\\
&= \frac{1}{(2\pi\sqrt{-1})^{|\Delta_+\setminus\Psi|}}\sum_{\lambda\in P\setminus H_\Delta}\prod_{\alpha\in\Psi}\frac{a_{\alpha}^{\langle \alpha^\vee,\lambda\rangle}}{(1-q^{\langle \alpha^\vee,\lambda\rangle})^{s_{\alpha}}}\\
&\quad\times\prod_{\alpha\in\Delta_+\setminus\Psi}\int_{\mathbb{T}} z_\alpha^{-\langle\alpha^\vee,\lambda\rangle}\psi_{s_\alpha}(a_\alpha z_\alpha; q)\,\frac{dz_\alpha}{z_\alpha}.
\end{aligned}$$

We now write $\lambda=\sum_{j=1}^r n_j\lambda_j$. Since

$$\int_{\mathbb{T}}\psi_{s_\alpha}(a_\alpha z_\alpha; q)\,\frac{dz_\alpha}{z_\alpha}=0,$$

we can extend the summation range $P \setminus H_\Delta$ to the set of all λ satisfying $n_j \neq 0$ $(1 \leq j \leq r)$. Thus we obtain

$$\begin{aligned} S(\boldsymbol{s}, \boldsymbol{a}, \Delta; q) &= \frac{1}{(2\pi\sqrt{-1})^{|\Delta_+ \setminus \Psi|}} \sum_{\substack{n_i \neq 0 \\ 1 \leq i \leq r}} \prod_{j=1}^{r} \frac{a_{\alpha_j}^{n_j}}{(1-q^{n_j})^{s_{\alpha_j}}} \\ &\quad \times \prod_{\alpha \in \Delta_+ \setminus \Psi} \int_{\mathbb{T}} z_\alpha^{-\langle \alpha^\vee, n_1\lambda_1 + \cdots + n_r\lambda_r \rangle} \psi_{s_\alpha}(a_\alpha z_\alpha; q) \frac{dz_\alpha}{z_\alpha} \\ &= \frac{1}{(2\pi\sqrt{-1})^{|\Delta_+ \setminus \Psi|}} \int_{\mathbb{T}^{|\Delta_+ \setminus \Psi|}} \left(\prod_{\alpha \in \Delta_+ \setminus \Psi} \psi_{s_\alpha}(a_\alpha z_\alpha; q) \right) \\ &\quad \times \prod_{j=1}^{r} \psi_{s_{\alpha_j}} \left(a_{\alpha_j} \prod_{\alpha \in \Delta_+ \setminus \Psi} z_\alpha^{-\langle \alpha^\vee, \lambda_j \rangle} \right) \prod_{\alpha \in \Delta_+ \setminus \Psi} \frac{dz_\alpha}{z_\alpha}, \end{aligned}$$

which completes the proof of the theorem. □

4. Properties of functions $\psi_k(a; q)$

In this section, we investigate basic properties of functions $\psi_k(a; q)$ for $k \in \mathbb{Z}_{>0}$. The results established in this section will be used in the next section.

PROPOSITION 4.1. *Let $k \in \mathbb{Z}_{>0}$. Then we have the following.*

(1) *The function $\psi_k(a; q)$ satisfies the following q-difference relation:*

$$\psi_k(qa; q) = \psi_k(a; q) - \psi_{k-1}(a; q),$$

where we put $\psi_0(a; q) = -1$.

(2) *The function $\psi_k(a; q)$ has the following symmetry:*

$$\psi_k(q^k a^{-1}; q) = (-1)^k \psi_k(a; q).$$

(3) *The function $\psi_k(a; q)$ can be written as follows:*

$$\psi_k(a; q) = \sum_{r=0}^{\infty} \binom{k+r-1}{r} \left(\frac{q^r a}{1-q^r a} + (-1)^k \frac{q^{r+k} a^{-1}}{1-q^{r+k} a^{-1}} \right). \tag{4.1}$$

This expression gives the meromorphic continuation of $\psi_k(a; q)$ to the whole complex plane. The function $\psi_k(a; q)$ is holomorphic except at simple poles $a = q^{\mathbb{Z}_{\leq 0}}$, $q^{k+\mathbb{Z}_{\geq 0}}$.

Proof. The claims (1) and (2) are clear from the definitions. The claim (3) follows from the binomial expansion given by

$$\frac{1}{(1-q^n)^k} = \sum_{r=0}^{\infty} \binom{k+r-1}{r} q^{nr} \quad (n > 0).$$ □

By using Proposition 4.1(3) repeatedly, we have

$$\psi_k(q^n a; q) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \psi_{k-i}(a; q) \tag{4.2}$$

for $n \geq 1$, where we put $\psi_k(a; q) = 0$ for $k \in \mathbb{Z}_{\leq -1}$.

The following proposition implies that the function $\psi_k(a;q)$ can be considered to be a q-analogue of the periodic Bernoulli polynomial $B_k(\{x\})$.

PROPOSITION 4.2. *Let* $k\in\mathbb{Z}_{>0}$. *Then, for* $t, x\in\mathbb{R}$ *and* $(t,x)\notin(\mathbb{Z}_{\le 0}\cup(k+\mathbb{Z}_{\ge 0}))\times\mathbb{Z}$, *we have*

$$\lim_{q\to 1}(1-q)^k\psi_k(q^t e^{2\pi\sqrt{-1}x};q)=-\frac{(2\pi\sqrt{-1})^k}{k!}B_k(\{x\}).$$

Proof. By (4.2) and Proposition 4.1(2), it enough to show the proposition for $0\le t<k$. When $0<t<k$, the proposition follows immediately from the following well-known Fourier series expansion of the periodic Bernoulli polynomial:

$$B_k(\{x\})=-k!\sum_{n\in\mathbb{Z}-\{0\}}\frac{e^{2\pi\sqrt{-1}x}}{(2\pi\sqrt{-1}n)^k}.$$

We now show the proposition for $t=0$ and $x\notin\mathbb{Z}$ by induction on k. By Proposition 4.1(3), we have

$$\psi_1(e^{2\pi\sqrt{-1}x};q)=2\sqrt{-1}\sin(2\pi x)\sum_{r=1}^{\infty}\frac{q^r}{q^{2r}-2\cos(2\pi x)q^r+1}+\frac{e^{2\pi\sqrt{-1}x}}{1-e^{2\pi\sqrt{-1}x}}.$$

It follows that

$$\begin{aligned}\lim_{q\to 1}(1-q)\psi_1(e^{2\pi\sqrt{-1}x};q)&=2\sqrt{-1}\sin(2\pi x)\int_0^1\frac{du}{u^2-2\cos(2\pi x)u+1}\\&=2\pi\sqrt{-1}\int_0^\infty\frac{\sin(2\pi x)}{\cosh(2\pi t)-\cos(2\pi x)}\,dt\\&=-2\pi\sqrt{-1}B_1(\{x\}),\end{aligned}$$

where we put $u=e^{-2\pi t}$ in the second equality. In the last equality, we used the integral representation of the Bernoulli polynomial (see [**3**, (21), p. 38]). Thus we find that the proposition holds for $k=1$.

