ON QUALITATIVE PROBABILITY σ-ALGEBRAS

By C. Villegas

Instituto de Matemática y Estadística, Montevideo

1. Introduction and summary. The first clear and precise statement of the axioms of qualitative probability was given by de Finetti ([1], Section 13). A more detailed treatment, based however on more complex axioms for conditional qualitative probability, was given later by Koopman [5]. De Finetti and Koopman derived a probability measure from a qualitative probability under the assumption that, for any integer n, there are n mutually exclusive, equally probable events. L. J. Savage [6] has shown that this strong assumption is unnecessary. More precisely, he proves that if a qualitative probability is only fine and tight, then there is one and only one probability measure compatible with it.

No property equivalent to countable additivity has been used as yet in the development of qualitative probability theory. However, since the concept of countable additivity is of such fundamental importance in measure theory, it is to be expected that an equivalent property would be of interest in qualitative probability theory, and that in particular it would simplify the proof of the existence of compatible probability measures.

Such a property is introduced in this paper, under the name of monotone continuity. It is shown that, if a qualitative probability is atomless and monotonely continuous, then there is one and only one probability measure compatible with it, and this probability measure is countably additive. It is also proved that any fine and tight qualitative probability can be extended to a monotonely continuous qualitative probability, and therefore, contrary to what might be expected, there is no loss in generality if we consider only qualitative probabilities which are monotonely continuous.

At the present time there is still a controversy over the interpretation which should be given to the word probability in the scientific and technical literature. Although the present writer subscribes to the opinion that this interpretation may be different in different contexts, in this paper we do not enter into this controversy. We simply remark that a qualitative probability, as a numerical one, may be interpreted either as an objective or as a subjective probability, and therefore the following axiomatic theory is compatible with both interpretations of probability.

2. Events. In this section we shall review the basic concepts and results concerning events. Following Glivenko [2], Koopman [5] and Halmos [3] we shall consider the *events* as undefined terms, which may be subject to the usual operations of *union*, *intersection* and *complementation*, satisfying the axioms of a Boolean algebra [7]. The union and intersection of two events A and B will be

Received 18 September 1963; revised 9 March 1964.

denoted respectively by $A \cup B$ and $A \cap B$, and the complement of A by A^c. If, intuitively speaking, A implies B, we shall write $A \subset B$ (or $B \supset A$), and we shall say also that A is contained in B (or that B contains A). More formally, $A \subset B$ is defined by $A \cap B \equiv A$ (or, equivalently, by $A \cup B \equiv B$). In this paper the identity of events will be always denoted by the sign $\equiv$, leaving the sign $=$ for the relation "equally probable" to be introduced later. The relation $\subset$ is a partial order with a first element O, which is called the *impossible event*, and a last element, which is called the *sure event*. Two events will be called *incompatible*, or *mutually exclusive*, if their intersection is the impossible event. A fundamental result in the theory of Boolean algebras is that every Boolean algebra is isomorph to an algebra of sets of a space Ω. This result is due to Stone [8] and his construction of Ω has the following simple interpretation. A possible state ω of the algebra of events $\mathfrak{A}$ is a classification of the events into two classes: the events which "*occur*" and the events which "*do not occur*". This classification is such that,

(i) if A occurs, then A^c does not occur;
(ii) if $A \subset B$ and A occurs, then B occurs;
(iii) if $A \subset B$ and B does not occur, then A does not occur either.

