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RESEARCH PAPER

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On q-Calculus and Starlike Functions

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Abstract

We consider the class $\mathcal{S}^*(\zeta,\alpha)$, $0 \le \alpha < 1$, of normalized analytic functions f such that

$$\Re\mathrm{e}\left\{\frac{z\mathrm{d}_\zeta f(z)}{f(z)}\right\} > \alpha, \quad |z|<1,$$

where $\mathrm{d}_\zeta f$ is the convolution operator

$$\mathrm{d}_\zeta f(z) = \frac{1}{z}\left\{f(z) * \frac{z}{(1-\zeta z)(1-z)}\right\},$$

where ζ is complex, $|\zeta| \le 1$. For $\zeta = 1$ the operator becomes the derivative f', while for real $\zeta = q$, $0<q<1$, we obtain the Jackson q-derivative $\mathrm{d}_q f$.

Keywords Analytic functions · Convex functions · Starlike functions · q-Calculus · q-Derivative operator · Convolution operator

Mathematics Subject Classification 30C45

1 Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : \ |z|<1\}$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0)=0, f'(0)=1$, i.e.

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \tag{1}$$

and let $\mathcal{S} \subset \mathcal{A}$ be the class of functions which are univalent in $\mathbb{D}$. Moreover, we shall use the following notations:

$$J_{ST}(f;z) := \frac{zf'(z)}{f(z)}, \quad J_{CV}(f;z) := 1 + \frac{zf''(z)}{f'(z)}. \tag{2}$$

Let a function $f \in \mathcal{H}$ be univalent in $\mathbb{D}$ with the normalization $f(0)=0$. Then f maps $\mathbb{D}$ onto a starlike domain with respect to the origin, if and only if

$$\Re\mathrm{e}\{J_{ST}(f;z)\} > 0, \quad z \in \mathbb{D}, \tag{3}$$

and such function f is said to be starlike in $\mathbb{D}$ with respect to the origin (or briefly starlike). Furthermore, a function f maps $\mathbb{D}$ onto a convex domain E, if and only if

$$\Re\mathrm{e}\{J_{CV}(f;z)\} > 0, \quad z \in \mathbb{D}, \tag{4}$$

and such function f is said to be convex in $\mathbb{D}$ (or briefly convex). Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$, if and only if the linear segment joining w_0 to any other point $w \in E$ lies entirely in E, while a set E is said to be convex, if and only if it is starlike with

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respect to each of its points, that is, if and only if the linear segment joining any two points of E lies entirely in E. It is well known that if an analytic function f satisfies condition (3) and $f(0) = 0, f'(0) \neq 0$, then f is univalent and starlike in $\mathbb{D}$. By $\mathcal{S}^*$ and $\mathcal{K}$, we denote the subclasses of $\mathcal{A}$ which consist of starlike univalent functions and convex univalent functions in $\mathbb{D}$, respectively. It is known that for $f \in \mathcal{A}$ condition (4) is sufficient for starlikeness of f. The following condition

$$|J_{CV}(f;z) - 1| < 2, \quad z \in \mathbb{D}$$

is also sufficient for starlikeness of f.

Robertson introduced in Robertson (1936) the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α, $\alpha < 1$, which are defined by:

$$\begin{aligned} \mathcal{S}^*(\alpha) &:= \{f \in \mathcal{A} : \ \Re\mathfrak{e}\{J_{ST}(f;z)\} > \alpha, \ z \in \mathbb{D}\}, \\ \mathcal{K}(\alpha) &:= \{f \in \mathcal{A} : \ \Re\mathfrak{e}\{J_{CV}(f;z)\} > \alpha, \ z \in \mathbb{D}\} \\ &= \{f \in \mathcal{A} : \ zf'(z) \in \mathcal{S}^*(\alpha), \ z \in \mathbb{D}\}. \end{aligned}$$

