ON q-TH DERIVATIVE OF VECTOR BUNDLES

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To Professor KIYOSHI NOSHIRO on the occasion of his 60th birthday

(0. 0) In the present note we shall be concerned with the improvement of fundamental definitions in higher order enumerative geometry which has been recently given by W. F. Pohl. Pohl's definition of q-th derivative of vector bundle is very complicated. We shall give a simpler and more reasonable definition of the q-th derivative of vector bundle in terms of sheaf theory and simplify the proofs in [P]. We shall also give a definition of higher order singularity of map.

1. The sheaf D_q

(1. 1) Let (X, O_X) be an object of a category C of ringed spaces over ground field k. Ringed space over k is a pair (X, O_X) of topological space X and sheaf O_X of k-algebras over X. (cf. [G]).

The following are most important examples of the category C;

(i) the category of differentiable-manifolds; k is the field R of real numbers.

(ii) the category of complex manifolds; k is the field $\mathbf{C}$ of complex numbers.

(iii) the category of algebraic k-schemes (or varieties defined over k).

(1. 2) Let F_1 be the sheaf of derivations of O_X over k, i.e.

$$F_1 = \{t \in \mathrm{Hom}_k(O_X, O_X) \mid t(ab) = a\cdot t(b) + bt(a) \text{ for any } a, b \in O_X \text{ and } t(c) = 0 \text{ for any } c \in k\}$$

where $\mathrm{Hom}_k(O_X, O_X)$ denotes the sheaf of k-homomorphisms of O_X into itself.

Then, F_1 is a sheaf of k-modules. Tensoring O_X with F_1 over k, we get a sheaf of O_X-modules $F = F_1 \otimes_k O_X$.

Although we may also consider F_1 as a sheaf of O_X-modules, we do not

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give O_X-modules structure to F_1. Hence O_X-modules structure of F is given only by base extension.

(1.3) Let $T(F)$ be a tensor algebra of F over O_X, i.e. $T(F) = \bigoplus_{n\geq 0} F^{\otimes n}$. $T(F)$ is a sheaf of graded rings. Let p be a characteristic of a field k. Then, $\mathfrak{A}$ is a sheaf of two-sided ideals of $T(F)$ generated by the elements:

$t\otimes a - at\otimes 1 - t(a)$, $(t_1\otimes 1)\otimes(t_2\otimes 1) - (t_2\otimes 1)\otimes(t_1\otimes 1) - [t_1, t_2]\otimes 1$, $(t\otimes 1)^{\otimes p} - t^p$, where $t, t_1, t_2 \in F_1$, $a\in O_X$ and $[t_1, t_2]$ denotes Lie bracket of t_1 and t_2.

The quotient D of $T(F)$ by $\mathfrak{A}$ is a sheaf of filtered rings. If we denote by D_q the q-th part of D, then $D_0 = O_X$, and we put $D_q/D_0 = F_q$ which is called the sheaf of germs of osculating vector fields of order q in [P]. We denote by $t_1a_1t_2a_2\cdots t_qa_q$ the canonical image in D of $(t_1\otimes a_1)\otimes(t_2\otimes a_2)\otimes\cdots\otimes(t_q\otimes a_q)$ and $t\cdot 1 = 1\cdot t$ is also written simply by t.

(1. 4) Let $S(F_1)$ be a symmetric algebra of F_1 over k, i.e. $S(F_1)$ is the quotient of tensor algebra $T(F_1)$ by the ideal $\langle t\otimes u - u\otimes t/t,\ u\in F_1\rangle$. Then $S(F_1)$ is a sheaf of graded rings. We denote by $S^q(F_1)$ the q-th degree part of $S(F_1)$. There is a canonical homomorphism

$$S_q : F_1^{\otimes q}\to S^q(F_1).$$

A homomorphism $m : F_1^{\otimes q}\to D_q$ is defined by $m(t_1\otimes t_2\otimes\cdots\otimes t_q) = t_1t_2\cdots t_q$. Then for $q\geq 1$, we have $m(t_1\otimes t_2\otimes\cdots\otimes t_q - t_1\otimes\cdots\otimes t_{i+1}\otimes t_i\otimes\cdots\otimes t_q) = t_1\cdots t_{i-1}[t_i, t_{i+1}]t_{i+2}\cdots t_q D_{q-1}$, hence m induces a homomorphism

$$\varphi_q : S^q(F_1)\to D_q/D_{q-1} \quad \text{for } q\geq 1.$$

PROPOSITION (1.5) *φ_q is an isomorphism, hence we have an exact sequence*

(1. 5. 1) $$0\longrightarrow D_{q-1}\xrightarrow{I_q} D_q\longrightarrow S^q(F_1)\longrightarrow 0$$

where I_q is a canonical inclusion.

