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ON QUASI-EINSTEIN ALMOST HYPERBOLIC HERMITIAN MANIFOLD WITH QUASI-CONSTANT CURVATURE

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Abstract. In 1954 almost hyperbolic Hermitian manifold introduced by P. Libermann were classified for the first time in 1988 by C. L. Bejan. Recently in 1998 C. L. Bejan and L. Ornea constructed an example of an almost hyperbolic Hermitian manifold. Object of present paper is to study properties of quasi-Einstein almost hyperbolic Hermitian manifold with quasi-constant curvature.

1. Introduction

Let us consider a differentiable manifold M_{2n} of class C^∞ endowed with a tensor field of type $(1,1)$ F such that for an arbitrary vector field X.

$$\overline{\overline{X}} = X \tag{1.1}$$

where $\overline{\overline{X}} \overset{\text{def}}{=} F(X)$.

Then F is called *an almost product structure*, and the differentiable manifold M_{2n} is called *an almost product manifold*.

On an almost product manifold M_{2n} if there exists a metric tensor g such that

$$g(\overline{X}, \overline{Y}) + g(X, Y) = 0 \tag{1.2}$$

Then we say that g is compatible with almost product structure and $\{F, g\}$ is called *almost hyperbolic Hermitian structure*. The manifold M_{2n} with an almost hyperbolic Hermitian structure is said to be *almost hyperbolic Hermitian manifold*.

A Riemannian manifold (M_n, g) $(n > 3)$ is siad to be quasi-constant curvature [1] $(QC)_n$ if its curvature tensor $'K$ of tye $(0,4)$ satisfies

$$\begin{aligned} 'K(X < Y < Z < W) = {} & a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + b[g(Y,Z)A(X)A(W) \\ & -g(X,Z)A(Y)A(W) + g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)] \end{aligned} \tag{1.3}$$

where a, b are scalars with $b \neq 0$ and A is non zero 1-form defined as

$$A(X) = g(X, U) \qquad \forall X, \tag{1.4}$$

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And U is a unit vector field known as generator of the manifold.

A non flat Riemannian manifold (M_n, g) is said to be quasi-Einstein [2] if it's Riccitensor i.e. Ric of type $(0, 2)$ is not identically zero and satisfies

$$Ric(X, Y) = ag(X, Y) + bA(X)A(Y) \tag{1.5}$$

where a, b are scalars with $b \neq 0$ and A is non zero 1-form defined as

$$A(X) = g(X, U) \qquad \forall X, \tag{1.6}$$

And U is a unit vector field.

2. Scalar curvature

Theorem 2.1. *On an almost hyperbolic Hermitian manifold with quasi-constant curvature, the associated scalars a and b are given by the relation*

$$a = \frac{(2n-3)b}{n(n-2)}. \tag{2.1}$$

Proof. From (1.3), we have

$$\begin{aligned} K(X, Y, Z) = a[g(Y, Z)X - g(X, Z)Y] + b[g(Y, Z)A(X)U - g(X, Z)A(Y)U \\ +A(Y)A(Z)X - A(X)A(Z)Y] \end{aligned} \tag{2.2}$$

and from (2.2) we have

$$\begin{aligned} K(\bar{X}, \bar{Y}, Z) = a[g(\bar{Y}, Z)\bar{X} - g(\bar{X}, Z)\bar{Y}] + b[g(\bar{Y}, Z)A(\bar{X})U - g(\bar{X}, Z)A(\bar{Y})U \\ +A(\bar{Y})A(Z)\bar{X} - A(\bar{X})A(Z)\bar{Y}] \end{aligned} \tag{2.3}$$

From (2.2), (2.3) and using (1.1) with (1.2) i.e. $[{}'F(X, Y) + {}'F(Y, X) = 0 = g(\bar{X}, Y) + (X, \bar{Y})]$ and [3]

$${}'K(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = {}'K(X, Y, Z, W) = {}'K(X, Y, \bar{Z}, \bar{W}). \tag{2.4}$$