If $0 \leq \alpha < 1$, then a function in either of these sets is univalent; if $\alpha < 0$, it may fail to be univalent. In particular, we have $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$. It is known as the old Strohhäcker result (Strrohhäcker 1933) that $\mathcal{K}(0) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0)$. Furthermore, note that if $f \in \mathcal{K}(\alpha)$, then $f \in \mathcal{S}^*(\delta(\alpha))$, see Wilken and Feng (1980), where

$$\delta(\alpha) = \begin{cases} (1-2\alpha)/(2^{2-2\alpha} - 2) & \text{for} \quad \alpha \neq 1/2, \\ 1/\log 4 & \text{for} \quad \alpha = 1/2. \end{cases}$$

Robertson (1985) proved that if $f \in \mathcal{A}$ with $f(z)/z \neq 0$, $z \in \mathbb{D}$, and if there exists k, $0 < k \leq 2$, such that

$$|J_{CV}(f;z) - 1| \leq k|J_{ST}(f;z)|, \quad z \in \mathbb{D},$$

then $f \in \mathcal{S}^*(2/(2+k))$. In Mocanu (1986), it was proved that if $f \in \mathcal{A}$ with $f(z)f'(z)/z \neq 0$ and

$$|J_{CV}(f;z)| \leq \sqrt{2}|J_{ST}(f;z) + 1|, \quad z \in \mathbb{D},$$

then $f \in \mathcal{S}^*$. Several more complicated sufficient conditions for starlikeness and convexity are collected in Chapter 5 of Miller and Mocanu (2000).

Jackson (1908, 1910) introduced and studied the q-derivative, $0 < q < 1$, as

$$d_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad z \neq 0 \tag{5}$$

and $d_q f(0) = f'(0)$. Thus, from (5) for a function f given by (1) we have

$$z d_q f(z) = z + \sum_{n=2}^{\infty} [n]_q a_n z^n, \tag{6}$$

where

$$[n]_q = \frac{1-q^n}{1-q}, \quad n = 2, 3, \ldots.$$

Let us recall also the definition of the convolution. The convolution, or the Hadamard product, of two power series

$$f_1(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad f_2(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

convergent in $\mathbb{D}$ is the function $f_3 = f_1 * f_2$ with power series

$$f_3(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

2 ζ-Derivative Operator

The function h_ζ of the form

$$h_\zeta(z) = \frac{z}{(1-\zeta z)(1-z)} = \sum_{n=1}^{\infty} \frac{1-\zeta^n}{1-\zeta} z^n, \quad z \in \mathbb{D}$$

is starlike for all complex ζ, $|\zeta| < 1$. It is easy to check that if $\zeta \to 1^-$, then the function h_ζ becomes the well-known Koebe function

$$h_1(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n, \quad z \in \mathbb{D}.$$

For each $f \in \mathcal{A}$, we can express its derivative in terms of the Koebe function as

$$f'(z) = \frac{1}{z}\{f(z) * h_1(z)\} = \frac{1}{z}\left\{f(z) * \frac{z}{(1-z)^2}\right\}, \quad z \in \mathbb{D}, \tag{7}$$

where $*$ denotes the Hadamard product, or convolution, of power series. It is natural to consider the following generalization of (7) for $\zeta \in \mathbb{C}$, $|\zeta| \leq 1$

$$d_\zeta f(z) = \frac{1}{z}\{f(z) * h_\zeta(z)\} = \frac{1}{z}\left\{f(z) * \frac{z}{(1-\zeta z)(1-z)}\right\}. \tag{8}$$

For $\zeta = 1$, convolution operator (8) becomes the derivative f', while for real $\zeta = q$, $0 < q < 1$, we obtain the Jackson q-derivative of f, namely $d_q f(z)$, which is defined in (5). Therefore, for f given by (1) we have:

$$\begin{aligned} \mathrm{d}_\zeta f(z) &= \frac{1}{z}\left\{ f(z) * \frac{z}{(1-\zeta z)(1-z)} \right\} \\ &= \frac{1}{z}\left\{ \left(z + \sum_{n=2}^{\infty} a_n z^n \right) * \left(\sum_{n=1}^{\infty} \frac{1-\zeta^n}{1-\zeta}\, z^n \right) \right\} \\ &= \frac{1}{z}\left\{ z + \sum_{n=2}^{\infty} \frac{1-\zeta^n}{1-\zeta}\, a_n z^n \right\} \\ &= \frac{1}{z}\left\{ z + \sum_{n=2}^{\infty} [n]_\zeta a_n z^n \right\}, \end{aligned} \tag{9}$$

where

$$[n]_\zeta = \frac{1-\zeta^n}{1-\zeta}, \quad n = 2, 3, \ldots.$$

For these reasons, we can also look at Jackson's q-derivative $\mathrm{d}_q f$ from (5), as a special case of convolution operator (8).

