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ON QUADRUPLES OF PROJECTORS CONNECTED BY A LINEAR RELATION

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We describe the set of $\gamma \in \mathbb{R}$ for which there exist quadruples of projectors P_i for a fixed collection of numbers $\alpha_i \in \mathbb{R}_+$, $i = \overline{1, 4}$, such that $\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = \gamma I$.

1. Introduction

Let $M_i = \{0 = \alpha_0^{(i)} < \alpha_1^{(i)} < \ldots < \alpha_{m_i}^{(i)}\}$, $i = \overline{1, n}$, be given sets in $\mathbb{R}_+$. Collections of self-adjoint operators $A_i = A_i^*$ with spectra $\sigma(A_i) \subset M_i$ and the sum divisible by a scalar operator were studied in many papers (see, e.g., the references in [1]). Considering these operators as representations of the generators of an involutive algebra, we obtain the equivalent problem of the description of irreducible $*$-representations of the algebra

$$\mathcal{A}_{M_1, M_2, \ldots, M_n; \gamma} = \mathbb{C}\left\langle a_1 \ldots a_n \;\middle|\; a_i = a_i^*,\ R_i(a_i) = 0,\ a_1 + a_2 + \ldots + a_n = \gamma e \right\rangle,$$

where R_i is an annihilating polynomial of the corresponding generator a_i. This algebra is isomorphic to the algebra generated by the collection of projectors

$$\mathcal{P}_{M_1, M_2, \ldots, M_n; \gamma} = \mathbb{C}\left\langle p_1^{(1)}, \ldots, p_{m_1}^{(1)}, \ldots, p_1^{(n)}, \ldots, p_{m_n}^{(n)} \;\middle|\; p_i^{(k)} = p_i^{(k)2} = p_i^{(k)*},\ \sum_{i=1}^{n} \sum_{k=1}^{m_i} \alpha_k^{(i)} p_k^{(i)} = \gamma e,\ p_j^{(i)} p_k^{(i)} = 0 \right\rangle.$$

We can associate every algebra of this type with a graph Γ that has n branches meeting at a single vertex (the root of the graph). Each ith branch of the graph contains m_i vertices marked by numbers $\alpha_i^{(k)}$, $k = \overline{1, m_i}$. We associate the root of the graph with the number γ (for more details on the relationship between the problem considered and representations of graphs, see [2]). The vector $\chi = (\alpha_1^{(1)}, \ldots, \alpha_{m_1}^{(1)}; \ldots; \alpha_n^{(1)}, \ldots, \alpha_{m_n}^{(n)})$ is called the character of the algebra $\mathcal{P}_{M_1, M_2, \ldots, M_n; \gamma}$. The algebra $\mathcal{P}_{M_1, M_2, \ldots, M_n; \gamma}$ is uniquely defined by its graph, character χ, and number γ. In what follows, we denote it by $\mathcal{P}_{\Gamma, \chi, \gamma}$. It was shown in [3] that, independently of χ and γ, the following assertions are true:

(i) the algebra $\mathcal{P}_{\Gamma, \chi, \gamma}$ is finite-dimensional if Γ is a Dynkin diagram of the type D_n, E_6, E_7, or E_8;

(ii) $\mathcal{P}_{\Gamma, \chi, \gamma}$ is an infinite-dimensional algebra of polynomial growth if Γ is an extended Dynkin diagram of the type $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$;

(iii) for the other graphs, the algebra $\mathcal{P}_{\Gamma, \chi, \gamma}$ contains a free algebra with two self-adjoint generators.

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In the investigation of $*$-representations of these algebras, the following problems naturally arise:

1a. Describe the set of pairs $(\chi;\gamma)$ for which the algebra $\mathcal{P}_{\Gamma,\chi,\gamma}$ has a $*$-representation; we denote this set by Σ_Γ.

1b. For every character χ, describe the set of γ for which the algebra $\mathcal{P}_{\Gamma,\chi,\gamma}$ has a $*$-representation; we denote this set by $\Sigma_{\Gamma,\chi}$.