We have

$$\begin{aligned} &a[g(Y, Z)X - g(X, Z)Y] + b[g(Y, Z)A(X)U - g(X, Z)A(Y)U + A(Y)A(Z)X - A(X)A(Z)Y] \\ &= a[{}'F(Y, Z)\bar{X} - {}'F(X, Z)\bar{Y}] \\ &\quad + b[{}'F(Y, Z)A(\bar{X})U - {}'F(X, Z)A(\bar{Y})U + A(\bar{Y})A(Z)\bar{X} - A(\bar{X})A(Z)\bar{Y}] \end{aligned} \tag{2.5}$$

On contracting (2.6) with respect to X and Y respectively, we have

$$n(n-2)a + b(2n-3) = 0. \tag{2.6}$$

Theorem 2.2. *On an almost hyperbolic Hermitian manifold with quasi-constant curvature, the scalar curvature r is given by*

$$r = n[(n-1)a - 2b] \tag{2.7}$$

provided quasi-constant curvature satisfies (2.4).

Proof. From (1.3), we have

$$'K(\bar{X},\bar{Y},\bar{Z},\bar{W}) = a[g(\bar{Y},\bar{Z})g(\bar{X},\bar{W}) - g(\bar{X},\bar{Z})g(\bar{Y},\bar{W})] + b[g(\bar{Y},\bar{Z})A(\bar{X})A(\bar{W}) \\ -g(\bar{X},\bar{Z})A(\bar{Y})A(\bar{W}) + g(\bar{X},\bar{W})A(\bar{Y})A(\bar{Z}) - g(\bar{Y},\bar{W})A(\bar{X})A(\bar{Z})]. \tag{2.8}$$

Using (2.4) and (2.1) in (2.9), we have

$$'K(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] - b[g(Y,Z)A(\bar{X})A(\bar{W}) \\ -g(X,Z)A(\bar{Y})A(\bar{W}) + g(X,W)A(\bar{Y})A(\bar{Z}) - g(Y,W)A(\bar{X})A(\bar{Z})]. \tag{2.9}$$

From (2.9), we have

$$K(X,Y,Z,W) = a[g(Y,Z)X - g(X,Z)Y] - b[g(Y,Z)A(\bar{X})\bar{U} - g(X,Z)A(\bar{Y})\bar{U}] \\ -b[A(\bar{Y})A(\bar{Z})X - A(\bar{X})A(\bar{Z})Y] \tag{2.10}$$

On contracting (2.10) with respect to X and using (1.4) and (1.1) with (1.2) i.e. $['F(X,Y) + 'F(Y,X) = 0 = g(\bar{X},Y) + (X,\bar{Y})]$, we have

$$Ric(Y,Z) = [(n-1)a - b]g(Y,Z) - nb[A(\bar{Y})A(\bar{Z})]. \tag{2.11}$$

Again constracting (2.11) with respect to Y, we have (2.7).

3. Weyl's projective curvature tensor

Theorem 3.1. *On quasi - Einstein almost hyperbolic Hermitian manifold with quasi-constrant curvature, the Wely's projective curvature tensor satisfies*

$$^*W(\bar{Y},\bar{Z}) + {}^*W(Y,Z) = b(n-2)\{A(\bar{Y})A(\bar{Z}) + A(Y)A(Z)\} \\ \textit{where } {}^*W(Y,Z) = (\textit{trace } w)(Y,Z). \tag{3.1}$$

Proof. From [4], we have

$$W(X,Y,Z) = K(X,Y,Z) + \frac{1}{(n-1)}\{Ric(X,Z)Y - Ric(Y,Z)X\}. \tag{3.2}$$

And from (1.3), we have

$$K(X,Y,Z) = a[g(Y,Z)X - g(X,Z)Y] + b[g(Y,Z)A(X)U \\ -g(X,Z)A(Y)U + A(Y)A(Z)X - A(X)A(Z)Y]. \tag{3.3}$$