Definition 1 [11] Let $f \in \mathcal{A}$. For given ζ, $|\zeta| \le 1$, we say that f is in the class $\mathcal{S}^*(\zeta, \alpha)$ of ζ-starlike functions of order α, $0 \le \alpha < 1$, if

$$\Re\mathrm{e}\left\{ \frac{z\mathrm{d}_\zeta f(z)}{f(z)} \right\} > \alpha, \quad z \in \mathbb{D}, \tag{10}$$

where the operator d_ζ is defined in (8).

Remark 1 For $\zeta = 1$, condition (10) becomes

$$\Re\mathrm{e}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad z \in \mathbb{D}, \tag{11}$$

and the class $\mathcal{S}^*(1, \alpha)$ becomes the well-known class $\mathcal{S}^*(\alpha)$ of starlike functions of order α, while for $\zeta = 0$ we have $\mathcal{S}^*(0, \alpha) = \mathcal{A}$. For $\zeta \ne 1$, condition (10) becomes

$$\Re\mathrm{e}\left\{ \frac{f(\zeta z) - f(z)}{(\zeta - 1)z} \frac{z}{f(z)} \right\} > \alpha, \quad z \in \mathbb{D}. \tag{12}$$

Remark 2 It is known that condition (11) follows the univalence of f, whenever $f \in \mathcal{A}$. If $\zeta \ne 1$, then condition (10) does not follow that f is univalent in $\mathbb{D}$. For example, it is known that $f(z) = z + (2/3)z^2$ is not univalent in $\mathbb{D}$, while $f \in \mathcal{S}^*(1/2, 0)$ because for this function f we have

$$\Re\mathrm{e}\left\{ \frac{z\mathrm{d}_\zeta f(z)}{f(z)} \right\} = \Re\mathrm{e}\left\{ \frac{1+z}{1+(2/3)z} \right\} > 0, \quad z \in \mathbb{D}.$$

Theorem 1 *The function* $g(z) = z + az^2$ *is in the class* $\mathcal{S}^*(\zeta, \alpha)$, *if and only if*

$$\Re\mathrm{e}\left\{ \frac{1 - |a|^2(\zeta + 1) - |a||\zeta|}{1 - |a|^2} \right\} > \alpha. \tag{13}$$

Proof We have

$$\Re\mathrm{e}\left\{ \frac{z\mathrm{d}_\zeta g(z)}{g(z)} \right\} = \Re\mathrm{e}\left\{ \frac{1 + a(\zeta + 1)z}{1 + az} \right\}.$$

The function

$$z \mapsto \frac{1 + a(\zeta + 1)z}{1 + az}$$

maps $\mathbb{D}$ onto a disc at S with radius R, where

$$S = \frac{1 - |a|^2(\zeta + 1)}{1 - |a|^2}, \quad R = \frac{|a||\zeta|}{1 - |a|^2}.$$

Therefore, $g \in \mathcal{S}^*(\zeta, \alpha)$, if and only if $\Re\mathrm{e}(S - R) > \alpha$, which gives (13). □

Lemma 1 *Ruscheweyh and Sheil-Small* (1973) *If* $f \in \mathcal{S}^*(1/2)$ *and* $g \in \mathcal{S}^*(1/2)$ [*or if* $f \in \mathcal{K}$ *and* $g \in \mathcal{S}^*$], *then*

$$\frac{f(z) * g(z)F(z)}{f(z) * g(z)} \in \overline{\mathrm{co}}\{F(\mathbb{D})\}, \quad z \in \mathbb{D}, \tag{14}$$

where $F \in \mathcal{H}$ *and* $\overline{\mathrm{co}}\{F(\mathbb{D})\}$ *denotes the closed convex hull of* $F(\mathbb{D})$.