Such classifications always exist; but all known proofs of this fact are based on the axiom of choice or its equivalents. Among all possible states ω there is one which is the *real state*; it is the classification into the sets which really occur and the sets which do not really occur. The set Ω of all possible states ω is the *Stone space* of the algebra $\mathfrak{A}$. Given an event A, let $\bar{A}$ be the set of all states ω for which A occurs. Since the sure event always occurs, the corresponding set is the Stone space itself. Let $\bar{\mathfrak{A}} \equiv \{\bar{A} : A \varepsilon \mathfrak{A}\}$ be the family of all such sets of possible states. (The words set, family and collection will be used as synonyms in this paper.) Note that every $\bar{A}$ is a set of Ω, but not necessarily every set of Ω belongs to $\bar{\mathfrak{A}}$. However, $\bar{\mathfrak{A}}$ is closed with respect to the usual set-theoretical operations of union, intersection and complementation, so that, with respect to them, $\bar{\mathfrak{A}}$ is a Boolean algebra of sets. It can be proved that the natural correspondence $A \rightarrow \bar{A}$ is an isomorphism between the two Boolean algebras $\mathfrak{A}$ and $\bar{\mathfrak{A}}$. We may identify therefore the elements of $\mathfrak{A}$ with the corresponding elements of $\bar{\mathfrak{A}}$, that is, we can think of the events as sets of states. In particular, the sure event can be considered as the Stone space Ω itself. It can be shown that the algebra of events $\mathfrak{A}$, considered as an algebra of sets of the Stone space Ω, is not a σ-algebra of sets; more precisely, the union of any denumerable collection of disjoint sets of $\mathfrak{A}$ does not belong to $\mathfrak{A}$ [7]. However, there is no loss of generality in assuming that the algebra of events $\mathfrak{A}$ is a σ-algebra of sets of some space Ω, because it can always be extended to one. Consider, in effect, the case in which we are given a Boolean algebra of events $\mathfrak{A}_0$. Let as before $\bar{\mathfrak{A}}_0$ be the corresponding algebra of sets in the Stone space Ω. Consider the σ-algebra of sets $\mathfrak{A}$ generated by $\bar{\mathfrak{A}}_0$; i.e., the smallest σ-algebra of sets of Ω which contains $\bar{\mathfrak{A}}_0$. This is an extension of $\mathfrak{A}_0$, and, since the elements of this σ-algebra are also sets of states, they have also the same simple interpretation.

3. Qualitative probability. If, intuitively speaking, A and B are equally probable events, we shall write $A = B$. If A is strictly less probable (strictly more probable) than B, then we shall write $A < B$ (resp. $A > B$). In both cases we shall write $A \leqq B$ (resp. $A \geqq B$) and we shall say that A is less probable (resp. more probable) than B. More formally, a relation $\leqq$ between events will be called a *qualitative probability*, if it satisfies the following axioms (in which $A < B$ is defined by $A \leqq B$ but not $B \geqq A$):

AXIOM OF PRE-ORDER Q1. The relation $\leqq$ is a total pre-order between events, with O as the first element and Ω as the last one. That is,

(a) if A and B are events, then either $A \leqq B$ or $B \leqq A$;

(b) for any event A, $A \leqq A$;

(c) if $A \leqq B$ and $B \leqq C$, then $A \leqq C$;

(d) $O < \Omega$, and for any event A, $O \leqq A \leqq \Omega$.

AXIOM OF MONOTONY Q2. If $B_1 \cap B_2 \equiv O$, then from $A_1 \leqq B_1$, $A_2 \leqq B_2$ it follows that $A_1 \cup A_2 \leqq B_1 \cup B_2$ and if, in one of the first two inequalities, the sign $\leqq$ is replaced by $<$, then the last one holds with the sign $\leqq$ replaced by $<$.

These axioms are equivalent to those proposed by de Finetti [1]. For the proof, see [6], p. 32, Exercise 5a. Formally, the relation $A = B$, meaning that A and B are equally probable, is defined by $A \leqq B$ and $B \leqq A$. If $A = O$, we shall say that A is a *null* or an *almost impossible event*, and if $A = \Omega$, we shall say that A is an *almost sure event.* Two events will be called *almost incompatible*, if they have a null intersection. If $\mathcal{A}$ is an algebra of events and $\leqq$ is a qualitative probability defined on $\mathcal{A}$, the pair $(\mathcal{A}, \leqq)$ will be called a *qualitative probability algebra*. The following propositions are easy consequences of the Axioms ([6], p. 32):

(1) *Strenghtened transitivity.* If $A \leqq B$ and $B < C$, then $A < C$.

(2) *Compatibility with the inclusion.* If $A \subset B$, then $A \leqq B$.

(3) If $A_2 \subset A_1$ and $B_2 \subset B_1$, then from $A_1 \leqq B_1$, $A_2 \geqq B_2$ it follows that $A_1 - A_2 \leqq B_1 - B_2$.

(4) *Duality.* If $A \leqq B$, then $B^c \leqq A^c$.