Using (3.3) and (1.5) in (3.2), we have

$$W(X,Y,Z) = \left\{\frac{a(n-2)}{(n-1)}\right\}[g(Y,Z)X - g(X,Z)Y] + \left\{\frac{b(n-2)}{(n-1)}\right\}[A(Y)A(Z)X - A(X)A(Z)Y] + b[g(Y,Z)A(X)U - g(X,Z)A(Y)U] \quad (3.4)$$

Let

$$^{*}W(Y,Z) = (trac\, w)(Y,Z). \quad (3.5)$$

Contracting (3.4) with respect to X and using (3.5), we have

$$^{*}W(Y,Z) = [a(n-2)+b]g(Y,Z) + b(n-2)A(Y)A(Z) \quad (3.6)$$

and

$$^{*}W(\bar{Y},\bar{Z}) = [a(n-2)+b]g(\bar{Y},\bar{Z}) + b(n-2)A(\bar{Y})A(\bar{Z}). \quad (3.7)$$

From (3.6) and (3.7) with (1.2) we have (3.1).

Theorem 3.2. *On quasi - Einstein almost hyperbolic Hermitian manifold with quasi-constrant curvature, the Wely's projective curvature tensor satisfies*

$$W(\bar{X},\bar{Y},\bar{Z}) = -(W(\bar{X},Y,Z) + W(X,\bar{Y},Z) + W(X,Y,\bar{Z})) \quad (3.8)$$

and

$$W(X,Y,Z) = -(W(X,\bar{Y},\bar{Z}) + W(\bar{X},Y,\bar{Z}) + W(\bar{X},\bar{Y},Z)). \quad (3.9)$$

If

$$A(Y)A(Z)\bar{X} + A(\bar{Y})A(Z)X + A(Y)A(\bar{Z})X = A(\bar{Y})A(\bar{Z})X. \quad (3.10)$$

Proof. From (3.4), we have (1.2)

$$W(\bar{X},\bar{Y},\bar{Z}) = \left\{\frac{a(n-2)}{(n-1)}\right\}[-g(Y,Z)\bar{X} + g(X,Z)\bar{Y}] + \left\{\frac{b(n-2)}{(n-1)}\right\}[A(\bar{Y})A(\bar{Z})\bar{X} - A(\bar{X})A(\bar{Z})\bar{Y}] + b[-g(Y,Z)A(\bar{X})U + g(X,Z)A(\bar{Y})U]. \quad (3.11)$$

And also we have

$$W(\bar{X},Y,Z) = \left\{\frac{a(n-2)}{(n-1)}\right\}[g(Y,Z)\bar{X} + g(\bar{X},Z)Y] + \left\{\frac{b(n-2)}{(n-1)}\right\}[A(Y)A(Z)\bar{X} - A(\bar{X})A(Z)Y] + b[g(Y,Z)A(\bar{X})U - g(\bar{X},Z)A(Y)U] \quad (3.12)$$

$$W(X,\bar{Y},Z) = \left\{\frac{a(n-2)}{(n-1)}\right\}[g(\bar{Y},Z)X - g(X,Z)\bar{Y}] + \left\{\frac{b(n-2)}{(n-1)}\right\}[A(\bar{Y})A(Z)X - A(X)A(Z)\bar{Y}] + b[g(\bar{Y},Z)A(X)U - g(X,Z)A(\bar{Y})U] \quad (3.13)$$

and

$$W(X,Y,\bar{Z}) = \left\{\frac{a(n-2)}{(n-1)}\right\}[g(Y,\bar{Z})X + g(X,\bar{Z})Y] + \left\{\frac{b(n-2)}{(n-1)}\right\}[A(Y)A(\bar{Z})X - A(X)A(\bar{Z})Y] + b[g(Y,\bar{Z})A(X)U - g(X,\bar{Z})A(Y)U]. \quad (3.14)$$

Adding (3.12). (3.13), (3.14) and using (3.10) with (1.1) with (1.2) i.e. $['F(X,Y) + 'F(Y,X) = 0 = g(\bar{X},Y) + (X,\bar{Y})]$, we get (3.8) and from (3.8), we get (3.9).