Theorem 2 *If f is in the class* $\mathcal{S}^*(1/2)$ *of starlike functions of order 1/2, then for all* ζ, $|\zeta| \le 1$, *f is in the class* $\mathcal{S}^*(\zeta, 1/(1+|\zeta|)$ *of ζ-starlike functions of order* $1/(1+|\zeta|)$.

Proof The case $|\zeta| = 1$ is trivial. Assume that $|\zeta| < 1$. Note that

$$g(z) = \frac{z}{1-z} \in \mathcal{S}^*(1/2), \quad z \in \mathbb{D}.$$

Therefore, Lemma 1 gives

$$\frac{f(z) * \frac{z}{1-z} \frac{\frac{z}{(1-\zeta z)(1-z)}}{\frac{z}{1-z}}}{f(z) * \frac{z}{1-z}} \in \overline{\mathrm{co}}\{F(\mathbb{D})\}, \tag{15}$$

where

$$F(z) = \frac{\frac{z}{(1-\zeta z)(1-z)}}{\frac{z}{1-z}} = \frac{1}{1-\zeta z}, \quad \Re\mathrm{e}\left\{ \frac{1}{1-\zeta z} \right\} > \frac{1}{1+|\zeta|}, \quad z \in \mathbb{D},$$

for all ζ, $|\zeta| \le 1$. Hence, (15) becomes

$$\Re\mathrm{e}\left\{ \frac{f(z) * \frac{z}{(1-\zeta z)(1-z)}}{f(z) * \frac{z}{1-z}} \right\} > \frac{1}{1+|\zeta|}, \quad z \in \mathbb{D},$$

or equivalently

$$\Re\mathrm{e}\left\{ \frac{z\mathrm{d}_\zeta f(z)}{f(z)} \right\} > \frac{1}{1+|\zeta|}, \quad z \in \mathbb{D},$$

in view of (10), this means that f is ζ-starlike functions of order $1/(1+|\zeta|)$. □

Since $1/2 \le 1/(1+|\zeta|)$, for all ζ, $|\zeta| \le 1$, Theorem 2 implies the following corollary.

Corollary 1 *If f is in the class* $\mathcal{S}^*(1/2)$ *of starlike functions of order* 1/2, *then for all* ζ, $|\zeta|\leq 1$, *f is in the class* $\mathcal{S}^*(\zeta,1/2)$ *of* ζ*-starlike functions of order* 1/2.

Corollary 1 provides some examples of ζ-starlike functions of order 1/2, for example

$$g\in\mathcal{S}^*(1/2)\ \Rightarrow\ g\in\mathcal{S}^*(\zeta,1/2),\qquad \text{for all } \zeta,\ |\zeta|\leq 1.$$

It is known that $\mathcal{K}\subset\mathcal{S}^*(1/2)$; therefore, Corollary 1 leads to the following result.

Corollary 2 *If f is in the class* $\mathcal{K}$ *of convex univalent functions, then for all* ζ, $|\zeta|\leq 1$, *f is in the class* $\mathcal{S}^*(\zeta,1/2)$ *of* ζ*-starlike functions of order* 1/2.

We can look for the smallest α such that for all ζ, $|\zeta|\leq 1$, we have $\mathcal{K}\subset\mathcal{S}^*(\zeta,\alpha)$. From Corollary 2, we have $0<\alpha\leq 1/2$ and it is known that for $\zeta=1$ the order of α-starlikeness in the class of convex functions is 1/2, so 1/2 is the solution of this problem. However, we may consider this problem for a given ζ.