2. For every pair $(\chi;\gamma)\in\Sigma_\Gamma$, describe all irreducible (up to unitary equivalence) $*$-representations of the algebra $\mathcal{P}_{\Gamma,\chi,\gamma}$.

The structure of the sets Σ_Γ and $\Sigma_{\Gamma,\chi}$ and $*$-representations of $\mathcal{P}_{\Gamma,\chi,\gamma}$ essentially depends on the graph. In [4], problems 1 and 2 were considered for ordinary Dynkin diagrams. In the case where Γ is an extended Dynkin diagram, several authors described the sets $\Sigma_{\Gamma,\chi}$ for special characters (see the references in [1]). A complete description of the set $\Sigma_{\tilde{D}_4}$ is given in [5]. However, despite the fact that the set $\Sigma_{\tilde{D}_4,\chi}$ is a subset of $\Sigma_{\tilde{D}_4}$, it is a difficult to obtain its description from the results of [5].

In the present paper, we give a direct description of the set $\Sigma_{\tilde{D}_4,\chi}$. We investigate in what cases this set is infinite [a necessary and sufficient condition is given, namely, all components of the character $\chi = (\alpha_1;\alpha_2;\alpha_3;\alpha_4)$ must satisfy the inequality $\alpha_i < (\alpha_1+\alpha_2+\alpha_3+\alpha_4)/2$ (Sec. 3)], which enables us, by analogy with [5], to investigate in what cases the algebra $\mathcal{P}_{\tilde{D}_4,\chi,\gamma}$ has a representation on the hyperplane $\gamma = (\alpha_1+\alpha_2+\alpha_3+\alpha_4)/2$ (Sec. 3). In Sec. 5, we describe the structure of the set $\Sigma_{\tilde{D}_4,\chi}$ for the special character $\chi_\delta = (1,1,\delta,\delta)$.

1. Auxiliary Statements

Recall that the set of possible values γ for which there exist triples of projectors P_1, P_2, P_3 such that $\alpha_1P_1+\alpha_2P_2+\alpha_3P_3 = \gamma I$ for a fixed collection $\alpha_i\in\mathbb{R}$, $i = 1, 2, 3$, arranged in ascending order is described by the relation (see [4])

$$\Sigma_{D_4,(\alpha_1,\alpha_2,\alpha_3)} = \{0\}\cup\left\{\sum_{i\in J}\alpha_i,\ J\subset\{1,2,3\}\right\}\cup\{(\alpha_1+\alpha_2+\alpha_3)/2\}. \tag{1}$$

Assume that $\alpha_3 < \alpha_1+\alpha_2$. Otherwise the set does not contain the point $(\alpha_1+\alpha_2+\alpha_3)/2$.

Proposition 1. *If a collection of projectors* $P_1, P_2, \ldots, P_n$ *satisfies the equality*

$$\alpha_1P_1+\alpha_2P_2+\ldots+\alpha_{n-1}P_{n-1}+P_n = I,$$

then the projector P_n *commutes with the other projectors, i.e.,* $[P_n,P_i] = 0$, $i = \overline{1,n-1}$.

Proof. Consider the operator $Q = I - P_n$. Multiplying the equality $\alpha_1 P_1 + \alpha_2 P_2 + \ldots + \alpha_{n-1} P_{n-1} = Q$ by P_n from the right and from the left and taking into account that $QP_n = P_n Q = 0$, we get

$$\alpha_1 P_n P_1 P_n + \alpha_2 P_n P_2 P_n + \ldots + \alpha_3 P_n P_{n-1} P_n = 0.$$

Since all operators $P_n P_i P_n$ are nonnegative, we have $P_n P_i P_n = 0$, $i = \overline{1, n-1}$. One can verify that $P_n P_i Q P_i P_n = 0$. Therefore, $P_n P_i Q = Q P_i P_n = 0$. Taking into account that $P_n P_i Q = P_n P_i (I - P_n) = P_n P_i$, we get $P_n P_i = 0$. By analogy, one can show that $P_i P_n = 0$. This means that $[P_n, P_i] = 0$, $i = \overline{1, n-1}$.