DEFINITION. We shall say that a qualitative probability, defined on a σ-algebra of sets of a space Ω, is *monotonely continuous*, if, given a monotone increasing sequence of events $A_n \uparrow A$ (that is, a sequence $A_1 \subset A_2 \subset \cdots$ converging to A), and an event B such that, for every n, $A_n \leqq B$, then $A \leqq B$. If $\mathcal{A}$ is a σ-algebra of sets of a space Ω, and $\leqq$ is a monotonely continuous qualitative probability defined on $\mathcal{A}$, then the pair $(\mathcal{A}, \leqq)$ will be called a *qualitative probability σ-algebra.*

THEOREM 1. *A necessary and sufficient condition in order that $\leqq$ be monotonely continuous, is that given a monotone increasing sequence of events $A_n \uparrow A$, and an event $B < A$, there exists an integer $N > 0$ such that, for $n \geqq N$, we have $B < A_n$.*

(The proof is left to the reader.)

In a qualitative probability σ-algebra the following propositions are true.

PROPOSITION 1. If $A_n \uparrow A$ and $B_n \uparrow B$ (or, alternatively, $A_n \uparrow A$ and $B_n \downarrow B$), then from $A_n \leqq B_n$, $n = 1, 2, \cdots$ it follows that $A \leqq B$.

PROPOSITION 2. If $A_n \uparrow A$ and $B_n \uparrow B$ (or, alternatively, $A_n \uparrow A$ and $B_n \downarrow B$), then from $A_n = B_n$, $n = 1, 2, \cdots$ it follows that $A = B$.

PROPOSITION 3. If $B_n \cap B_m = O$ for all $m \neq n$, then from $A_n \leqq B_n$, $n = 1, 2, \cdots$ it follows that

$$\bigcup^{\infty} A_n \leqq \bigcup^{\infty} B_n .$$

LEMMA 1. *If* $(\mathfrak{A}, \leqq)$ *is a qualitative probability* σ*-algebra, and if* $\{A_i : A_i \varepsilon \mathfrak{A}, i \varepsilon I\}$ *is an infinite collection of almost incompatible events, then, given an event* $A > O$, *there is only a finite number of* A_i*'s which are more probable than* A.

PROOF. Assume, on the contrary, that there is a sequence of almost incompatible events $\{A_{i_n} : n = 1, 2, \cdots\}$ such that for all n we have $A_{i_n} \geqq A$. Consider the sequence $S_m \equiv \bigcup_{n=m}^{\infty} A_{i_n}$. Obviously $S_m \downarrow S = O$, $S_m \geqq A$, and therefore, by the dual of the definition of monotone continuity, $O \geqq A$, contrary to the hypothesis.

COROLLARY. *If* $\leqq$ *is monotonely continuous, and if* $\{A_i : i \varepsilon I\}$ *is an infinite collection of almost incompatible, equally probable events, then all the* A_i*'s are null events.*

LEMMA 2. *If* $(\mathfrak{A}, \leqq)$ *is a qualitative probability* σ*-algebra, and* $\{A_n\}$ *is a monotone decreasing sequence of events, such that* $A_{n+1} \leqq A_n - A_{n+1}$, *then* A_n *converges to a null event.*

PROOF. Since the events $A_n - A_{n+1}$ are incompatible, from $\bigcap_{n=1}^{\infty} A_n \leqq A_{n+1} \leqq A_n - A_{n+1}$, the conclusion $\bigcap_{n=1}^{\infty} A_n = O$ follows immediately by Lemma 1.

We shall say that an event $A > O$ is an *atom* if it contains no other event B such that $O < B < A$. A qualitative probability algebra (and the corresponding qualitative probability) is *atomless* if it has no atoms.

LEMMA 3. *If a qualitative probability* σ*-algebra is atomless, then given an event* $A > O$, *there is a monotone decreasing sequence of non-null events* $\{A_n\}$, *whose first element is* $A_1 \equiv A$, *and which converges to a null event.*

PROOF. Since $A_1 \equiv A > O$ and there are no atoms, there is an event $B_1 \subset A_1$ such that $O < B_1 < A_1$. If $B_1 \leqq A_1 - B_1$, then we define $A_2 \equiv B_1$. In the contrary case, we define $A_2 \equiv A_1 - B_1$ and in both cases we have $A_2 \leqq A_1 - A_2$. Proceeding indefinitely in this way, we can define a sequence of events such that $A \equiv A_1 \supset A_2 \supset \cdots$ and $A_{n+1} \leqq A_n - A_{n+1}$, and therefore, by the previous lemma, this sequence converges to a null event.