We next assume that the proposition is true for $k\ge 1$. By Proposition 4.1(1) and (2), we have

$$\psi_{k+1}(e^{2\pi\sqrt{-1}x};q)=(-1)^{k+1}\psi_{k+1}(q^k e^{-2\pi\sqrt{-1}x};q)+\psi_k(e^{2\pi\sqrt{-1}x};q).$$

Thus the induction hypothesis implies that

$$\begin{aligned}\lim_{q\to 1}(1-q)^{k+1}\psi_{k+1}(e^{2\pi\sqrt{-1}x};q)&=(-1)^{k+1}\left(-\frac{(2\pi\sqrt{-1})^{k+1}}{(k+1)!}B_{k+1}(\{-x\})\right)\\&=-\frac{(2\pi\sqrt{-1})^{k+1}}{(k+1)!}B_{k+1}(\{x\}),\end{aligned}$$

which proves the proposition for $k+1$. We thus finish the proof of the proposition. □

By definition, the generating function of the Bernoulli polynomials $B_k(x)$ is given by

$$\frac{te^{xt}}{e^t-1}.$$

Meanwhile the generating function of the functions $\psi_k(a;q)$ becomes the Kronecker function.

PROPOSITION 4.3. *We define the theta function $\theta(a; q)$ by*

$$\theta(a; q) := \prod_{m=0}^{\infty}(1 - aq^m)(1 - a^{-1}q^{m+1})$$

and the Kronecker function $F(\alpha, a; q)$ by

$$F(\alpha, a; q) := \frac{\theta'(1; q)\theta(a\alpha; q)}{\theta(a; q)\theta(\alpha; q)}$$

for $a, \alpha \in \mathbb{C}$. Then, for $a \in \mathbb{C}$ satisfying $q < |a| < 1$, the Kronecker function $F(\alpha, a; q)$ is expanded into a Laurent series around $\alpha = 1$, as follows:

$$F(\alpha, a; q) = \frac{1}{\alpha - 1} + \sum_{k=0}^{\infty}(-1)^k \psi_k(q^k a; q)(\alpha - 1)^k.$$

Proof. See [**8**, Proposition 2.2]. □

The following proposition will play an important role in the next section.

PROPOSITION 4.4. *For $k_1, k_2 \in \mathbb{Z}_{>0}$, we have the following:*

$$\begin{aligned}
&\psi_{k_1}(a_1; q)\psi_{k_2}(a_2; q) \\
&= \sum_{k=0}^{k_1} \binom{k_1 + k_2 - k - 1}{k_2 - 1} (-1)^{k_1 - k} \psi_k(a_1 a_2; q)\psi_{k_1+k_2-k}(q^{k_1-k}a_2; q) \\
&\quad + \sum_{l=0}^{k_2} \binom{k_1 + k_2 - l - 1}{k_1 - 1} (-1)^{k_2 - l} \psi_l(a_1 a_2; q)\psi_{k_1+k_2-l}(q^{k_2-l}a_1; q) + \psi_{k_1+k_2}(a_1 a_2; q).
\end{aligned}$$

Proof. It is known that the Kronecker function satisfies the following Fay's identity (see [**2**] or [**8**, Theorem 2.3]):

$$\begin{aligned}
F(\alpha_1, a_1; q)F(\alpha_2, a_2; q) &= F(\alpha_1, a_1 a_2; q)F(\alpha_1^{-1}\alpha_2, a_2; q) \\
&\quad + F(\alpha_2, a_1 a_2; q)F(\alpha_1 \alpha_2^{-1}, a_1; q).
\end{aligned}$$

We now expand both sides into Laurent series of $\alpha_1 - 1$ and $\alpha_2 - 1$, and then compare the coefficients of $(\alpha_1 - 1)^{k_1 - 1}$ and $(\alpha_2 - 1)^{k_2 - 1}$. Then, by Proposition 4.3, we obtain the proposition. □

5. The cases of $\Delta = \Delta(A_1), \Delta(A_2), \Delta(A_3)$

In this section, we show that, when $\Delta = \Delta(A_1), \Delta(A_2), \Delta(A_3)$ and all components of the vector $\boldsymbol{s}$ are positive integers, the 'Weyl group symmetric' linear combination of the functions $\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta; q)$ introduced in Section 3 can be written in terms of the functions ψ_k. By letting $q \to 1$ in this result, we obtain explicit expressions for Witten's volume formulas (3.1) of A_1, A_2 and A_3 types.

Example 5.1. Let $r = 1$. By putting $s = k (k \in \mathbb{Z}_{>1})$ in Theorem 3.3, we have

$$S(k, a, \Delta(A_1); q) = \psi_k(a; q).$$

By definition, it holds that

$$S(s, a, \Delta(A_1); q) = \zeta_1(s, a, \Delta(A_1); q) + (-1)^{-s}\zeta_1(s, q^s a^{-1}, \Delta(A_1); q).$$

Thus we obtain

$$\zeta_1(k, a, \Delta(A_1); q) + (-1)^{-k}\zeta_1(k, q^k a^{-1}, \Delta(A_1); q) = \psi_k(a; q). \tag{5.1}$$

In particular, when $k = 2m$ ($m \geq 1$) and $a = q^m$, we have

$$\zeta_1(2m, q^m, \Delta(A_1); q) = \frac{1}{2}\psi_{2m}(q^m; q).$$

We now put $k = 2m (m \geq 1)$, $a = q^t (0 < t < 2m)$, multiply both sides by $(1 - q)^{2m}$ and take the limit as $q \to 1$ in (5.1). Then, by Proposition 4.2, we obtain the well-known formula

$$\zeta(2m) = (-1)^{m+1}\frac{B_{2m}(2\pi)^{2m}}{2(2m)!},$$

which is due to Euler. Here $B_n := B_n(0)$ denotes the nth Bernoulli number.

We next consider the case where $\Delta = \Delta(A_2)$. Then for $\boldsymbol{k} \in \mathbb{Z}^3_{>0}$, the linear combination $S(\boldsymbol{k}, \boldsymbol{a}, \Delta(A_2); q)$ can be written in terms of the functions ψ_k, as follows.

THEOREM 5.2. *We have*

$$\begin{aligned} &S(\boldsymbol{k}, \boldsymbol{a}, \Delta(A_2); q) \\ &= (-1)^{k_{12}} \sum_{k=0}^{k_{13}} \binom{k_{12} + k_{13} - k - 1}{k_{12} - 1} \psi_k(a_{12}a_{13}; q)\psi_{k_{12}+k_{13}+k_{23}-k}(a_{12}^{-1}a_{23}q^{k_{12}}; q) \\ &\quad + \sum_{l=0}^{k_{12}} \binom{k_{12} + k_{13} - l - 1}{k_{13} - 1} (-1)^{k_{12}-l}\psi_l(a_{12}a_{13}; q)\psi_{k_{12}+k_{13}+k_{23}-l}(a_{13}a_{23}q^{k_{12}-l}; q). \end{aligned}$$

Proof. By Theorem 3.3, we have

$$\begin{aligned} S(\boldsymbol{k}, \boldsymbol{a}, \Delta(A_2); q) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} &\psi_{k_{13}}(a_{13}z_{13}; q)\psi_{k_{12}}(a_{12}z_{13}^{-1}; q) \\ &\times \psi_{k_{23}}(a_{23}z_{13}^{-1}; q)\,\frac{dz_{13}}{z_{13}}. \end{aligned}$$

Proposition 4.4 gives

$$\begin{aligned} &\psi_{k_{13}}(a_{13}z_{13}; q)\psi_{k_{12}}(a_{12}z_{13}^{-1}; q) \\ &\quad = \sum_{k=0}^{k_{13}} \binom{k_{12} + k_{13} - k - 1}{k_{12} - 1} (-1)^{k_{13}-k}\psi_k(a_{12}a_{13}; q)\psi_{k_{12}+k_{13}-k}(q^{k_{13}-k}a_{12}z_{13}^{-1}; q) \\ &\qquad + \sum_{l=0}^{k_{12}} \binom{k_{12} + k_{13} - l - 1}{k_{13} - 1} (-1)^{k_{12}-l}\psi_l(a_{12}a_{13}; q)\psi_{k_{12}+k_{13}-l}(q^{k_{12}-l}a_{13}z_{13}; q) \\ &\qquad + \psi_{k_{12}+k_{13}}(a_{12}a_{13}; q). \end{aligned}$$