Open problem. For given ζ, $|\zeta|\leq 1$, find the smallest α such that

$$\mathcal{K}\subset\mathcal{S}^*(\zeta,\alpha).$$

Recall here another definition of q-starlike functions of order α. Namely, making use of q-derivative (6), Argawal and Sahoo in Agrawal and Sahoo (2017) introduced the class $\mathcal{S}^*_q(\alpha)$. A function $f\in\mathcal{A}$ belongs to the class $\mathcal{S}^*_q(\alpha)$, $0\leq\alpha<1$, if

$$\left|\frac{z\mathrm{d}_q f(z)}{f(z)}-\frac{1-\alpha q}{1-q}\right|\leq\frac{1-\alpha}{1-q},\quad z\in\mathbb{D}. \tag{16}$$

If $q\to 1^-$ the class $\mathcal{S}^*_q(\alpha)$ reduces to the class $\mathcal{S}^*(\alpha)$. If $\alpha=0$, the class $\mathcal{S}^*_q(\alpha)$ coincides with the class $\mathcal{S}^*_q(0)=\mathcal{S}^*_q$, which was first introduced by Ismail et al. (1990) and was considered in Abu-Risha et al. (2007), Agrawal and Sahoo (2014), Annaby and Mansour (2012), Aouf and Seoudy (2019), Raghavendar and Swaminathan (2012), Rønning (1994), Sahoo and Sharma (2015), Seoudy and Aouf (2014, 2016). Moreover, only for $\alpha=0$ the classes $\mathcal{S}^*_q(\alpha)$ and $\mathcal{S}^*(q,\alpha)$ are equal one to another. In other cases, i.e. for $0<\alpha<1$, condition (16) follows

$$\Re\mathfrak{e}\left\{\frac{z\mathrm{d}_q f(z)}{f(z)}\right\}>\alpha,\quad z\in\mathbb{D}.$$

Therefore,

$$0<q<1\ \Rightarrow\ \mathcal{S}^*_q(\alpha)\subset\mathcal{S}^*(q,\alpha).$$

Lemma 2 *If f is in the class* $\mathcal{K}$ *of convex univalent functions, then we have*

$$\Re\mathfrak{e}\left\{\frac{(1-\zeta)\mathrm{d}_\zeta f(z)}{f'(z)}\right\}>0,\quad z,\zeta\in\mathbb{D}, \tag{17}$$

Proof It is known that if $f\in\mathcal{K}$, then $f(\mathbb{D})$ is starlike with respect to each of its points, so we have

$$\Re\mathfrak{e}\left\{\frac{zf'(z)}{f(z)-f(x)}\right\}>0,\quad |x|<|z|<1.$$

This implies

$$\Re\mathfrak{e}\left\{\frac{f(z)-f(\zeta z)}{zf'(z)}\right\}>0,\quad z,\zeta\in\mathbb{D}. \tag{18}$$

On the other hand,

$$\frac{f(z)-f(\zeta z)}{zf'(z)}=\frac{f(z)-f(\zeta z)}{(1-\zeta)z}\frac{(1-\zeta)z}{zf'(z)}=\frac{(1-\zeta)\mathrm{d}_\zeta f(z)}{f'(z)}. \tag{19}$$

Finally, from (18) and (19), we get (17). □

Theorem 3 *If f and h are in the class* $\mathcal{K}$ *of convex univalent functions, then we have*

$$\Re\mathfrak{e}\left\{\frac{h(z)*(1-\zeta)z\mathrm{d}_\zeta f(z)}{h(z)*zf'(z)}\right\}>0,\quad z,\zeta\in\mathbb{D}. \tag{20}$$

Proof From the hypothesis, we have $h\in\mathcal{K}$ and $zf'(z)\in\mathcal{S}^*$, $z\in\mathbb{D}$, so by Lemma 1 we have

$$\frac{h(z)*(1-\zeta)z\mathrm{d}_\zeta f(z)}{h(z)*zf'(z)}=\frac{h(z)*zf'(z)\frac{(1-\zeta)\mathrm{d}_\zeta f(z)}{f'(z)}}{h(z)*zf'(z)}\in\overline{\mathrm{co}}\{F(\mathbb{D})\},$$

where

$$F(z)=\frac{(1-\zeta)\mathrm{d}_\zeta f(z)}{f'(z)},\quad z\in\mathbb{D}.$$

By Lemma 2, we get $\Re\mathfrak{e}\{F(z)\}>0$, which implies (20). □

Corollary 3 If f is in the class $\mathcal{K}$ of convex univalent functions, then we have

$$\Re\mathfrak{e}\left\{\frac{(1-\zeta)\int_0^z\mathrm{d}_\zeta f(t)\mathrm{d}t}{f(z)}\right\}>0,\quad z,\zeta\in\mathbb{D}. \tag{21}$$