Corollary 1. *There exists a collection of projectors* P_i, $i = \overline{1, 4}$, *such that* $\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = \alpha_4 I$.

2. Quadruple of Projectors and Coxeter Functors

Let a collection of numbers $\alpha_i \in \mathbb{R}$, $i = \overline{1, 4}$, be given. Our aim is to describe the set of γ for which there exist quadruples of projectors P_i, $i = \overline{1, 4}$, such that $\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = \gamma I$.

We can associate these quadruples of projectors with the associative $\mathbb{C}$-algebra $\mathcal{P}_{\tilde{D}_4, \chi, \gamma}$ generated by the generators $\{p_i\}_{i=1}^4$ and the relations

$$p_i = p_i^2 = p_i^*,$$

$$\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \alpha_4 p_4 = \gamma e,$$

where $\chi = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is the character of the algebra (we assume that $\alpha_1 \le \alpha_2 \le \alpha_3 \le \alpha_4$). Then the problem can be reformulated as follows: For every character, describe the set $\Sigma_{\tilde{D}_4, \chi}$ of γ for which the algebra $\mathcal{P}_{\tilde{D}_4, \chi, \gamma}$ has a $*$-representation. Let χ_i denote the ith component of the character χ and let $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$.

The set $\Sigma_{\tilde{D}_4, \chi}$ possesses the following properties (see [6]):

(i) $\Sigma_{\tilde{D}_4, \chi} \subset [0, \alpha]$;

(ii) $\Sigma_{\tilde{D}_4, \chi} \ni \sum_{i \in J} \alpha_i$, $J \subset \{0, 1, 2, 3, 4\}$;

(iii) $\tau \in \Sigma_{\tilde{D}_4, \chi} \Leftrightarrow \alpha - \tau \in \Sigma_{\tilde{D}_4, \chi}$.

Since the set $\Sigma_{\tilde{D}_4, \chi}$ is symmetric with respect to $\alpha/2$ [property (iii)], we study the set $\Sigma_{\tilde{D}_4, \chi} \cap [0; \alpha/2)$.

To investigate the set $\Sigma_{\tilde{D}_4, \chi}$, we use the method of Coxeter functors (introduced in [6]), which establish an equivalence between the categories of $*$-representations $\operatorname{Rep} \mathcal{P}_{\tilde{D}_4, \chi, \gamma}$ for different values of the parameters χ and γ. The linear T and hyperbolic S functors were constructed in [6]. The action of these functors between the categories generates the action on the pair $(\chi; \gamma)$:

$$S\colon (\chi;\gamma) \mapsto (\gamma-\alpha_1,\ \gamma-\alpha_2,\ \gamma-\alpha_3,\ \gamma-\alpha_4;\ \gamma),$$

$$T\colon (\chi;\gamma) \mapsto (\alpha_1,\alpha_2,\alpha_3,\alpha_4;\ \alpha-\gamma).$$

Let $(\chi^{(n)};\gamma^{(n)}) = (ST)^n(\chi;\gamma)$ and $\lambda = \alpha/2-\gamma$. Then the following statement is true:

Proposition 2. *The components of the character* $\chi^{(n)}$ *and the number* $\gamma^{(n)}$ *are determined by the formulas*

$$\chi_i^{(2n-1)} = \frac{\alpha}{2}-\alpha_i-(2n-1)\lambda, \qquad \chi_i^{(2n)} = \alpha_i-2n\lambda, \quad i = \overline{1,4}, \tag{2}$$

$$\gamma^{(n)} = \frac{\alpha}{2} - (2n+1)\lambda, \quad n\in\mathbb{N}. \tag{3}$$

To prove this proposition, it is necessary to write the action of the functor ST on the pair $(\chi;\gamma)$ and to use the method of mathematical induction.