Thus we find that

$$
\begin{aligned}
&S(\boldsymbol{k}, \boldsymbol{a}, \Delta(A_2); q)\\
&= \sum_{k=0}^{k_{13}} \binom{k_{12}+k_{13}-k-1}{k_{12}-1} (-1)^{k_{13}-k} \psi_k(a_{12}a_{13}; q)\\
&\quad \times \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \psi_{k_{23}}(a_{23}z_{13}^{-1}; q) \psi_{k_{12}+k_{13}-k}(q^{k_{13}-k} a_{12} z_{13}^{-1}; q) \frac{dz_{13}}{z_{13}}\\
&\quad + \sum_{l=0}^{k_{12}} \binom{k_{12}+k_{13}-l-1}{k_{13}-1} (-1)^{k_{12}-l} \psi_l(a_{12}a_{13}; q)\\
&\quad \times \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \psi_{k_{23}}(a_{23}z_{13}^{-1}; q) \psi_{k_{12}+k_{13}-l}(q^{k_{12}-l} a_{13} z_{13}; q) \frac{dz_{13}}{z_{13}}\\
&= (-1)^{k_{12}} \sum_{k=0}^{k_{13}} \binom{k_{12}+k_{13}-k-1}{k_{12}-1} \psi_k(a_{12}a_{13}; q) \psi_{k_{12}+k_{13}+k_{23}-k}(a_{12}^{-1} a_{23} q^{k_{12}}; q)\\
&\quad + \sum_{l=0}^{k_{12}} \binom{k_{12}+k_{13}-l-1}{k_{13}-1} (-1)^{k_{12}-l} \psi_l(a_{12}a_{13}; q) \psi_{k_{12}+k_{13}+k_{23}-l}(a_{13}a_{23} q^{k_{12}-l}; q),
\end{aligned}
$$

which completes the proof of the theorem. □

When $k_{12} = k_{13} = k_{23} = 2m$, Theorem 5.2 becomes

$$
\begin{aligned}
S((2m), \boldsymbol{a}, \Delta(A_2); q) = &\sum_{i=0}^{2m} \binom{4m-i-1}{2m-1} \psi_i(a_{12}a_{13}; q)\\
&\times (\psi_{6m-i}(a_{12}^{-1} a_{23} q^{2m}; q) + \psi_{6m-i}(a_{13}^{-1} a_{23}^{-1} q^{4m}; q)). \qquad (5.2)
\end{aligned}
$$

In particular, by letting $a_{12}, a_{13}, a_{23} \to q^m$ in (5.2), we obtain

$$
\begin{aligned}
6\zeta_2((2m), (q^m), \Delta(A_2); q) = &\sum_{i=0}^{2m} \binom{4m-i-1}{2m-1} \Bigg(2\psi_{i,2m}(q^{2m}; q) \psi_{6m-i}(q^{2m}; q)\\
&+ \binom{2m-1}{i-1} q^m \psi'_{6m-i}(q^{5m-i}; q) \Bigg),
\end{aligned}
$$

where we put

$$
\psi_{k,l}(a; q) := \psi_k(a; q) + (-1)^{k+1} \binom{l-1}{k-1} \frac{q^l a^{-1}}{1 - q^l a^{-1}}
$$

for $l \in \mathbb{Z}_{\geq 1}$.

Let us consider what is obtained by letting $q \to 1$ in (5.2). We put

$$
a_{12} = q^t e^{2\pi\sqrt{-1}x_{12}}, \quad a_{13} = q^t e^{2\pi\sqrt{-1}x_{13}}, \quad a_{23} = q^t e^{2\pi\sqrt{-1}x_{23}},
$$

where t, x_{12}, x_{13} and x_{23} satisfy the following conditions:

$$
\begin{gathered}
0 < t < 2m, \quad x_{12}, x_{13}, x_{23} \in \mathbb{R},\\
x_{12} + x_{13}, x_{12} - x_{23}, x_{13} + x_{23} \notin \mathbb{Z}.
\end{gathered}
$$

We now multiply both sides of (5.2) by $(1-q)^{6m}$ and take the limit as $q \to 1$. Then, by Proposition 4.2, we have

$$\sum_{w \in W} \zeta_2((2m), (x_\alpha)_{\alpha \in \Delta_+}, \Delta(A_2))$$
$$= (2\pi\sqrt{-1})^{6m} \sum_{i=0}^{2m} \binom{4m-i-1}{2m-1} \frac{B_i(\{x_{12}+x_{13}\})}{i!}$$
$$\times \left(\frac{B_{6m-i}(\{x_{23}-x_{12}\})}{(6m-i)!} + \frac{B_{6m-i}(\{-x_{13}-x_{23}\})}{(6m-i)!} \right), \tag{5.3}$$

where we put

$$\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta) := \sum_{\lambda \in P_+} \prod_{\alpha \in \Delta_+} \frac{a_\alpha^{\langle \alpha^\vee, \lambda+\rho \rangle}}{\langle \alpha^\vee, \lambda+\rho \rangle^{s_\alpha}}.$$

When $a_\alpha = 1$ for all $\alpha \in \Delta_+$, $\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta)$ is equal to the zeta function of the root system $\zeta_r(\boldsymbol{s}, \Delta)$. By letting $x_{12}, x_{13}, x_{23} \to 0$ in (5.3), we obtain the following result discovered independently by Zagier, Garoufalidis and Weinstein (see [**14**]):

$$6\zeta_2((2m), \Delta(A_2)) = 8 \sum_{\substack{i=0 \\ i:\text{even}}}^{2m} \binom{4m-i-1}{2m-1} \zeta(i)\zeta(6m-i).$$

This result is an explicit expression for Witten's volume formula (3.1) of A_2 type.

Finally, we consider the case where $\Delta = \Delta(A_3)$. Proposition 4.4 yields the following theorem.

THEOREM 5.3. *We have*

$$S((2m), \boldsymbol{a}, \Delta(A_3); q) = \sum_{i=0}^{2m} \binom{4m-i-1}{2m-1} (A(\boldsymbol{a}; q) + B(\boldsymbol{a}; q) + C(\boldsymbol{a}; q) + D(\boldsymbol{a}; q)),$$

where $A(\boldsymbol{a}; q)$, $B(\boldsymbol{a}; q)$, $C(\boldsymbol{a}; q)$ *and* $D(\boldsymbol{a}; q)$ *are given by the following:*

$$A(\boldsymbol{a}; q) := \sum_{\substack{0 \le j \le 2m \\ 0 \le t \le 4m+i-j}} \binom{2m+i-j-1}{i-1} \binom{6m+i-j-t-1}{2m-1}$$
$$\times \psi_j(a_{12}a_{13}a_{14}; q) \psi_t(a_{12}^{-1}a_{13}^{-1}a_{24}a_{34}q^i; q)$$
$$\times (\psi_{12m-j-t}(a_{12}a_{23}^{-1}a_{24}^{-1}q^{6m-i}; q) + \psi_{12m-j-t}(a_{13}a_{23}a_{34}^{-1}q^{4m-i}; q)),$$