4. Concircular curvature tensor

Theorem 4.1. *On quasi - Einstein almost hyperbolic Hermitian manifold with quasi-constant curvature, the concircular curvature tensor satisfies*

$$C(\bar{X},\bar{Y},\bar{Z}) = -(C(\bar{X},Y,Z) + C(X,\bar{Y},Z) + C(X,Y,\bar{Z})) \tag{4.1}$$

and

$$C(X,Y,Z) = -(C(X,\bar{Y},\bar{Z}) + C(\bar{X},Y,\bar{Z}) + C(\bar{X},\bar{Y},Z)) \tag{4.2}$$

provided (3.10) *holds.*

Proof. From [4], we have

$$C(X,Y,Z) = K(X,Y,Z) + \left[\frac{r}{n(n-1)}\right]\{Ric(Y,Z)x - Ric(X,Z)Y\}. \tag{4.3}$$

On contracting (1.5) with respect to X and Y respectively, we have

$$r = na + b \tag{4.4}$$

Using (3.3) and (3.4) in (4.3), we have

$$C(X,Y,Z) = a[g(Y,Z)X - g(X,Z)Y] + \left\{\frac{na+b}{n(n-1)}\right\}[g(Y,Z)X - g(X,Z)Y]$$
$$+b[g(Y,Z)A(X)U - g(X,Z)A(Y)U + A(Y)A(Z)X - A(X)A(Z)Y]. \tag{4.5}$$

From (4.5), we have

$$C(X,Y,Z) = \left\{\frac{a(n-2)}{(n-1)}\right\}[g(Y,Z)X - g(X,Z)Y] + b[g(Y,Z)A(X)U - g(X,Z)A(Y)U$$
$$+A(Y)A(Z)X - A(X)A(Z)Y - g(Y,Z)\bar{X} - g(X,Z)\bar{Y}]. \tag{4.6}$$

From (4.6) with (1.2), we have

$$C(\bar{X},\bar{Y},\bar{Z}) = \left\{\frac{a(n-2)}{(n-1)}\right\}[-g(Y,Z)\bar{X} + g(X,Z)\bar{Y}] + b[-g(Y,Z)A(\bar{X})U + g(X,Z)A(\bar{Y})U$$
$$+A(\bar{Y})A(\bar{Z})\bar{X} - A(\bar{X})A(\bar{Z})\bar{Y} + g(Y,Z)\bar{X} - g(X,Z)\bar{Y}] \tag{4.7}$$

and also replacing X, Y, Z by $\bar{X}, \bar{Y}, \bar{Z}$ respectively, we have

$$C(\bar{X},Y,Z) = \left\{\frac{a(n-2)}{(n-1)}\right\}[g(Y,Z)\bar{X} + g(\bar{X},Z)Y] + b[g(Y,Z)A(\bar{X})U - g(\bar{X},Z)A(Y)U$$
$$+A(Y)A(Z)\bar{X} - A(\bar{X})A(Z)Y - g(Y,Z)\bar{X} + g(\bar{X},Z)Y] \tag{4.8}$$

$$C(X,\bar{Y},Z) = \left\{\frac{a(n-2)}{(n-1)}\right\}[g(\bar{Y},Z)X - g(X,Z)\bar{Y}] + b[g(\bar{Y},Z)A(X)U - g(X,Z)A(\bar{Y})U$$
$$+A(\bar{Y})A(Z)X - A(X)A(Z)\bar{Y} - g(\bar{Y},Z)X + g(X,Z)\bar{Y}] \tag{4.9}$$

and

$$C(X,Y,\bar{Z}) = \Big\{\frac{a(n-2)}{(n-1)}\Big\}[g(Y,\bar{Z})X - g(X,\bar{Z})Y] + b[g(Y,\bar{Z})A(X)U - g(X,\bar{Z})A(Y)U$$
$$+A(Y)A(\bar{Z})X - A(X)A(\bar{Z})Y - g(Y,\bar{Z})X + g(X,\bar{Z})Y] \qquad (4.10)$$

Adding (4.8). (4.9) and (4.10) and using (4.7) with (1.1) and (1.2) i.e. $['F(X,Y) + 'F(Y,X) = 0 = g(\bar{X},Y) + (X,\bar{Y})]$, we get (4.1) and from (4.1), we get (4.2).