Corollary 2. *For any* $\gamma\in[0,\alpha/2)$, *there exists* $n\in\mathbb{N}$ *such that either one of the components of the character* $\chi^{(n)}$ *or the number* $\gamma^{(n)}$ *is less than or equal to zero.*

Proof. Since $\lambda > 0$ for any $\gamma\in[0,\alpha/2)$, it follows from relations (2) and (3) that the sequences $\{\chi_i^{(2n)}\}_{n=1}^{\infty}$, $\{\chi_i^{(2n-1)}\}_{n=1}^{\infty}$, and $\{\gamma_i^{(n)}\}_{n=1}^{\infty}$ are infinitely decreasing, and, hence, there exists n for which Proposition 2 is true.

Theorem 1. *The number* $\gamma\in[0,\alpha/2)$ *belongs to the set* $\Sigma_{\tilde{D}_4,\chi}$ *if and only if there exist* $n\in\mathbb{Z}_+$ *and* $j\in\{1,2,3,4\}$ *such that the following two conditions are satisfied:*

$$\chi_j^{(n)} \le 0, \quad \chi_i^{(k)} > 0, \quad \gamma^{(k)} \ge 0 \quad \forall k < n, \tag{4}$$

$$\gamma^{(n-1)} \in \Sigma_{D_4,\chi^*}, \tag{5}$$

where the character χ^* *is defined by the triple of coefficients* $\chi_i^{(n-1)}$, $i = \overline{1,4}$, $i \ne j$.

Proof. The proof of the theorem follows from Corollary 2 and the functoriality of the mapping (ST) used for the construction of the corresponding sequences.

3. Infinite Sets $\Sigma_{\tilde{D}_4,\chi}$ and Representations on a Hyperplane

Theorem 2. *The set* $\Sigma_{\tilde{D}_4,\chi}$ *contains an infinite subset* Σ_∞ *with limit point* $\alpha/2$ *if and only if* $\alpha_i < \alpha/2$, $i = \overline{1,4}$. *If this condition is satisfied, then the following assertions are true:*

(i) $\Sigma_\infty = \left\{\frac{\alpha}{2} - \frac{\alpha_1}{2n} \;\middle|\; n \in \mathbb{N}\right\}$ *if* $\alpha_2 + \alpha_3 > \alpha_1 + \alpha_4$;

(ii) $\Sigma_\infty = \left\{\frac{\alpha}{2} - \frac{\alpha - 2\alpha_4}{2(2n-1)} \;\middle|\; n \in \mathbb{N}\right\}$ *if* $\alpha_2 + \alpha_3 < \alpha_1 + \alpha_4$;

(iii) $\Sigma_\infty = \left\{\frac{\alpha}{2} - \frac{\alpha_1}{n} \;\middle|\; n \in \mathbb{N}\right\}$ *if* $\alpha_2 + \alpha_3 = \alpha_1 + \alpha_4$.

Proof. *Necessity.* If one of the coefficients satisfies the inequality $\alpha_i \geq \alpha/2$, then the corresponding projector P_i commutes with the other projectors and, hence, is equal to either 0 or I in an irreducible representation. In this case, the problem reduces to a triple (or, correspondingly, to a smaller number) of projectors. Therefore, by virtue of Theorem 1, the set $\Sigma_{\tilde{D}_4,\chi}$ is finite.