THEOREM 2. *If a qualitative probability* σ*-algebra is atomless, then every family* $\mathfrak{F}$ *of almost incompatible, non-null events is at most denumerable.*

PROOF. Let $\{C_n : n = 1, 2, \cdots\}$ be a monotone decreasing sequence of non-null events whose intersection is null. Such sequences exist by the Lemma 3. If an event is less probable than any C_n, then by monotone continuity it is a null event. Hence, if $\mathfrak{F}$ is a collection of almost incompatible, non-null events, all events of $\mathfrak{F}$ are more probable than some C_n. But, by Lemma 1, given an integer n, there can be only a finite number of events in $\mathfrak{F}$ which are more probable than C_n and consequently the family $\mathfrak{F}$ is at most a denumerable family.

The following lemma, and the theorem which follows it were communicated to the author by J. J. Schäffer. Although they are not an essential link in our arguments, they are included here for the sake of completeness.

LEMMA 4. *In a qualitative probability σ-algebra there is at most a denumerable number of atoms, and they can be ordered in a sequence $\{A_i : i = 1, 2, \cdots\}$ such that for any integer n, $A_n \geqq A_{n+1}$.*

PROOF. Assume that we have already selected the atoms $A_1, \cdots, A_n$ in such a way that they are more probable than any of the remaining ones. Let A be one of the remaining atoms. By the Lemma 1, there is only a finite number of remaining atoms which are more probable than A, and therefore, we can select as the element A_{n+1} of our sequence one atom which is more probable than any of the remaining ones. Assume that, following this procedure, we have selected an infinite sequence of atoms. Then, there can be no other atoms left, because if an atom is not in the sequence, all the atoms of the sequence would be more probable than it, which is impossible by the Lemma 1.

THEOREM 3. *Every family $\mathfrak{F}$ of almost incompatible, non-null events of a qualitative probability σ-algebra is at most a denumerable family.*

PROOF. Let Ω_0 be the union of all the atoms and let $\Omega_1 \equiv \Omega - \Omega_0$. By partitioning every event in $\mathfrak{F}$ in one part contained in Ω_0 and one part contained in Ω_1, we have a new family $\mathfrak{F}'$. Let $\mathfrak{F}_0'$, $\mathfrak{F}_1'$ be the collections of all non-null events in $\mathfrak{F}'$ which are contained in Ω_0 and in Ω_1 respectively. By the previous lemma, $\mathfrak{F}_0'$ is at most a denumerable family, and by Theorem 2 the same holds for $\mathfrak{F}_1'$. Hence, $\mathfrak{F}'$ and a fortiori $\mathfrak{F}$ is at most a denumerable family.

DEFINITION. A *partition* of an event A is a collection of almost incompatible, non-null events whose union is A. An *incomplete partition* of an event A is a collection of almost incompatible, non-null events contained in A. Note that, by the previous theorem, the partitions and incomplete partitions of an event are at most denumerable collections of events. If the union B of all the elements of an incomplete partition of A is less probable than $A - B$, then the incomplete partition will be called a *minor incomplete partition.*

THEOREM 4. *If a qualitative probability σ-algebra is atomless, then every event can be partitioned into two equally probable events.*