Theorem 4.2. *On quasi - Einstein almost hyperbolic Hermitian manifold with quasi-constant curvature, the concircular curvature tensor satisfies*

$$^*C(\bar{Y},\bar{Z}) + {}^*C(Y,Z) = b(n-2)\{A(\bar{Y})A(\bar{Z}) + A(Y)A(Z)\} \qquad (4.11)$$

where

$$^*C(Y,Z) = (trac\,C)(Y,Z)\}. \qquad (4.12)$$

Proof. Contracting (4.6) with respect to X and using (4.12), we have

$$^*C(Y,Z) = (n-2)[(a-b)g(Y,Z) + A(Y)A(Z)] \qquad (4.13)$$

and

$$^*C(\bar{Y},\bar{Z}) = (n-2)[(a-b)g(\bar{Y},\bar{Z}) + A(\bar{Y})A(\bar{Z})]. \qquad (4.14)$$

From (4.13) and (4.14) with (1.2) we have (4.11).

5. Conformal curvature tensor

Theorem 5.1. *On quasi - Einstein almost hyperbolic Hermitian manifold is conformally flat, and then the scalar curvature r satisfies*

$$r = \Big[\frac{na}{(n-1)} - \frac{(n-1)b}{(n-2)}\Big]. \qquad (5.1)$$

Proof. Conformal curvature tensor $V(X,Y,$ and $Z)$ is defined as [4]

$$V(X,Y,Z) = K(X,Y,Z) - \frac{1}{(n-1)}\{Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)RX$$
$$-g(X,Z)RY\} - \Big[\frac{r}{(n-1)(n-2)}\Big]\{g(Y,Z)X - g(X,Z)Y\}. \qquad (5.2)$$

Let

$$V(X,Y,Z) = 0. \qquad (5.3)$$

Then we have

$$K(X,Y,Z) = \frac{1}{(n-1)}\{Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)RX - g(X,Z)RY\}$$
$$-\Big[\frac{r}{(n-1)(n-2)}\Big]\{g(Y,Z)X - g(X,Z)Y\}. \qquad (5.4)$$

Using (1.1) and (4.4) in (5.4), we have

$$K(X,Y,Z) = \frac{a}{(n-1)}[g(Y,Z)X - g(X,Z)Y] + \frac{b}{(n-1)(n-2)}[(n-1)\{g(Y,Z)A(X)U - g(X,Z)A(Y)U + A(Y)A(Z)X - A(X)A(Z)Y\} - \{g(Y,Z)X - g(X,Z)Y\}]. \tag{5.5}$$

From (5.5). we have

$$K(\bar{X},\bar{Y},Z) = \frac{a}{(n-1)}[g(\bar{Y},Z)\bar{X} - g(\bar{X},Z)\bar{Y}] + \frac{b}{(n-1)(n-2)}[(n-1)\{g(\bar{Y},Z)A(\bar{X})U - g(\bar{X},Z)A(\bar{Y})U + A(\bar{Y})A(Z)\bar{X} - A(\bar{X})A(Z)\bar{Y}\} - \{g(\bar{Y},Z)\bar{X} - g(\bar{X},Z)\bar{Y}\}]. \tag{5.6}$$