Sufficiency. Assume, e.g., that $\alpha_2 + \alpha_3 > \alpha_1 + \alpha_4$. We show that, for every $\gamma = \alpha/2 - \alpha_1/(2n)$, $n \in \mathbb{N}$, there exists a representation of the algebra $\mathcal{P}_{\tilde{D}_4,\chi,\gamma}$. For this γ, we have $\chi_1^{(2n)} = 0$, $\chi_i^{(2n-1)} > 0$, $i = \overline{1,4}$, and $\chi_1^{(2n-1)} = \gamma^{(2n-1)}$ ($\chi_1^{(2n-1)}$ is equal to zero at the next step). According to Corollary 1, this algebra has a representation, and, hence, the initial algebra also has a representation. The case $\alpha_2 + \alpha_3 < \alpha_1 + \alpha_4$ is proved by analogy. If $\alpha_2 + \alpha_3 = \alpha_1 + \alpha_4$, then these two sets are infinite. With regard for the equalities

$$\frac{\alpha}{2} - \frac{\alpha - 2\alpha_4}{2(2n-1)} = \frac{\alpha}{2} - \frac{2\alpha_1 + 2\alpha_4 - 2\alpha_4}{2(2n-1)} = \frac{\alpha}{2} - \frac{\alpha_1}{2n-1},$$

we get

$$\Sigma_\infty = \left\{\frac{\alpha}{2} - \frac{\alpha_1}{n} \;\middle|\; n \in \mathbb{N}\right\}.$$

Remark 1. By analogy, we can show that, in the case where the condition $\alpha_i < \alpha/2$, $i = \overline{1,4}$, is satisfied, parallel with the infinite set Σ_∞, $\Sigma_{\tilde{D}_4,\chi}$ also contains a finite set Σ_0 defined by the following rule:

(i) $\Sigma_0 = \left\{\frac{\alpha}{2} - \frac{\alpha - 2\alpha_4}{2(2n-1)} \;\middle|\; n < \frac{\alpha_1}{\alpha_2 + \alpha_3 - \alpha_1 - \alpha_4},\; n \in \mathbb{N}\right\}$ if $\alpha_2 + \alpha_3 > \alpha_1 + \alpha_4$;

(ii) $\Sigma_0 = \left\{\frac{\alpha}{2} - \frac{\alpha_1}{2n} \;\middle|\; n < \frac{\alpha_1}{\alpha_1 + \alpha_4 - \alpha_2 - \alpha_3},\; n \in \mathbb{N}\right\}$ if $\alpha_2 + \alpha_3 < \alpha_1 + \alpha_4$;

(iii) $\Sigma_0 = \varnothing$ if $\alpha_2 + \alpha_3 = \alpha_1 + \alpha_4$.

Theorem 3. *Suppose that the numbers* $\alpha_i \in \mathbb{R}$, $i = \overline{1,4}$, *are such that* $\alpha_i < \alpha/2$. *Then there exists a collection of projectors* P_1, P_2, P_3, *and* P_4 *such that* $\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = \alpha I/2$.

Proof. It is necessary to show that the set $\Sigma_{\tilde{D}_4,\chi}$ contains the point $\alpha/2$. According to the Shulman theorem [7], the set $\Sigma_{\tilde{D}_4,\chi}$ is closed. The character $\chi = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfies the conditions of Theorem 2. Consequently, the set $\Sigma_{\tilde{D}_4,\chi}$ contains the infinite subset Σ_∞, and, hence, by virtue of closedness, it also contains the limit point of the series $\alpha/2$.

Note that this theorem was presented in a somewhat different form by Kirichenko (see, e.g., [5]).

4. Subsets in the Set $\Sigma_{\tilde{D}_4,\chi}$

As shown in the proof of Theorem 2, the problem reduces to the case of a smaller number of projectors if at least one component of the character satisfies the inequality $\chi_i \geq \alpha/2$. Therefore, in what follows, we assume without loss of generality that $\chi_i < \alpha/2$, $i = \overline{1, 4}$.

To describe other sets, we use Theorem 1. Let $\gamma \in \Sigma_{\tilde{D}_4,\chi}$ and let k be such that condition (4) is satisfied. Two cases are possible, namely, $k = 2n$ and $k = 2n-1$.