PROOF. Given an event $A > O$, consider the set S of all minor incomplete partitions of A. Let S be partially ordered by inclusion. That is, if $\mathfrak{B}$, $\mathfrak{C}$ are two minor incomplete partitions, we say that $\mathfrak{B} \leqq \mathfrak{C}$ if every event of $\mathfrak{B}$ is also an event of $\mathfrak{C}$. The family $\mathfrak{U}$ of all the elements in a chain of minor incomplete partitions of A is itself a minor incomplete partition of A, and is therefore an upper bound in S for the chain. Then, by Zorn's lemma, there is a maximal partition in S, that is, a minor incomplete partition $\mathfrak{B}$ which is not properly contained in any other minor incomplete partition. Hence, if B is the union of all the elements of $\mathfrak{B}$, and $\{C_n\}$ is a monotone decreasing sequence of non-null events contained in $C \equiv A - B$, and converging to a null event, we have for every n, $B \cup C_n > A - B \cup C_n$, and therefore, by monotone continuity, $B \geqq C$. Since $\mathfrak{B}$ is a minor incomplete partition $B \leqq C$ and therefore $B = C$. Consider a sequence of random variables $X_1, X_2, \cdots$ defined on the same space Ω and with values on the same space E. If E is a finite space, we shall say that the sequence $\{X_n\}$ is a sequence of *finitely-valued random variables.* If $x_1, \cdots, x_n$ are points in E, then we shall denote by $\langle x_1, \cdots, x_n\rangle$ the event $\{\omega: X_i(\omega) = x_i, i = 1, \cdots, n\}$. A sequence of finitely-valued random variables $\{X_n\}$ will be called a *uniform sequence* if, for any given m, all the events $\langle x_1, \cdots, x_m\rangle$ are equally probable. Given an infinite sequence $x_1, x_2, \cdots$ of points of the finite space E, we shall denote by $\langle x_1, x_2, \cdots\rangle$ the event $\{\omega: X_i(\omega) = x_i, i = 1, 2, \cdots\}$. This event is obviously the intersection of all the events $X_n^{-1}(x_n) \equiv \{\omega: X_n(\omega) = x_n\}$. If $\{X_n\}$ is a uniform sequence, then by monotone continuity all the events $\langle x_1, x_2, \cdots\rangle$ are equally probable, and therefore, by the corollary to Lemma 1, all of them are null events.

Consider a real random variable X with values in the real closed interval $[0, 1]$. Let I, J be two intervals (not necessarily closed) contained in $[0, 1]$ and let $|I|, |J|$ be their lengths. The random variable X will be called a *uniform random variable* if $X^{-1}(I) \leqq X^{-1}(J)$ is equivalent to $|I| \leqq |J|$. From the corollary to Lemma 1 it follows that for any $0 \leqq a \leqq 1$, $X^{-1}(a)$ is a null event. It follows also that if $0 = a_0 < a_1 < \cdots < a_n = 1$, then the events $X^{-1}[a_0, a_1], \cdots, X^{-1}[a_{n-1}, a_n]$ constitute a finite partition of Ω.

THEOREM 5. *In a qualitative probability σ-algebra the following propositions are equivalent*:

(i) *there are no atoms;*

(ii) *every event can be partitioned into two equally probable events;*

(iii) *there is a uniform sequence of two-valued random variables;*

(iv) *there is a uniform random variable.*

PROOF. By Theorem 4, (ii) follows from (i). We shall prove now that (iii) follows from (ii). Divide Ω into two equally probable events and define $X_1(\omega) = 0$ on one of them and $X_1(\omega) = 1$ on the other. The two events considered are, therefore, $\langle 0\rangle$ and $\langle 1\rangle$. Dividing each of them into two equally probable events, and defining X_2 in an obvious manner, we obtain the events $\langle 0, 0\rangle$, $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, and $\langle 1, 1\rangle$. If this procedure is continued, a uniform sequence of two-valued random variables is defined.

We shall prove now that (iv) follows from (iii). Let $\{X_n\}$ be a uniform sequence of two-valued random variables. For each ω define $X(\omega)$ as the number which in the binary system has the expression $0.x_1x_2 \cdots$, where $x_n = X_n(\omega)$. For a given integer n, the family of all events $\langle x_1, \cdots, x_n\rangle$ constitute a *uniform partition* of Ω, that is, a partition all the elements of which are equally probable. If a, b are numbers which belong to $[0, 1]$ and have finite binary expressions $0.a_1 \cdots a_n$, $0.b_1 \cdots b_n$, then $X^{-1}[a, b]$ is an event equally probable than the union of all the events $\langle x_1, \cdots, x_n\rangle$ corresponding to numbers $0.x_1 \cdots x_n$ which belong to the semi-closed interval $[a, b)$. It follows then that $|I| \leqq |J|$ is equivalent to $X^{-1}(I) \leqq X^{-1}(J)$ for two intervals I, J whose extreme points are numbers with finite binary expressions, and by monotone continuity the same holds for any intervals I, J.