$$B(\boldsymbol{a}; q) := \sum_{\substack{0 \le j \le 2m \\ 0 \le u \le 2m}} \binom{2m+i-j-1}{i-1} \binom{6m+i-j-u-1}{4m+i-j-1}$$
$$\times \psi_j(a_{12}a_{13}a_{14}; q) \psi_u(a_{12}a_{13}a_{24}^{-1}a_{34}^{-1}q^{u-i}; q)$$
$$\times (\psi_{12m-j-u}(a_{13}^{-1}a_{23}^{-1}a_{34}q^{6m-u}; q) + \psi_{12m-j-u}(a_{12}^{-1}a_{23}a_{24}q^{4m-u}; q)),$$

$$C(\boldsymbol{a};q) := \sum_{\substack{0\le k\le i\\ 0\le v\le 4m+i-k}} \binom{2m+i-k-1}{i-k}\binom{6m+i-k-v-1}{2m-1}$$
$$\times \psi_k(a_{12}a_{13}a_{14};q)\psi_v(a_{14}^{-1}a_{24}^{-1}a_{34}^{-1}q^{k+v-i};q)$$
$$\times (\psi_{12m-k-v}(a_{12}^{-1}a_{23}a_{24}q^{6m+i-k-v};q) + \psi_{12m-k-v}(a_{13}^{-1}a_{23}^{-1}a_{34}q^{8m+i-k-v};q)),$$
$$D(\boldsymbol{a};q) := \sum_{\substack{0\le k\le i\\ 0\le w\le 2m}} \binom{2m+i-k-1}{i-k}\binom{6m+i-k-w-1}{4m+i-k-1}$$
$$\times \psi_k(a_{12}a_{13}a_{14};q)\psi_w(q^{i-k}a_{14}a_{24}a_{34};q)$$
$$\times (\psi_{12m-k-w}(a_{12}^{-1}a_{14}^{-1}a_{23}a_{34}^{-1}q^{6m};q) + \psi_{12m-k-w}(a_{13}a_{14}a_{23}a_{24}q^{4m-k-w};q)).$$

Proof. Theorem 3.3 implies that

$$S((2m),\boldsymbol{a},\Delta(A_3);q) = \int_{\mathbb{T}^2} S((2m),(a_{12}z_{14}^{-1},a_{13},a_{23}z_{14}^{-1}z_{24}^{-1}),\Delta(A_2);q)$$
$$\times \psi_{2m}(a_{14}z_{14};q)\psi_{2m}(a_{24}z_{24};q)\psi_{2m}(a_{34}z_{14}^{-1}z_{24}^{-1};q)\,\frac{dz_{14}\,dz_{24}}{z_{14}z_{24}}.$$

By Theorem 5.2, we have

$$S((2m),\boldsymbol{a},\Delta(A_3);q)$$
$$= \sum_{i=0}^{2m}\binom{4m-i-1}{2m-1}\frac{1}{(2\pi\sqrt{-1})^2}\int_{\mathbb{T}^2}\psi_i(a_{12}a_{13}z_{14}^{-1};q)$$
$$\times (\psi_{6m-i}(a_{12}^{-1}a_{23}z_{24}^{-1}q^{2m};q) + (-1)^i\psi_{6m-i}(a_{13}a_{23}z_{14}^{-1}z_{24}^{-1}q^{2m-i};q))$$
$$\times \psi_{2m}(a_{14}z_{14};q)\psi_{2m}(a_{24}z_{24};q)\psi_{2m}(a_{34}z_{14}^{-1}z_{24}^{-1};q)\,\frac{dz_{14}\,dz_{24}}{z_{14}z_{24}}$$
$$= \sum_{i=0}^{2m}\binom{4m-i-1}{2m-1}(I_1+(-1)^i I_2),$$

where we put

$$I_1 := \frac{1}{(2\pi\sqrt{-1})^2}\int_{\mathbb{T}^2}\psi_i(a_{12}a_{13}z_{14}^{-1};q)\psi_{6m-i}(a_{12}^{-1}a_{23}z_{24}^{-1}q^{2m};q)$$
$$\times \psi_{2m}(a_{14}z_{14};q)\psi_{2m}(a_{24}z_{24};q)\psi_{2m}(a_{34}z_{14}^{-1}z_{24}^{-1};q)\,\frac{dz_{14}\,dz_{24}}{z_{14}z_{24}},$$
$$I_2 := \frac{1}{(2\pi\sqrt{-1})^2}\int_{\mathbb{T}^2}\psi_i(a_{12}a_{13}z_{14}^{-1};q)\psi_{6m-i}(a_{13}a_{23}z_{14}^{-1}z_{24}^{-1};q)$$
$$\times \psi_{2m}(a_{14}z_{14};q)\psi_{2m}(a_{24}z_{24};q)\psi_{2m}(a_{34}z_{14}^{-1}z_{24}^{-1};q)\,\frac{dz_{14}\,dz_{24}}{z_{14}z_{24}}.$$

Let us calculate the integral I_1 by using Proposition 4.4 repeatedly. Since

$$\begin{aligned}
&\psi_i(a_{12}a_{13}z_{14}^{-1};q)\psi_{2m}(a_{14}z_{14};q)\\
&\quad=\sum_{k=0}^{i}\binom{i+2m-k-1}{2m-1}(-1)^{i-k}\psi_k(a_{12}a_{13}a_{14};q)\psi_{i+2m-k}(q^{i-k}a_{14}z_{14};q)\\
&\qquad+\sum_{l=0}^{2m}\binom{i+2m-l-1}{i-1}(-1)^{l}\psi_l(a_{12}a_{13}a_{14};q)\psi_{i+2m-l}(q^{2m-l}a_{12}a_{13}z_{14}^{-1};q),
\end{aligned}$$

we have

$$\begin{aligned}
I_1=&\sum_{k=0}^{i}\binom{i+2m-k-1}{2m-1}(-1)^{i-k}\psi_k(a_{12}a_{13}a_{14};q)\\
&\times\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\psi_{i+4m-k}(a_{14}a_{34}z_{24}^{-1}q^{i-k};q)\\
&\times\psi_{6m-i}(a_{12}^{-1}a_{23}z_{24}^{-1}q^{2m};q)\psi_{2m}(a_{24}z_{24};q)\,\frac{dz_{24}}{z_{24}}\\
&+\sum_{l=0}^{2m}\binom{i+2m-l-1}{i-1}(-1)^{l}\psi_l(a_{12}a_{13}a_{14};q)\\
&\times\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\psi_{i+4m-l}(a_{12}a_{13}a_{34}^{-1}z_{24}q^{4m-l};q)\\
&\times\psi_{6m-i}(a_{12}^{-1}a_{23}z_{24}^{-1}q^{2m};q)\psi_{2m}(a_{24}z_{24};q)\,\frac{dz_{24}}{z_{24}}.
\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}
&\psi_{i+4m-k}(a_{14}a_{34}z_{24}^{-1}q^{i-k};q)\psi_{2m}(a_{24}z_{24};q)\\
&\quad=\sum_{t=0}^{i+4m-k}\binom{6m+i-k-u-1}{2m-1}(-1)^{i+4m-k-t}\psi_t(a_{14}a_{24}a_{34}q^{i-k};q)\\
&\qquad\times\psi_{6m+i-k-t}(a_{24}z_{24}q^{i+4m-k-t};q)\\
&\qquad+\sum_{u=0}^{2m}\binom{6m+i-k-u-1}{4m+i-k-1}(-1)^{2m-u}\psi_u(a_{14}a_{24}a_{34}q^{i-k};q)\\
&\qquad\times\psi_{6m+i-k-u}(a_{14}a_{34}q^{i+2m-k-u};q),
\end{aligned}$$