1. *Case* $k = 2n$. Using relations (2) and (3), we can rewrite condition (4) in the form of the following system of inequalities:

$$\lambda > \frac{\alpha_1}{2n}, \quad \lambda < \frac{\alpha - 2\alpha_4}{2(2n-1)},$$
$$\lambda < \frac{\alpha_1}{2(n-1)}, \quad \lambda \leq \frac{\alpha}{2(4n-1)}, \tag{6}$$

where, as above, $\lambda = \alpha/2 - \gamma$. By virtue of Theorem 1, condition (5) can be rewritten as follows:

$$0 = \frac{\alpha}{2} - (4n-1)\lambda,$$

$$\frac{\alpha}{2} - \alpha_i - (2n-1)\lambda = \frac{\alpha}{2} - (4n-1)\lambda,$$

$$\frac{\alpha}{2} - \alpha_i - (2n-1)\lambda + \frac{\alpha}{2} - \alpha_j - (2n-1)\lambda = \frac{\alpha}{2} - (4n-1)\lambda, \quad i, j = 2, 3, 4, \quad i \neq j, \tag{7}$$

$$\sum_{i=2}^{4}\left(\frac{\alpha}{2} - \alpha_i - (2n-1)\lambda\right) = \frac{\alpha}{2} - (4n-1)\lambda,$$

$$\sum_{i=2}^{4}\left(\frac{\alpha}{2} - \alpha_i - (2n-1)\lambda\right) = 2\left(\frac{\alpha}{2} - (4n-1)\lambda\right).$$

Solving the system of inequalities (6) for every λ that satisfies one of the equations in (7), we obtain the following subsets in $\Sigma_{\tilde{D}_4,\chi}$:

$$\Sigma_1 = \left\{\frac{\alpha}{2} - \frac{\alpha}{2(4n-1)} \;\middle|\; n < \frac{\alpha_4}{4\alpha_4 - \alpha},\ n < \frac{\alpha - \alpha_1}{\alpha - 4\alpha_1},\ n \in \mathbb{N}\right\},$$

$$\Sigma_2^i = \left\{ \frac{\alpha}{2} - \frac{\alpha_i}{2n} \;\middle|\; n < \frac{\alpha_i}{2\alpha_i + 2\alpha_4 - \alpha},\; n < \frac{\alpha_i}{\alpha_i - \alpha_1},\; n < \frac{\alpha_i}{4\alpha_i - \alpha},\; n \in \mathbb{N} \right\},$$

$$\Sigma_3 = \left\{ \frac{\alpha}{2} - \frac{\alpha - 2\alpha_1}{2(2n+1)} \;\middle|\; n < \frac{\alpha - \alpha_1}{\alpha - 4\alpha_1},\; n < \frac{\alpha_2 + \alpha_3}{2(\alpha_4 - \alpha_1)},\; n(4\alpha_i - \alpha) < \alpha_i,\; n \in \mathbb{N} \right\}, \quad i = 2, 3, 4.$$

2. *Case* $k = 2n + 1$. Reasoning as in the previous case, we obtain the system of inequalities

$$\lambda > \frac{\alpha - 2\alpha_4}{2(2n+1)}, \qquad \lambda < \frac{\alpha_1}{2n},$$
$$\lambda < \frac{\alpha - 2\alpha_4}{2(2n-1)}, \qquad \lambda \leq \frac{\alpha}{2(4n+1)} \tag{8}$$

and the equations

$$0 = \frac{\alpha}{2} - (4n+1)\lambda,$$

$$\alpha_i - 2n\lambda = \frac{\alpha}{2} - (4n+1)\lambda,$$

$$\alpha_i - 2n\lambda + \alpha_j - 2n\lambda = \frac{\alpha}{2} - (4n+1)\lambda, \tag{9}$$

$$\sum_{i=1}^{3} \alpha_i - 2n\lambda = \frac{\alpha}{2} - (4n+1)\lambda,$$

$$\sum_{i=1}^{3} \alpha_i - 2n\lambda = 2\left(\frac{\alpha}{2} - (4n+1)\lambda \right), \qquad i, j = 1, 2, 3, \qquad i \neq j.$$