We shall prove now that (i) follows from (iv), or, equivalently, that, if there is an atom, then there is no uniform random variable. Let $A > O$ be an atom, and let X be a real random variable, whose values lie on $[0, 1]$. For any x, $0 \leqq x \leqq 1$, either $A \cap X^{-1}[0, x)$ or $A \cap X^{-1}(x, 1]$ is a null event, where $[0, x)$ denotes the interval closed at 0 and open at x, and similarly, $(x, 1]$ denotes the interval open at x and closed at 1. By a continuity argument, there is a number c, $0 \leqq c \leqq 1$, such that, if $a < c$, then $A \cap X^{-1}[0, a)$ is a null event, and, if $c < b$, then $A \cap X^{-1}(b, 1]$ is a null event. Hence $A \cap X^{-1}[a, b]$ is equally probable to A, and therefore, $X^{-1}[a, b] \geqq A$, for all a, b such that $a < c < b$. By monotone continuity $X^{-1}(c) \geqq A > O$, and therefore X is not uniform, since, for any uniform random variable, $X^{-1}(c) = O$.

4. Probability measures. A *probability measure* is a function which to every event A assigns a number $P(A)$, called its (numerical) probability, and which satisfies the following axioms:

P1. $P(O) = 0$, $P(\Omega) = 1$, and for any event A, $0 \leqq P(A) \leqq 1$.

P2. If $A \cap B \equiv O$, then $P(A \cup B) = P(A) + P(B)$.

If P is a probability measure defined on the algebra of events $\mathcal{A}$, the pair $(\mathcal{A}, P)$ is called a *probability algebra.* If $\mathcal{A}$ is a σ-algebra of sets of a space Ω, we shall say that the probability measure P is *countably additive* if, in addition to P1 and P2 it satisfies also the axiom

P3. If $\{A_i : i = 1, 2, \cdots\}$ is a sequence of events such that, for $i \neq j$, $A_i \cap A_j \equiv O$, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

The Axioms P1, P2 and P3 are essentially equal to those proposed by Kolmogorov (1933), and they provide the basis for the modern theory of probability. It follows easily that, if a probability measure is countably additive, and $\{A_i\}$ is a monotone sequence of events, then $\lim P(A_i) = P(\lim A_i)$. The pair $(\mathcal{A}, P)$ in which $\mathcal{A}$ is a σ-algebra of sets and P is a countably additive probability measure, is called a probability σ-algebra. A *probability space* is a triple $(\Omega, \mathcal{A}, P)$ in which Ω is a set, $\mathcal{A}$ is a σ-algebra of sets of Ω, and P is a countably additive probability measure defined on $\mathcal{A}$. We shall say that a probability algebra $(\mathcal{A}, P)$ is an extension of a probability algebra $(\mathcal{A}_0, P_0)$ if $\mathcal{A}_0$ is isomorph to a subalgebra $\mathcal{A}'$ of $\mathcal{A}$, and P_0 and P take the same values on corresponding elements of $\mathcal{A}_0$ and $\mathcal{A}'$.

THEOREM 1. *Any probability algebra can be extended to a probability σ-algebra.*

PROOF. For the proof of this theorem see ([7] Section 46).

REMARK. Note that, since the countably additive probability measure P whose existence is assured by this theorem can be defined by writing

$$P(A) = \sup \{P_0(A_0): A_0 \subset A, A_0 \varepsilon \mathcal{A}_0\}$$

for any $A \varepsilon \mathcal{A}$, it follows that, if the given probability measure P_0 is atomless, then P is also atomless.

We shall say that a probability measure P is *compatible* with a qualitative probability $\leqq$, if and only if $A \leqq B$ is equivalent to $P(A) \leqq P(B)$, for any pair of events A, B. It can be easily seen that, given a probability measure P defined on an algebra $\mathcal{A}$, the relation defined by $A \leqq B$ if $P(A) \leqq P(B)$ is a qualitative probability compatible with the given probability measure P.