and

$$\begin{aligned}
&\psi_{i+4m-l}(a_{12}a_{13}a_{34}^{-1}z_{24}q^{4m-l};q)\psi_{2m}(a_{24}z_{24};q)\\
&\quad=\sum_{v=0}^{4m-l+i}\binom{6m+i-l-v-1}{2m-1}(-1)^{v}\psi_v(a_{12}^{-1}a_{13}^{-1}a_{24}a_{34}q^{i};q)\\
&\qquad\times\psi_{6m-l+i-v}(a_{24}z_{24}q^{4m-l+i-v};q)
\end{aligned}$$

$$+\sum_{w=0}^{2m}\binom{6m+i-l-w-1}{4m+i-l-1}(-1)^{w+i-l}\psi_w(a_{12}^{-1}a_{13}^{-1}a_{24}a_{34}q^i;q)$$
$$\times\psi_{6m+i-l-w}(a_{12}^{-1}a_{13}^{-1}a_{34}z_{24}^{-1}q^{2m-w+i};q).$$

Thus we find that

$$I_1=\sum_{i=0}^{2m}\binom{4m-i-1}{2m-1}(A_0(\boldsymbol{a};q)+B_0(\boldsymbol{a};q)+C_0(\boldsymbol{a};q)+D_0(\boldsymbol{a};q)),$$

where we put

$$A_0(\boldsymbol{a};q):=\sum_{\substack{0\le j\le 2m\\0\le t\le 4m+i-j}}\binom{2m+i-j-1}{i-1}\binom{6m+i-j-t-1}{2m-1}$$
$$\times\psi_j(a_{12}a_{13}a_{14};q)\psi_t(a_{12}^{-1}a_{13}^{-1}a_{24}a_{34}q^i;q)\psi_{12m-j-t}(a_{12}a_{23}^{-1}a_{24}^{-1}q^{6m-i};q),$$
$$B_0(\boldsymbol{a};q):=\sum_{\substack{0\le j\le 2m\\0\le u\le 2m}}\binom{2m+i-j-1}{i-1}\binom{6m+i-j-u-1}{4m+i-j-1}$$
$$\times\psi_j(a_{12}a_{13}a_{14};q)\psi_u(a_{12}a_{13}a_{24}^{-1}a_{34}^{-1}q^{u-i};q)$$
$$\times\psi_{12m-j-u}(a_{13}^{-1}a_{23}^{-1}a_{34}q^{6m-u};q),$$
$$C_0(\boldsymbol{a};q):=\sum_{\substack{0\le k\le i\\0\le v\le 4m+i-k}}\binom{2m+i-k-1}{i-k}\binom{6m+i-k-v-1}{2m-1}$$
$$\times\psi_k(a_{12}a_{13}a_{14};q)\psi_v(a_{14}^{-1}a_{24}^{-1}a_{34}^{-1}q^{k+v-i};q)$$
$$\times\psi_{12m-k-v}(a_{12}^{-1}a_{23}a_{24}q^{6m+i-k-v};q),$$
$$D_0(\boldsymbol{a};q):=\sum_{\substack{0\le k\le i\\0\le w\le 2m}}\binom{2m+i-k1}{i-k}\binom{6m+i-k-w-1}{4m+i-k-1}$$
$$\times\psi_k(a_{12}a_{13}a_{14};q)\psi_w(a_{14}a_{24}a_{34}q^{i-k};q)\psi_{12m-k-w}(a_{12}^{-1}a_{14}^{-1}a_{23}a_{34}^{-1}q^{6m};q).$$

Since the integral I_2 can be calculated similarly, we finish the proof the theorem. □

We now put $a_{ij}=q^te^{2\pi\sqrt{-1}x_{ij}}$ $(0<t<2m,\ x_{ij}\in\mathbb{R})$ in Theorem 5.3. By setting x_{ij} appropriately, multiplying both sides by $(1-q)^{12m}$ and letting $q\to 1$, we obtain the following:

$$\sum_{w\in W}\zeta_3((2m),(x_\alpha)_{\alpha\in\Delta_+},\Delta(A_2))$$
$$=-(2\pi\sqrt{-1})^{12m}(A((x_\alpha)_{\alpha\in\Delta_+})+B((x_\alpha)_{\alpha\in\Delta_+})+C((x_\alpha)_{\alpha\in\Delta_+})+D((x_\alpha)_{\alpha\in\Delta_+})),\quad(5.4)$$

where we put

$$A((x_\alpha)_{\alpha\in\Delta_+}) := \sum_{\substack{0\le j\le 2m\\ 0\le t\le 4m+i-j}} \binom{2m+i-j-1}{i-1}\binom{6m+i-j-t-1}{2m-1} \times \frac{B_j(\{x_{12}+x_{13}+x_{14}\})}{j!}\frac{B_t(\{-x_{12}-x_{13}+x_{24}+x_{34}\})}{t!} \times \frac{B_{12m-j-t}(\{x_{12}-x_{23}-x_{24}\})}{(12mj-t)!},$$

$$B_0((x_\alpha)_{\alpha\in\Delta_+}) := \sum_{\substack{0\le j\le 2m\\ 0\le u\le 2m}} \binom{2m+i-j-1}{i-1}\binom{6m+i-j-u-1}{4m+i-j-1} \times \frac{B_j(\{x_{12}+x_{13}+x_{14}\})}{j!}\frac{B_u(\{x_{12}+x_{13}-x_{24}-x_{34}\})}{u!} \times \frac{B_{12m-j-u}(\{-x_{13}-x_{23}+x_{34}\})}{(12m-j-u)!},$$

$$C_0((x_\alpha)_{\alpha\in\Delta_+}) := \sum_{\substack{0\le k\le i\\ 0\le v\le 4m+i-k}} \binom{2m+i-k-1}{i-k}\binom{6m+i-k-v-1}{2m-1}, \times \frac{B_k(\{x_{12}+x_{13}+x_{14}\})}{k!}\frac{B_v(\{-x_{14}-x_{24}-x_{34}\})}{v!} \times \frac{B_{12m-k-v}(\{-x_{12}+x_{23}+x_{24}\})}{(12m-k-v)!},$$

$$D_0((x_\alpha)_{\alpha\in\Delta_+}) := \sum_{\substack{0\le k\le i\\ 0\le w\le 2m}} \binom{2m+i-k1}{i-k}\binom{6m+i-k-w-1}{4m+i-k-1} \times \frac{B_k(\{x_{12}+x_{13}+x_{14}\})}{k!}\frac{B_w(\{x_{12}+x_{13}+x_{14}\})}{w!} \times \frac{B_{12m-k-w}(\{-x_{12}-x_{14}+x_{23}-x_{34}\})}{(12m-k-w)!}.$$