From (5.5) and (5.6), using (2.4), we have

$$\begin{aligned}
&\frac{a}{(n-1)}[g(Y,Z)X - g(X,Z)Y] + \frac{b}{(n-1)(n-2)}[(n-1)\{g(Y,Z)A(X)U \\
&\quad - g(X,Z)A(Y)U + A(Y)A(Z)X - A(X)A(Z)Y\} - \{g(Y,Z)X - g(X,Z)Y\}] \\
&\quad = \frac{a}{(n-1)}[g(\bar{Y},Z)\bar{X} - g(\bar{X},Z)\bar{Y}] + \frac{b}{(n-1)(n-2)}[(n-1)\{g(\bar{Y},Z)A(\bar{X})U \\
&\quad - g(\bar{X},Z)A(\bar{Y})U + A(\bar{Y})A(Z)\bar{X} - A(\bar{X})A(Z)\bar{Y}\} - \{g(\bar{Y},Z)\bar{X} - g(\bar{X},Z)\bar{Y}\}].
\end{aligned} \tag{5.7}$$

On contracting (5.7) with respect to X, we have

$$ag(Y,Z) + bA(Y)A(Z) = \frac{a}{(n-1)}g(Y,Z) + \frac{b}{(n-2)}[A(\bar{Y})A(\bar{Z}) + A(Y)A(Z) - g(Y,Z)] \tag{5.8}$$

Again contracting (5.8) with respect to Y, we get (5.1).

Theorem 5.2. *If quasi - Einstein almost hyperbolic Hermitian manifold with quasi-constant curvature is conformally flat, and satisfy* (3.10)*, then curvature tensor K satisfies*

$$K(\bar{X},\bar{Y},\bar{Z}) = -(K(\bar{X},Y,Z) + K(X,\bar{Y},Z) + K(X,Y,\bar{Z})). \tag{5.9}$$

Proof. From (5.5) with (1.2), we have

$$\begin{aligned}
K(\bar{X},\bar{Y},\bar{Z}) &= \frac{a}{(n-1)}[-g(Y,Z)\bar{X} + g(X,Z)\bar{Y}] \\
&\quad + \frac{b}{(n-1)(n-2)}[(n-1)\{-g(Y,Z)A(\bar{X})U + g(X,Z)A(\bar{Y})U + A(\bar{Y})A(\bar{Z})\bar{X} \\
&\quad - A(\bar{X})A(\bar{Z})\bar{Y}\} - \{-g(Y,Z)\bar{X} + g(X,Z)\bar{Y}\}].
\end{aligned}$$

And also we have

$$K(\bar{X},Y,Z)=\frac{a}{(n-1)}[g(Y,Z)\bar{X}+g(\bar{X},Z)Y]+\frac{b}{(n-1)(n-2)}[(n-1)\{g(Y,Z)A(\bar{X})U$$
$$+g(\bar{X},Z)A(Y)U+A(Y)A(Z)\bar{X}-A(\bar{X})A(Z)Y\}-\{g(Y,Z)\bar{X}+g(\bar{X},Z)\bar{Y}\}] \quad (5.10)$$

$$K(X,\bar{Y},Z)=\frac{a}{(n-1)}[g(\bar{Y},Z)X+g(X,Z)\bar{Y}]+\frac{b}{(n-1)(n-2)}[(n-1)\{g(\bar{Y},Z)A(X)U$$
$$-g(X,Z)A(\bar{Y})U+A(\bar{Y})A(Z)X-A(X)A(Z)\bar{Y}\}-\{g(\bar{Y},Z)\bar{X}+g(X,Z)\bar{Y}\}] \quad (5.11)$$

$$K(X,Y,\bar{Z})=\frac{a}{(n-1)}[g(Y,\bar{Z})X-g(X,\bar{Z})Y]+\frac{b}{(n-1)(n-2)}[(n-1)\{g(Y,\bar{Z})A(X)U$$
$$-g(X,\bar{Z})A(Y)U+A(Y)A(\bar{Z})X-A(X)A(\bar{Z})Y\}-\{g(Y,\bar{Z})\bar{X}-g(X,\bar{Z})Y\}]. \quad (5.12)$$

Adding (5.10), (5.11), (5.12) and using (5.10) with (3.10)and with (1.1) with (1.2) i.e. $[{}'F(X,Y)+{}'F(Y,X)=0=g(\bar{X},Y)+(X,\bar{Y})]$, we get (5.9).

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