THEOREM 2. *A necessary and sufficient condition in order that a probability measure P which is compatible with a qualitative probability $\leqq$, be countably additive, is that the qualitative probability be monotonely continuous.*

PROOF. Assume that P is countably additive. Consider a sequence of events $A_n \uparrow A$, and assume that for all n, $A_n \leqq B$. Since this is equivalent to $P(A_n) \leqq P(B)$, by countable additivity $P(A) \leqq P(B)$, which is equivalent to $A \leqq B$.

Conversely, assume that $\leqq$ is monotonely continuous. It is well known that, in order to prove that P is countably additive, it is sufficient to prove that P is continuous from above at O, i.e., that, if $A_n \downarrow O$, then $\lim P(A_n) = 0$. Consider the sequence $B_n \equiv A_n - A_{n+1}$. If there is an integer $N > 0$ such that for all $n \geqq N$, B_n is a null event, then, for all $n \geqq N$, we have $A_n = O$ and therefore $P(A_n) = 0$. Assume that, on the contrary, there is an infinite number of non-null events B_n. Then, since they are disjoint events, given a number $\epsilon > 0$, there is an integer $m > 0$ such that $0 < P(B_m) < \epsilon$, and by compatibility, $B_m > O$. Then, by the dual of the Theorem 1 in the previous section, there is an integer $N > 0$ such that, for $n > N$, $A_n \leqq B_m$ and therefore $P(A_n) \leqq \epsilon$.

THEOREM 3. *If a qualitative probability σ-algebra is atomless, then there is one and only one compatible probability measure, and it is countably additive.*

PROOF. Since the qualitative probability σ-algebra is atomless, by Theorem 5 of the previous section, there is a uniform random variable X. Given a number x, $0 \leqq x \leqq 1$, consider the event $F(x) \equiv X^{-1}[0, x]$. Obviously $F(0) = O$, $F(1) \equiv \Omega$ and $F(x)$ is a monotone function of x. It is also a continuous function, in the sense that, if $\{x_n\}$ is a monotone sequence converging to x, then $\lim F(x_n)$ and $F(x)$ are equally probable events, or, in symbols, $F(x) = \lim F(x_n)$. Then, by a continuity argument, it can be easily shown that, given an arbitrary event A, there is one and only one number a, $0 \leqq a \leqq 1$, such that $A = X^{-1}[0, a]$. Then, for any event A, we define $P(A)$ by the equality $A = X^{-1}[0, P(A)]$. It follows immediately from this definition that the Axiom P1 is satisfied and that P is compatible with the qualitative probability. From the compatibility it follows immediately that P is monotone, i.e., that, if $A \subset B$, then $P(A) \leqq P(B)$.

We shall prove now that the function P just defined satisfies also the Axiom P2, and is, therefore, a probability measure. Let A, B be two events such that $A \cap B \equiv 0$. By the definition of P,

$$A = X^{-1}[0, P(A)], \qquad A \cup B = X^{-1}[0, P(A \cup B)].$$

Note that, since P is monotone, $P(A) \leqq P(A \cup B)$ and therefore, by subtraction we get

$$B = X^{-1}[P(A), P(A \cup B)].$$

But by the definition of $P(B)$, $B = X^{-1}[0, P(B)]$, and since X is a uniform variable,

$$P(B) = P(A \cup B) - P(A),$$

that is, P is finitely additive.

To prove that P is the only probability measure compatible with the given qualitative probability, it is enough to note that, since X is a uniform random variable, the events

$$\{\omega: (i-1)/n \leqq X(\omega) \leqq i/n\}$$

$i = 1, \cdots, n$ constitute a uniform partition of Ω, and therefore, for any rational a, $0 \leqq a \leqq 1$, and any probability measure compatible with the given qualitative probability, $P\{\omega: 0 \leqq X(\omega) \leqq a\} = a$. Finally, by the previous theorem, the probability measure P is countably additive.