Solving the system of inequalities (8) for every λ that satisfies one of the equations (9), we obtain the following subsets in $\Sigma_{\tilde{D}_4,\chi}$:

$$\Sigma_4 = \left\{ \frac{\alpha}{2} - \frac{\alpha}{2(4n+1)} \;\middle|\; n < \frac{\alpha - \alpha_4}{4\alpha_4 - \alpha},\; n < \frac{\alpha_1}{\alpha - 4\alpha_1},\; n \in \mathbb{N} \bigcup \{0\} \right\},$$

$$\Sigma_5^i = \left\{ \frac{\alpha}{2} - \frac{\alpha - 2\alpha_i}{2(2n+1)} \;\middle|\; n < \frac{\alpha_1}{\alpha - 2\alpha_i - 2\alpha_1},\; n < \frac{\alpha_i}{\alpha - 4\alpha_i},\; n < \frac{\alpha - \alpha_4 - \alpha_i}{2(\alpha_4 - \alpha_i)},\; n \in \mathbb{N} \bigcup \{0\} \right\}, \quad i = 1, 2, 3.$$

Thus, the structure of the set $\Sigma_{\tilde{D}_4,\chi}$ is completely described by the following theorem:

Theorem 4. *The set of* γ *for which the algebra* $\mathcal{P}_{\tilde{D}_4,\chi,\gamma}$ *has a representation is described by the following relation:*

$$\Sigma_{\tilde{D}_4,\chi} \cap [0; \alpha/2) = \Sigma_\infty \cup \Sigma_0 \cup \Sigma_1 \cup \Sigma_2^i \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_5^j, \quad i = 2, 3, 4, \quad j = 1, 2, 3.$$

The entire set $\Sigma_{\tilde{D}_4,\chi}$ *is obtained by symmetric mapping with respect to* $\alpha/2$ *and adjunction of the point* $\alpha/2$.

5. The Set $\Sigma_{\tilde{D}_4,\chi}$ for the Character $\chi_\delta = (1, 1, \delta, \delta)$

The structure of the set $\Sigma_{\tilde{D}_4,\chi}$ is considerably simplified if $\chi = \chi_1 = (1, 1, 1, 1)$ or $\chi_\delta = (1, 1, \delta, \delta)$ (see [5, 8]). We now show how to construct the set $\Sigma_{\tilde{D}_4,\chi}$ for the character $\chi_\delta = (1, 1, \delta, \delta)$ by using Theorems 2 and 4.

Using Proposition 2, we get

$$\Sigma_\infty = \left\{ 1 + \delta - \frac{1}{2n} \;\middle|\; n \in \mathbb{N} \right\}, \qquad \Sigma_0 = \varnothing.$$

The sets Σ_2^i, Σ_5^j, $i = 2, 3, 4$, $j = 1, 2, 3$, and Σ_3 do not exist, and the sets Σ_1 and Σ_4 take the form

$$\Sigma_1 = \left\{ 1 + \delta - \frac{1+\delta}{4n-1} \;\middle|\; n < \frac{\delta}{2(\delta-1)},\; n \in \mathbb{N} \right\},$$

$$\Sigma_4 = \left\{ 1 + \delta - \frac{1+\delta}{4n+1} \;\middle|\; n < \frac{1}{2(\delta-1)},\; n \in \mathbb{N} \cup \{0\} \right\}.$$

Thus,

$$\Sigma_{\tilde{D}_4,(1,1,\delta,\delta)} \cap [0; \alpha/2) = \left\{ 1 + \delta - \frac{1}{2n} \;\middle|\; n \in \mathbb{N} \right\} \cup \left\{ 1 + \delta - \frac{1+\delta}{4n-1} \;\middle|\; n < \frac{\delta}{2(\delta-1)},\; n \in \mathbb{N} \right\}$$
$$\cup \left\{ 1 + \delta - \frac{1+\delta}{4n+1} \;\middle|\; n < \frac{\delta}{2(\delta-1)},\; n \in \mathbb{N} \cup \{0\} \right\}.$$

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