Proof (cf. [P]. Th. 2. 1)

COROLLARY (1.6) *If F_1 is locally free, then D_q is also locally free.*

Proof. If F_1 is locally free, then $S^q(F_1)$ is locally free and it is proved by induction on q.

2. The sheaf $\Delta_q M$

DEFINITION (2. 1) Let M be a sheaf of O_X-modules. The sheaf $\Delta_q M = D_q\otimes_{O_X} M$

is called the q-th derivative of M.

It is easily verified that this definition of q-th derivative of M is the same as Pohl's one (cf. [P]. § III. 1.). Δ_q is a functor from the category of sheaves of O_X-modules into itself.

ASSUMPTION (2.2) Assume that in category C, F_1 is always locally free. This assumption is satisfied, for example, if C is (i), (ii) of (1.1) or the category of non-singular algebraic varieties.

PROPOSITION (2.3) *Under the assumption* (2.2), *there is an exact sequence for* $q \geq 1$,

(2.3.1) $$0 \longrightarrow \Delta_{q-1}M \xrightarrow{I_q^M} \Delta_q M \longrightarrow \mathbf{S}^q(F_1) \otimes_{O_X} M \longrightarrow 0$$

where $I_q^M = I_q \otimes i_M$ *and* i_M *is identity of* M.

Proof. By (1.5.1), there is an exact sequence

$$0 \to D_{q-1} \to D_q \to \mathbf{S}^q(F_1) \to 0.$$

Tensoring M, we get an exact sequence

$$D_{q-1} \otimes M \to D_q \otimes M \to \mathbf{S}^q(F_1) \otimes_{O_X} M \to 0.$$

On the other hand, $\mathbf{S}^q(F_1)$ is locally free, and so is O_X-flat. Therefore (2.3.1) is exact (cf. [B] I, 2 Prop. 4).

PROPOSITION (2.4) *Under the assumption* (2.2), *the exact sequence* (2.3.1) *is an exact functor of* M. *In particular* Δ_q *is an exact functor.*

Proof. If F_1 is locally free, D_q and $\mathbf{S}^q(F_1)$ are also locally free.

(2.5) We define a homomorphism $\varphi_1 : F = F_1 \otimes_k O_X \to O_X$ by $\varphi_1(t \otimes a) = t(a)$, $t \in F_1$, $a \in O_X$. A homomorphism $\varphi_n : F^{\otimes n} \to O_X$ is defined to be $\varphi_n = \varphi_1 \circ (i_F \otimes \varphi_{n-1})$ where i_F is identity homomorphism of F. Let $\varphi_0 : O_X \to O_X$ be the identity. From $\{\varphi_n\}_{n=0}$, we get a homomorphism

$$\varphi : T(F) \to O_X.$$

It is clear that Ker $\varphi \supset \mathfrak{A}$. φ induces a homomorphism

$$\rho : D = T(F)/\mathfrak{A} \to O_X.$$

We denote by ρ_q the restriction of ρ on D_q, i.e. $\rho_q : D_q \to O_X$. Since $D_q \otimes_{O_X} O_X = D_q$, $\Delta_q O_X^N = D_q^N$.

Definition (2. 5. 1) We denote by ω_q the map $\rho_q^N = \rho_q + \cdots + \rho_q : \Delta_q O_X^N \to O_X^N$. The restriction of ω_q on $\Delta_{q-1} O_X^N$ is ω_{q-1}, i.e.

(2. 5. 2) $$\omega_{q-1} = \omega_q \circ I_q^{O_X^N}$$

because it holds $\rho_{q-1} = \rho_q \circ I_q$ (cf. [P]. Prop. 3. 9_q)
Let $\omega_0 : \Delta_0 O_X^N = O_X^N \to O_X^N$ be the identity. We get also

(2. 5. 3) $$\omega_1 \circ I_0^{O_X^N} = \text{identity},$$

i.e. ω_1 defines a splitting of (2. 3. 1) for $q=1$ (cf [P]. Prop. 3. 8).