By letting $x_{ij}\to 0$ in (5.4), we obtain the following result due to Gunnells and Sczech [**5**, Proposition 8.5], which can be regarded as an explicit expression for Witten's volume formula (3.1) of A_3 type:

$$24\zeta_3((2m),\,\Delta(A_3)) = 16\sum_{i=0}^{2m}\binom{4m-i-1}{2m-1}(A+B+C+D),$$

where we put

$$A := \sum_{\substack{0\le j\le 2m\\ 0\le t\le 4m+i-j\\ j,t\equiv 0 \bmod 2}} \binom{2m+i-j-1}{i-1}\binom{6m+i-j-t-1}{2m-1}\zeta(j)\zeta(t)\zeta(12m-j-t),$$

$$B := \sum_{\substack{0\le j\le 2m\\ 0\le u\le 2m\\ j,u\equiv 0 \bmod 2}} \binom{2m+i-j-1}{i-1}\binom{6m+i-j-u-1}{4m+i-j-1}\zeta(j)\zeta(u)\zeta(12m-j-u),$$

$$C := \sum_{\substack{0\le k\le i\\ 0\le v\le 4m+i-k\\ k,v\equiv 0 \bmod 2}} \binom{2m+i-k-1}{i-k}\binom{6m+i-k-v-1}{2m-1}\zeta(k)\zeta(v)\zeta(12m-k-v),$$

$$D := \sum_{\substack{0\le k\le i\\ 0\le w\le 2m\\ k,w\equiv 0 \bmod 2}} \binom{2m+i-k-1}{i-k}\binom{6m+i-k-w-1}{4m+i-k-1}\zeta(k)\zeta(w)\zeta(12m-k-w).$$

6. Two-parameter deformations of zeta functions of root systems

In this section, we establish a p-deformation of the q-analogue of the zeta function of a root system (thus considered to be a two-parameter deformation of the zeta function of a root system) and investigate its basic properties. We start with the following integral representation of the zeta function (2.1).

PROPOSITION 6.1. *Put* $c(a) := a/(1-a)$. *Then, when* $\mathrm{Re}\, s_\alpha > 0$, $|a_\alpha| < 1$ *for all* $\alpha \in \Delta_+$, *we have*

$$\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta; q) = \frac{1}{(2\pi\sqrt{-1})^{|\Delta_+|}} \int_{\mathbb{T}^{|\Delta_+|}} \prod_{i=1}^{r} c\Bigg(\prod_{\alpha\in\Delta_+} (a_\alpha t_\alpha z_\alpha)^{\langle\alpha^\vee,\lambda_i\rangle}\Bigg)$$

$$\times \prod_{\alpha\in\Delta_+} \psi_{s_\alpha}(t_\alpha^{-1} z_\alpha^{-1}; q) \prod_{\alpha\in\Delta_+} \frac{dz_\alpha}{z_\alpha}, \tag{6.1}$$

where $t_\alpha (\alpha \in \Delta_+)$ *are complex numbers satisfying* $1 < |t_\alpha| < |a_\alpha^{-1}|$.

This proposition follows immediately from the series expression of $c(a)$, given by

$$c(a) = \sum_{n=1}^{\infty} a^n \quad (|a| < 1).$$

By substituting the Kronecker function $F(\alpha, a; p)$ for the rational function $c(a)$ in (6.1), we define the two-parameter deformation of the zeta function of the root system Δ, as follows.

Definition 6.2. Let p be a complex number satisfying $0 < |p| < 1$ and assume that $\mathrm{Re}\, s_\alpha > 0$, $|q^{s_\alpha}| < |a_\alpha| < 1$ for all $\alpha \in \Delta_+$. We put $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_r)$. We define the two-parameter deformation of the zeta function of Δ by

$$\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\beta}, \Delta; p, q) := \frac{1}{(2\pi\sqrt{-1})^{|\Delta_+|}} \int_{\mathbb{T}^{|\Delta_+|}} \prod_{i=1}^{r} F\Bigg(\beta_i, \prod_{\alpha\in\Delta_+} (a_\alpha t_\alpha z_\alpha)^{\langle\alpha^\vee,\lambda_i\rangle}; p\Bigg)$$

$$\times \prod_{\alpha\in\Delta_+} \psi_{s_\alpha}(t_\alpha^{-1} z_\alpha^{-1}; q) \prod_{\alpha\in\Delta_+} \frac{dz_\alpha}{z_\alpha},$$

where t_α are complex numbers satisfying the following:

$$\max\{|p^{1/|\Delta_+|} a_\alpha^{-1}|, 1\} < |t_\alpha| < |a_\alpha^{-1}|.$$

PROPOSITION 6.3. *We have*

(1) $\lim_{p\to 0} \zeta_r(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\beta}, \Delta;\, p, q) = (-1)^r \zeta_r(\boldsymbol{s}, \boldsymbol{a}, \Delta;\, q)$.

(2) *The function* $\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\beta}, \Delta;\, p, q)$ *has the following series representation:*

$$\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\beta}, \Delta;\, p, q) = \sum_{\lambda\in P\setminus H_\Delta} \prod_{i=1}^{r} \frac{1}{p^{\langle\alpha_i^\vee,\lambda\rangle}\beta_i - 1} \prod_{\alpha\in\Delta_+} \frac{a_\alpha^{\langle\alpha^\vee,\lambda\rangle}}{(1-q^{\langle\alpha^\vee,\lambda\rangle})^{s_\alpha}}.$$

(3) *Let* $|p| < |\beta_i| < 1$ $(i = 1, \ldots, r)$. *We also denote* β_i *by* $\beta_\alpha (\alpha = \alpha_i)$. *Then, by the following expression, the function* $\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\beta}, \Delta;\, p, q)$ *becomes a meromorphic function of* $\boldsymbol{s}$ *and* $\boldsymbol{a}$*:*

$$\begin{aligned}
&\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\beta}, \Delta;\, p, q)\\
&\quad= \sum_{w\in W} (-1)^{|w^{-1}\Psi\cap\Delta_+|} \Bigg(\prod_{\alpha\in w^{-1}\Delta_+\cap\Delta_-} (-1)^{-s_{w\alpha}}\Bigg)\\
&\quad\times \sum_{\substack{r_\alpha=0\\ \alpha\in w^{-1}\Psi\cap\Delta_+}}^{\infty} \sum_{\substack{r_\alpha=1\\ \alpha\in w^{-1}\Psi\cap\Delta_-}}^{\infty} \sum_{\substack{t_\alpha=0\\ \alpha\in w^{-1}\Delta_+\cap\Delta_+}}^{\infty} \sum_{\substack{t_\alpha=0\\ \alpha\in w^{-1}\Delta_+\cap\Delta_-}}^{\infty}\\
&\quad\times \prod_{\alpha\in w^{-1}\Delta_+} \binom{s_{w\alpha}+t_\alpha-1}{t_\alpha} \prod_{\alpha\in w^{-1}\Psi\cap\Delta_+} \beta_{w\alpha}^{r_\alpha} \prod_{\alpha\in w^{-1}\Psi\cap\Delta_-} \beta_{w\alpha}^{-r_\alpha}\\
&\quad\times \prod_{i=1}^{r} \frac{\prod_{\alpha\in w^{-1}\Psi\cap\Delta_\pm} p^{\langle\pm r_\alpha\alpha^\vee,\lambda_i\rangle} \prod_{\alpha\in w^{-1}\Delta_+\cap\Delta_\pm} q^{\langle\pm t_\alpha\alpha^\vee,\lambda_i\rangle} \prod_{\alpha\in\Delta_+} a_\alpha^{\langle\alpha^\vee,\lambda_i\rangle}}{1-\prod_{\alpha\in w^{-1}\Psi\cap\Delta_\pm} p^{\langle\pm r_\alpha\alpha^\vee,\lambda_i\rangle} \prod_{\alpha\in w^{-1}\Delta_+\cap\Delta_\pm} q^{\langle\pm t_\alpha\alpha^\vee,\lambda_i\rangle} \prod_{\alpha\in\Delta_+} a_\alpha^{\langle\alpha^\vee,\lambda_i\rangle}},
\end{aligned}$$

where we put

$$\prod_{\alpha\in w^{-1}\Psi\cap\Delta_\pm} p^{\langle\pm r_\alpha\alpha^\vee,\lambda_i\rangle} := \Bigg(\prod_{\alpha\in w^{-1}\Psi\cap\Delta_+} p^{\langle r_\alpha\alpha^\vee,\lambda_i\rangle}\Bigg)\Bigg(\prod_{\alpha\in w^{-1}\Psi\cap\Delta_-} p^{\langle -r_\alpha\alpha^\vee,\lambda_i\rangle}\Bigg).$$