Proof It is known that the following function

$$H_1(z):=\log\left\{\frac{1}{1-z}\right\}=\sum_{n=1}^{\infty}\frac{z^n}{n},\quad z\in\mathbb{D},$$

belongs to the class $\mathcal{K}$. Furthermore, for $f\in\mathcal{H}$, we have

$$H_1(z)*f(z)=\int_0^z\frac{f(t)-f(0)}{t}\ \mathrm{d}t.$$

It is easy to check that

$$\frac{H_1(z)*(1-\zeta)z\mathrm{d}_\zeta f(z)}{H_1(z)*zf'(z)}=\frac{(1-\zeta)\int_0^z\mathrm{d}_\zeta f(t)\mathrm{d}t}{f(z)}. \tag{22}$$

From (22) and Theorem 3, we immediately get (21). □

Some further applications of Theorem 3 can be obtained in the same way as in Corollary 3 by choosing some other convex functions. The following functions are in the class $\mathcal{K}$:

$$H_2(z) := \frac{z}{1-z}, \quad H_3(z) := \frac{-2(z+\log(1-z))}{z}, \\ H_4(z) := \frac{1}{1-\zeta}\log\frac{1-\zeta z}{1-z}, \quad z,\zeta \in \mathbb{D}. \tag{23}$$

The above functions H_i, $i = 2, 3, 4$, generate the following corollaries.

Corollary 4 If f is in the class $\mathcal{K}$ of convex univalent functions, then we have

$$\Re\mathfrak{e}\left\{\frac{(1-\zeta)\mathrm{d}_\zeta f(z)}{f'(z)}\right\} > 0, \quad z,\zeta \in \mathbb{D}. \tag{24}$$

Proof For $f \in \mathcal{H}$, we have

$$H_2(z) * f(z) = f(z), \tag{25}$$

where H_2 is given by (23). Inequality (24) is obtained from (20) by putting $h(z) = H_2(z)$ and using (25). □

Corollary 5 If f is in the class $\mathcal{K}$ of convex univalent functions, then we have

$$\Re\mathfrak{e}\left\{\frac{(1-\zeta)\int_0^z t\mathrm{d}_\zeta f(t)\mathrm{d}t}{\int_0^z tf'(t)\mathrm{d}t}\right\} > 0, \quad z,\zeta \in \mathbb{D}. \tag{26}$$

Proof For $f \in \mathcal{H}$, we have

$$H_3(z) * f(z) = \left(\sum_{n=1}^{\infty}\frac{2z^n}{n+1}\right) * f(z) = \frac{2}{z}\int_0^z f(t)\mathrm{d}t, \tag{27}$$

where H_3 is given by (23). Inequality (26) is obtained from (20) by putting $h(z) = H_3(z)$ and using (27). □

Corollary 6 If f is in the class $\mathcal{K}$ of convex univalent functions, then we have

$$\Re\mathfrak{e}\left\{\frac{(1-\zeta)\int_0^z \mathrm{d}_\zeta(t\mathrm{d}_\zeta f(t))\mathrm{d}t}{\int_0^z \mathrm{d}_\zeta(tf'(t))\mathrm{d}t}\right\} > 0, \quad z,\zeta \in \mathbb{D}. \tag{28}$$

Proof For $f \in \mathcal{H}$, we have

$$H_4(z) * f(z) = \left(\sum_{n=1}^{\infty}\frac{1+\zeta+\cdots+\zeta^{n-1}}{n}z^n\right) * f(z) \\ = \int_0^z \mathrm{d}_\zeta f(t)\mathrm{d}t, \tag{29}$$

where H_4 is given by (23). Inequality (28) is obtained from (20) by putting $h(z) = H_4(z)$ and using (29). □

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