A qualitative probability algebra (and the corresponding qualitative probability) is *fine*, if given an event $A > O$ there is a finite partition of Ω all elements of which are less probable than A. Two events A, B are *almost equivalent*, if and only if, for all non-null A' and B' such that $A \cap A' \equiv B \cap B' \equiv O$, we have $A \cup A' \geqq B$ and $B \cup B' \geqq A$. A qualitative probability algebra (and the corresponding qualitative probability) is *tight*, if and only if every pair of almost equivalent events are equally probable. We shall say that a qualitative probability algebra $(\mathcal{A}, \leqq)$ is an *extension* of a qualitative probability algebra $(\mathcal{A}_0, \leqq_0)$, if $\mathcal{A}_0$ is isomorph to a subalgebra $\mathcal{A}'$ of $\mathcal{A}$, and $A \leqq_0 B$ is equivalent to $A' \leqq B'$, where A, B and A', B' are corresponding pairs of events in $\mathcal{A}_0$ and $\mathcal{A}'$.

THEOREM 4. *If a qualitative probability algebra is fine and tight, then it can be extended to a qualitative probability σ-algebra.*

PROOF. Let $(\mathcal{A}_0, \leqq_0)$ be the given qualitative probability algebra. Since by hypothesis it is fine and tight, by L. J. Savage's theorem [6] there is one and only one probability measure P_0 defined on $\mathcal{A}_0$ and compatible with $\leqq_0$. By Theorem 1, $(\mathcal{A}_0, P_0)$ can be extended to a probability σ-algebra $(\mathcal{A}, P)$. Define now a qualitative probability $\leqq$ on $\mathcal{A}$ by writing $A \leqq B$ if and only if $P(A) \leqq P(B)$. Since P is countably additive, from Theorem 2 it follows that $\leqq$ is monotonely continuous.

REMARK. If, in addition, $(\mathcal{A}_0, \leqq_0)$, and consequently, also $(\mathcal{A}_0, P_0)$, are atomless, then by the remark to Theorem 1 the probability σ-algebra $(\mathcal{A}, P)$ and consequently also $(\mathcal{A}, \leqq)$ are atomless. Note that, on the other hand, if a qualitative probability algebra is fine and it has an atom, then there is a finite uniform partition all elements of which are atoms, and therefore the same qualitative probability algebra is also a qualitative probability σ-algebra.

THEOREM 5. *If a qualitative probability σ-algebra is atomless, then it is fine and tight.*

PROOF. Let A and B be two almost equivalent events. Let A', B' be events such that $A \cap A' \equiv B \cap B' \equiv O$. Then, by Lemma 3 of the previous section, there are monotone sequences of non-null events, $\{A_n\}$, $\{B_n\}$, whose first elements are A', B' and which converge to null events. Hence we have, for all n, $A \cap A_n \equiv B \cap B_n \equiv O$, and, since A and B are almost equivalent events, we have

$A \cup A_n \geqq B$, $B \cup B_n \geqq A$, and letting $n \to \infty$, the conclusion $A = B$ follows immediately by the dual of the definition of monotone continuity, and therefore the qualitative probability is tight.

We shall prove now that it is also fine. Consider an event $A > O$. Then, if P is the probability measure compatible with the qualitative probability, $P(A) > 0$. Choose then an integer n such that $1/2^n \leqq P(A)$, and let $\mathcal{E}$ be a uniform partition of Ω with 2^n elements. If E_i is any element of $\mathcal{E}$, we have clearly $P(E_i) \leqq P(A)$ and by compatibility, $E_i \leqq A$.

REFERENCES

[1] DE FINETTI, BRUNO (1931). Sul significato soggetivo della probabilita. *Fund. Math.* **17** 298–329.

[2] GLIVENKO, B. I. (1939). *A Course in Probability Theory* (in Russian). Moscow and Leningrad.

[3] HALMOS, P. R. (1944). The foundations of probability. *Amer. Math. Monthly* **51** 497–510.

[4] KOLMOGOROFF, A. (1933). *Grundbegriffe der Wahrscheinlichkeitsrechnung.* Springer, Berlin.

[5] KOOPMAN, B. O. (1940). The axioms and algebra of intuitive probability. *Ann. of Math.* **41** 269–292.

[6] SAVAGE, L. J. (1954). *The Foundations of Statistics.* Wiley, New York.

[7] SIKORSKI, ROMAN (1960). *Boolean Algebras.* Springer, Berlin.

[8] STONE, M. H. (1936). The theory of representation for Boolean algebras. *Trans. Amer. Math. Soc.* **40** 37–111.