Proof. The claim (1) follows from the fact that $F(\alpha, a;\, p) \to -c(a) + 1/(\alpha-1)$ as $p \to 0$. The claim (2) is an immediate consequence of the following Laurent series expansion of the Kronecker function $F(\alpha, a;\, p)$ (see [**12**]):

$$F(\alpha, a;\, p) = \sum_{n\in\mathbb{Z}} \frac{a^n}{1-p^n} \quad (|p| < |a| < 1).$$

Let us prove the claim (3). Since there exist unique $w \in W$ and $\lambda' \in P_+$ satisfying $\lambda = w(\lambda' + \rho)$ for all $\lambda \in P \setminus H_\Delta$, claim (2) implies that

$$\begin{aligned}
\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\beta}, \Delta;\, p, q) &= \sum_{w\in W} \sum_{\lambda'\in P_+} \prod_{\alpha\in\Psi} \frac{1}{p^{\langle\alpha^\vee,w(\lambda+\rho)\rangle}\beta_\alpha - 1} \prod_{\alpha\in\Delta_+} \frac{a_\alpha^{\langle\alpha^\vee,w(\lambda+\rho)\rangle}}{(1-q^{\langle\alpha^\vee,w(\lambda'+\rho)\rangle})^{s_\alpha}}\\
&= \sum_{w\in W} \sum_{\lambda'\in P_+} \prod_{\alpha\in w^{-1}\Psi} \frac{1}{p^{\langle\alpha^\vee,\lambda+\rho\rangle}\beta_{w\alpha} - 1} \prod_{\alpha\in w^{-1}\Delta_+} \frac{a_{w\alpha}^{\langle\alpha^\vee,\lambda'+\rho\rangle}}{(1-q^{\langle\alpha^\vee,\lambda'+\rho\rangle})^{s_{w\alpha}}}.
\end{aligned}$$

By decomposing the products $\prod_{\alpha\in w^{-1}\Psi}$ and $\prod_{\alpha\in w^{-1}\Delta_+}$ into

$$\prod_{\alpha\in w^{-1}\Psi} = \prod_{\alpha\in w^{-1}\Psi\cap\Delta_+}\ \prod_{\alpha\in w^{-1}\Psi\cap\Delta_-},$$

$$\prod_{\alpha\in w^{-1}\Delta_+} = \prod_{\alpha\in w^{-1}\Delta_+\cap\Delta_+}\ \prod_{\alpha\in w^{-1}\Delta_+\cap\Delta_-}$$

and using the binomial expansion, we obtain the claim. □

Example 6.4. When $\Delta = \Delta(A_1)$, by Proposition 6.3(2), $\zeta_1(s, a, 1, \Delta(A_1); p, q)$ can be expressed as follows:

$$\zeta_1(1, a, 1, \Delta(A_1); p, q) = -\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{a^n}{(1-p^n)(1-q^n)}.$$

Thus the function $\zeta_1(1, qe^{2\pi\sqrt{-1}x}, 1, \Delta(A_1); p, q)$ has the following Taylor series expansion around $x = 0$:

$$\zeta_1(1, qe^{2\pi\sqrt{-1}x}, 1, \Delta(A_1); p, q) = -\sum_{k=1}^{\infty} \frac{(2\pi\sqrt{-1})^{k-1}}{(k-1)!} Z_k(p, q)x^{k-1},$$

where we put

$$Z_k(p, q) := \sum_{n=1}^{\infty} n^{k-1} \frac{q^n - (-1)^k p^n}{(1-p^n)(1-q^n)}$$

for $k \in \mathbb{Z}_{>0}$. The numbers $Z_k(p, q)$ are essentially the same as the elliptic zeta values, introduced by Felder and Varchenko [**4**].

Let us consider an analogy of Theorem 3.3 for $\zeta_r(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\beta}, \Delta; p, q)$. We define the p-deformation of $S(\boldsymbol{s}, \boldsymbol{a}, \Delta; q)$ by

$$S(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\beta}, \Delta; p, q) := \sum_{w\in W}\Bigg(\prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{-s_\alpha}\Bigg)\zeta_r(w^{-1}\boldsymbol{s}, w^{-1}\boldsymbol{a}, \boldsymbol{\beta}, \Delta; p, q).$$

By a similar argument used to prove Theorem 3.3, we obtain

$$S(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\beta}, \Delta; p, q) = \sum_{\lambda\in P\setminus H_\Delta} s(\lambda, \boldsymbol{\beta}, \Delta) \prod_{\alpha\in\Delta_+} \frac{a_\alpha^{\langle\alpha^\vee,\lambda\rangle}}{(1-q^{\langle\alpha^\vee,\lambda\rangle})^{s_\alpha}},$$

where we put

$$s(\lambda, \boldsymbol{\beta}, \Delta) := \sum_{w\in W}\prod_{i=1}^{r} \frac{1}{p^{\langle\alpha_i^\vee, w\lambda\rangle}\beta_i - 1}.$$

When $\Delta = \Delta(A_1)$, $\Delta(A_2)$, $\Delta(A_3)$, $s(\lambda, (1, \ldots, 1), \Delta)$ can be calculated, as follows.

THEOREM 6.5. *We have*

(1) *When $\Delta = \Delta(A_1)$, for $\lambda \in P \setminus H_\Delta$, we have*

$$s(\lambda, 1, \Delta(A_1)) = -1.$$

Thus it holds that

$$\lim_{\beta_1\to 1} S(s, a_{12}, \beta_1, \Delta(A_1); p, q) = -S(s, a_{12}, \Delta(A_1); q).$$

(2) *When $\Delta = \Delta(A_2)$, for $\lambda \in P \setminus H_\Delta$, we have*

$$s(\lambda, (1, 1), \Delta(A_2)) = 1.$$

Thus it holds that

$$\lim_{\beta 1, \beta_2 \to 1} S(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\beta}, \Delta(A_2); p, q) = S(\boldsymbol{s}, \boldsymbol{a}, \Delta(A_2); q).$$

(3) *When $\Delta = \Delta(A_3)$, for $\lambda \in P \setminus H_\Delta$, we have*

$$s(\lambda, (1, 1, 1), \Delta(A_3)) = -1.$$

Thus it holds that

$$\lim_{\beta_1, \beta_2, \beta_3 \to 1} S(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\beta}, \Delta(A_3); p, q) = -S(\boldsymbol{s}, \boldsymbol{a}, \Delta(A_2); q).$$