3. *Higher order singularities of maps*

(3. 1) Let $G_{n,r}$ be the Grassmanian manifold of $(r-1)$-dimensional projective subspaces in n-dimensional projective space. Given a suitable structure sheaf O_G on $G_{n,r}$, $G_{n,r}$ may be considered as an object of category (i), (ii) or (iii) of (1. 1). *$G_{n,r}$ is imbedded in some projective space by Plücker coordinates* $p = (\ldots, p_{i_1 \ldots i_r}, \ldots)$. Plücker coordinates $p_{i_1 \ldots i_r}$ satisfy the well-known relations

(3. 1. 1) $$\sum_{k=1}^{r+1} (-1)^k p_{i_1 \ldots i_{r-1} j_k} p_{j_1 \ldots \hat{j_k} \ldots j_{r+1}} = 0$$

for any $i_1, \ldots, i_{r-1}, j_1, \ldots, j_{r+1}$ such that $1 \leq i_1 < \cdots < i_{r-1} \leq n$, $1 \leq j_1 < \cdots < j_{n+1} \leq n$.

(3. 2) Let $V_{i_1 \ldots i_r}$ be the set of points $p = (\ldots, p_{i_1 \ldots i_r}, \ldots)$ of $G_{n,r}$ such that $p_{i_1 \ldots i_r} \neq 0$ Then $V_{i_1 \ldots i_r}$ is an (affine) open subset of $G_{n,r}$, and $\{V_{i_1 \ldots i_r}\}$ is an open covering of $G_{n,r}$. Let $\mathscr{A}_{i_1 \ldots i_r}$ be the restriction of the sheaf O_G^r on $V_{i_1 \ldots i_r}$. Let $v_1^{(i)}, \ldots, v_r^{(i)}$ and $v_1^{(j)}, \ldots, v_r^{(j)}$ be the canonical basis of $\mathscr{A}_{i_1 \ldots i_r}$ and $\mathscr{A}_{j_1 \ldots j_r}$ respectively. Now we define a homomorphism

(3. 2. 1) $$f_{(j)}^{(i)} = f_{j_1 \cdots j_r}^{i_1 \cdots i_r} : \mathscr{A}_{i_1 \ldots i_r} | V_{i_1 \ldots i_r} \cap V_{j_1 \ldots j_r} \to \mathscr{A}_{j_1 \ldots j_r} | V_{i_1 \ldots i_r} \cap V_{j_1 \ldots j_r},$$

by $f_{(j)}^{(i)}(v_\nu^{(i)}) = \sum_{\mu=1}^{r} a_{\nu\mu}^{(i)(j)} v_\mu^{(j)}$, where $a_{\nu\mu}^{(i)(j)} = (-1)^{r-\nu} \dfrac{p_{i_1 \ldots \hat{i_\nu} \ldots i_r j_\mu}}{p_{i_1 \ldots i_r}} \in \Gamma(V_{i_1 \ldots i_r}, O_G)$.

We shall show $f_{(k)}^{(j)} f_{(j)}^{(i)} = f_{(k)}^{(i)}$ on $V_{i_1 \ldots i_r} \cap V_{j_1 \ldots j_r} \cap V_{k_1 \ldots k_r}$ for any $i_1, \ldots, i_r$, $j_1, \ldots, j_r, k_1, \ldots, k_r$.

$$\begin{aligned} f_{(k)}^{(j)} f_{(j)}^{(i)}(v_\nu^{(i)}) &= \sum_{\mu=1}^{r} a_{\nu\mu}^{(i)(j)} f_{(k)}^{(j)}(v_\mu^{(j)}) \\ &= \sum_{\mu=1}^{r} a_{\nu\mu}^{(i)(j)} \Big(\sum_{\lambda=1}^{r} a_{\mu\lambda}^{(j)(k)} v_\lambda^{(k)} \Big) = \sum_{\lambda=1}^{r} \Big(\sum_{\mu=1}^{r} a_{\nu\mu}^{(i)(j)} a_{\mu\lambda}^{(j)(k)} \Big) v_\lambda^{(k)} \end{aligned}$$

$$= \sum_{\lambda=1}^{r} \frac{(-1)^{r-\nu}}{p_{i_1\ldots i_r}\cdot p_{j_1\ldots j_r}} (p_{i_1\ldots\hat{i}_\nu\ldots i_r j_1}\cdot(-1)^{r-1}$$