Proof. When $\Delta = \Delta(A_r)$, the Weyl group W becomes the symmetric group S_{r+1} of degree $r + 1$. The group $W = S_{r+1}$ acts on the space

$$V = \left\{ \sum_{i=1}^{r+1} x_i \boldsymbol{e}_i \;\middle|\; \sum_{i=1}^{r+1} x_i = 0 \right\}$$

by permutations of indices of the vectors $\boldsymbol{e}_i$. Thus, when $\Delta = \Delta(A_1)$, we have

$$s(\lambda, 1, \Delta(A_1)) = \frac{1}{p^{n_1} - 1} + \frac{1}{p^{-n_1} - 1} = -1$$

for $\lambda = n_1 \lambda_1$. Similarly, we have

$$\begin{aligned} s(\lambda, (1, 1), \Delta(A_2)) &= \frac{1}{(p^{n_1} - 1)(p^{n_2} - 1)} + \frac{1}{(p^{-n_1} - 1)(p^{n_1+n_2} - 1)} \\ &\quad + \frac{1}{(p^{-n_2} - 1)(p^{-n_1} - 1)} \\ &\quad + \frac{1}{(p^{n_1+n_2} - 1)(p^{-n_2} - 1)} + \frac{1}{(p^{-n_1-n_2} - 1)(p^{n_1} - 1)} \\ &\quad + \frac{1}{(p^{n_2} - 1)(p^{-n_1-n_2} - 1)} \\ &= 1 \end{aligned}$$

for $\lambda = n_1 \lambda_1 + n_2 \lambda_2$ and

$$\begin{aligned} &s(\lambda, (1, 1, 1), \Delta(A_3)) \\ &= \frac{1}{(p^{n_1} - 1)(p^{n_2} - 1)(p^{n_3} - 1)} + \frac{1}{(p^{-n_1} - 1)(p^{n_1+n_2} - 1)(p^{n_3} - 1)} \\ &\quad + \frac{1}{(p^{-n_2} - 1)(p^{-n_1} - 1)(p^{n_1+n_2+n_3} - 1)} + \frac{1}{(p^{-n_2-n_3} - 1)(p^{n_2} - 1)(p^{-n_1-n_2} - 1)} \\ &\quad + \frac{1}{(p^{n_1+n_2} - 1)(p^{-n_2} - 1)(p^{n_2+n_3} - 1)} + \frac{1}{(p^{n_1+n_2+n_3} - 1)(p^{-n_3} - 1)(p^{-n_2} - 1)} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(p^{n_1}-1)(p^{n_2+n_3}-1)(p^{-n_3}-1)}+\frac{1}{(p^{n_2}-1)(p^{-n_1-n_2}-1)(p^{n_1+n_2+n_3}-1)}\\
&+\frac{1}{(p^{-n_1-n_2}-1)(p^{n_1}-1)(p^{n_2+n_3}-1)}+\frac{1}{(p^{n_2+n_3}-1)(p^{-n_3}-1)(p^{-n_1-n_2}-1)}\\
&+\frac{1}{(p^{-n_1-n_2-n_3}-1)(p^{n_1+n_2}-1)(p^{-n_2}-1)}\\
&+\frac{1}{(p^{-n_2}-1)(p^{n_2+n_3}-1)(p^{-n_1-n_2-n_3}-1)}\\
&+\frac{1}{(p^{-n_2-n_3}-1)(p^{-n_1}-1)(p^{n_1+n_2}-1)}+\frac{1}{(p^{n_1+n_2}-1)(p^{n_3}-1)(p^{-n_2-n_3}-1)}\\
&+\frac{1}{(p^{n_1+n_2+n_3}-1)(p^{-n_2-n_3}-1)(p^{n_2}-1)}+\frac{1}{(p^{n_2}-1)(p^{n_3}-1)(p^{-n_1-n_2-n_3}-1)}\\
&+\frac{1}{(p^{n_2+n_3}-1)(p^{-n_1-n_2-n_3}-1)(p^{n_1+n_2}-1)}+\frac{1}{(p^{n_3}-1)(p^{-n_2-n_3}-1)(p^{-n_1}-1)}\\
&+\frac{1}{(p^{-n_1-n_2}-1)(p^{n_1+n_2+n_3}-1)(p^{-n_2-n_3}-1)}+\frac{1}{(p^{-n_3}-1)(p^{-n_1-n_2}-1)(p^{n_1}-1)}\\
&+\frac{1}{(p^{-n_1-n_2-n_3}-1)(p^{n_1}-1)(p^{n_2}-1)}+\frac{1}{(p^{-l}-1)(p^{n_1+n_2+n_3}-1)(p^{-n_3}-1)}\\
&+\frac{1}{(p^{n_3}-1)(p^{-n_1-n_2-n_3}-1)(p^{n_1}-1)}+\frac{1}{(p^{-n_3}-1)(p^{-n_2}-1)(p^{-n_1}-1)}\\
&=-1
\end{aligned}
$$

for $\lambda = n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3$. Thus we finish the proof of the theorem. □

Acknowledgements. The author would like to express his gratitude to Professor Yasushi Komori, Professor Yoshihiro Takeyama and Dr. Shotaro Kawata for their valuable comments and suggestions. He also would like to thank the referee for careful reading of the manuscript and constructive comments. This work is supported by JSPS KAKENHI Grant Number JP20K14289.

References

[1] N. Bourbaki. Groupes et Algebres de Lie, Chapitres 4, 5 et 6. Hermann, Paris, 1968.

[2] F. Brown and A. Levin. Multiple elliptic polylogarithms. aXiv: 1110.6917.

[3] A. Erdélyi. Higher Transcendental Functions. Vol. 1. McGraw-Hill, New York, 1953.

[4] G. Felder and A. Varchenko. Even powers of divisors and elliptic zeta values. J. Reine Angew. Math. **579** (2005), 195–201.

[5] P. E. Gunnells and R. Sczech. Evaluation of Dedekind sums, Eisenstein cocycles, and special values of L-functions. Duke Math. J. **118**(2) (2003), 229–260.

[6] J. E. Humphreys. Introduction to Lie Algebras and Representation Theory (Graduate Texts in Mathematics, 9). Springer, New York, 1972.

[7] M. Kaneko, N. Kurokawa and M. Wakayama. A variation of Euler's approach to values of the Riemann zeta function. Kyushu J. Math **57**(1) (2003), 175–192.

[8] M. Kato. On certain two-parameter deformations of multiple zeta values. Res. Number Theory **6**(3), (2020), 30, (22 pp).

[9] Y. Komori, K. Matsumoto and H. Tsumura. On Witten multiple zeta-functions associated with semisimple Lie algebras. II. J. Math. Soc. Japan **62** (2010), 355–394. Idem. III. In Multiple Dirichlet Series, L-functions and Automorphic Forms. Eds. D. Bump et al. (Progress in Mathametics, 300). Birkhäuser, Basel, 2012, pp. 223–286. Idem IV. Glasgow Math. J. **53** (2011), 185–206. Idem. V. ibid. **57** (2015), 107–130.

[10] K. Matsumoto and H. Tsumura. On Witten multiple zeta functions associated with semisimple Lie algebras I. Ann. Inst. Fourier **56** (2006), 1457–1504.

[11] H. Samelson. Notes on Lie Algebras (Universitext). Springer, Berlin, 1990.

[12] A. Weil. Elliptic Functions According to Eisenstein and Kronecker. Springer, Berlin, 1976.

[13] E. Witten. On quantum gauge theories in two demensions. Comm. Math. Phys. **141**(1) (1991), 153–209.

[14] D. Zagier. Values of zeta functions and their applications. In First European Congress of Mathematics. Vol. II. Eds. A. Joseph et al. (Progress in Mathematics, 120). Birkhäuser, Basel, 1994, pp. 497–512.

[15] J. Zhao. Multiple q-zeta functions and multiple q-polylogarithms. Ramanujan J. **14**(2) (2007), 189–221.

Masaki Kato
Department of General Education
National Institute of Technology
Toyama College
13 Hongo-machi, Toyama city
Toyama 939-8630
Japan
(E-mail: mkato@nc-toyama.ac.jp)