$$p_{j_2\ldots j_r k_\lambda} + p_{i_1\ldots\hat{i}_\nu\ldots i_r j_2}(-1)^{r-2} p_{j_1 j_3\ldots j_r k_\lambda} +$$

$$\cdots + p_{i_1\ldots\hat{i}_\nu\ldots i_r j_r}\cdot p_{j_1\ldots j_{r-1}k_\lambda}) v_\lambda^{(k)}$$

$$= \sum_{\lambda=1}^{r} \frac{(-1)^{r-\nu} p_{i_1\ldots\hat{i}_\nu\ldots i_r k_\lambda}\cdot p_{j_1\ldots j_r}}{p_{i_1\ldots i_r}\cdot p_{j_1\ldots j_r}} v_\lambda^{(k)} \qquad \text{(by (3.1.1))}$$

$$= \sum_{\lambda=1}^{r} a_{\nu\mu}^{(i)(k)} v_\lambda^{(k)} = f_{(k)}^{(i)}(v_{(\nu)}^{(i)}).$$

It is clear that $f_{(i)}^{(i)}$ = identity, hence $f_{(j)}^{(i)} = (f_{(i)}^{(j)})^{-1}$ on $V_{i_1\ldots i_r} \cap V_{j_1,\ldots,j_r}$, i.e. the homomorphism $f_{(j)}^{(i)}$ of (3. 2. 1) is an isomorphism. Then we can patch $\mathscr{A}_{i_1\ldots i_r}$ together with the isomorphism (3. 2. 1) and we obtain a sheaf of O_G-modules $\mathscr{E}_{n,r}$ over $G_{n,r}$ (cf [G] Chap. 0, (3. 3. 1)). $\mathscr{E}_{n,r}$ is locally free of rank r.

Remark (3. 2. 2) $\mathscr{E}_{n,r}$ is the sheaf of germs of cross-sections of the universal r-plane bundle over $G_{n,r}$.

(3. 3) We shall define a homomorphism σ of $\mathscr{E}_{n,r}$ into O_G^{n+1}. Firstly we define a homomorphism

$$\sigma_{i_1\ldots i_r} : \mathscr{A}_{i_1\ldots i_r} \simeq \mathscr{E}_{n,r} | V_{i_1\ldots i_r} \to O_G^{n+1} | V_{i_1\ldots i_r}.$$

Let $u_0, \ldots, u_n$ be the canonical basis of O_G^{n+1}. Then $\sigma_{i_1\ldots i_r}$ is defined by $\sigma_{i_1\ldots i_r}(v_\nu^{(i)}) = \sum_{\mu=0}^{n} b_{\nu\mu}\cdot u_\mu$ where $b_{\nu\mu} = (-1)^{r-\nu} \frac{p_{i_1\ldots\hat{i}_\nu\ldots i_r}}{p_{i_1\ldots i_r}} \in \Gamma(V_{i_1\ldots i_r}, O_G)$.

By virtue of (3. 1. 1), it is easily verified that

$$\sigma_{i_1\ldots i_r} | V_{i_1\ldots i_r} \cap V_{j_1\ldots j_r} = \sigma_{j_1\ldots j_r} | V_{i_1\ldots i_r} \cap V_{j_1\ldots j_r}.$$

Patching $\sigma_{i_1\ldots i_r}$ together, we get a homomorphism

(3. 3. 1) $$\sigma : \mathscr{E}_{n,r} \to O_G^{n+1}.$$

(3. 4) Let f be a morphism of X into $G_{n,r}$. By virtue of the homomorphism σ in (3. 3. 1), we obtain a homomorphism

$$f^*(\sigma) : f^*(\mathscr{E}_{n,r}) \to O_X^{n+1} = f^*(O_G^{n+1})$$

(where f^* denotes the reciprocal image) and consequently a homomorphism

(3. 4. 1) $$\Delta_q f^*(\sigma) : \Delta_q f^*(\mathscr{E}_{n,r}) \to \Delta_q(O_X^{n+1}).$$

DEFINITION (3. 5) We say that the morphism $f : X \to G_{n,r}$ is *non-singular of order q at $x \in X$*, if the homomorphism

$$\omega_q \circ \Delta_q f^*(\sigma) : \Delta_q f^*(\mathscr{E}_{n,r}) \to O_X^{n+1},$$

which is the composition of $\Delta_q f^*(\sigma)$ of (3. 4. 1) and ω_q of (2.5.1), is injective at X.

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